The Chowla-Selberg formula. A historical sketch.

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Introduction

The Chowla-Selberg formula is an identity, which computes certain elliptic integrals in terms of special values of Euler's Gamma function.

A (complete) *elliptic integral* (of the first kind) is an integral of the type

$$K(k) := \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2(\theta)}} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

where k is a real number. A related (but less interesting) integral is the (complete elliptic) integral (of the second kind)

$$T(k) := \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2(\theta)}\mathrm{d}\theta = \int_0^1 \sqrt{(1-t^2)(1-k^2t^2)}\mathrm{d}t$$

Examples

- If $k = \sqrt{1 - b^2/a^2}$ then $4aT(k)$ is

$$4a \int_0^{\pi/2} \sqrt{1 - (1 - b^2/a^2)\sin^2(\theta)}\,\mathrm{d}\theta = 4 \int_0^{\pi/2} \sqrt{a^2 \sin^2(x) + b^2 \sin^2(x)}\,\mathrm{d}x$$

This is the arclength of the *ellipse* with (semi-) minor and major axes a and b.

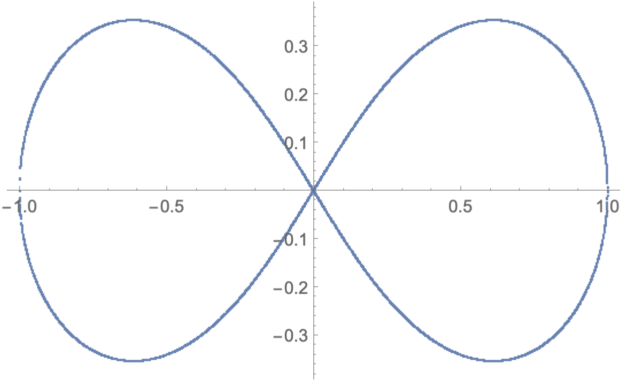
- If $k = 1/\sqrt{2}$, then

$$2\sqrt{2}K(k) = 2\sqrt{2} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - \frac{1}{2}\sin^2(\theta)}}\,\mathrm{d}\theta$$

is the arclength of the Bernoulli *lemniscate* given in polar coordinates by

$$r^2 = cos(2\theta)$$

Bernoulli Lemniscate



The integral computing the length of the ellipse was first considered by Wallis in 1655, who wrote down a power series development for it as he was unable to compute it in elementary terms.

The lemniscate first appeared in 1694 in the work of the Bernoulli brothers, as a solution to a problem posed by Leibniz in 1689 (the problem of *paracentric isochrones*). It can be described as the locus

$$\{z \in \mathbb{C} \mid |z - 1/\sqrt{2}| \cdot |z + 1/\sqrt{2}| = 1/2\}$$

It is Jakob Bernoulli who dubbed the above curve above *lemniscate*, a word with a Greek root meaning "ribbon of wool".

The duplication formula

The change of variable $u = \sqrt{1-t^2}$ gives the identity

$$K(1/\sqrt{2}) := 2\sqrt{2}\int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-t^2/2)}} = 4\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^4}}$$

In this form, the lemniscatic integral was studied by G. Fagnano. In 1718, he proved the intriguing identity

$$\int_0^z \frac{\mathrm{d}t}{\sqrt{1-t^4}} = 2\int_0^u \frac{\mathrm{d}t}{\sqrt{1-t^4}}$$

valid for any two $z, u \in [0,1]$ such that $z = \frac{2u\sqrt{1-u^4}}{1+u^4}$.

Figure: G. Fagnano (1682-1766) lived in Sinigaglia, where he was a magistrate. Most of his work was in Euclidean geometry. He was elected to the Royal Society in 1723. His tombstone bears the inscription *Veritas Deo* ∞ *gloria*.

Note the analogy between this identity and the duplication formula satisfied by the arcsin function (which computes the arclength of the circle):

$$\arcsin(z) = \int_0^z \frac{\mathrm{d}t}{\sqrt{1-t^2}} = 2\int_0^u \frac{\mathrm{d}t}{\sqrt{1-t^2}} = 2\arcsin(u) \quad (*)$$

valid if $z = 2u\sqrt{1-u^2}$ and $u \leq 1/\sqrt{2}$.

In both cases, z and u are related by a *polynomial* relation.

This theme was picked up by Euler, who was sent a copy of Fagnano complete works in 1751.

This led to Euler's *addition formula* for elliptic integrals of the first kind, which in particular generalises Fagnano's identity to any $k \in (0, 1)$.

The theory of elliptic functions

Legendre continued the study of elliptic integrals in his treatises *Exercices de calcul intégral* (published from 1811 to 1819) and *Traité des fonctions elliptiques et des intégrales Euleriennes* (published from 1825 to 1828).

Abel also turned to the study of elliptic integrals around 1827 and he was the first one to consider the *inverse* $\mathrm{sn}(z)$ of the function

$$z \mapsto \int_0^z \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

This inverse – the *elliptic sin* – turns out to be meromorphic on the complex plane, and it is doubly periodic if $k \neq 0$. It satisfies the differential equation

$$\left(\frac{\mathrm{d}\,\mathrm{sn}(z)}{\mathrm{d}z}\right)^2 = (1-\mathrm{sn}(z)^2)(1-k^2\mathrm{sn}(z)^2).$$

Jacobi studied these functions in depth in 1829 in his treatise *Fundamenta nova theoriae functionum ellipticarum* and a different presentation of the theory was given by Weierstrass in the 1850s.

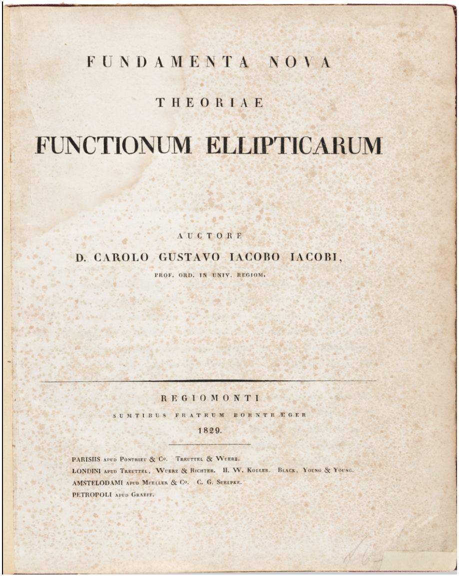
The upshot of this theory is that

• The elliptic integral $K(k)$ is associated with the algebraic curve $y^2 = (1 - x^2)(1 - k^2x^2)$.

• If $k \in (0, 1)$ this (projectivised) curve can be identified via the elliptic sin and its derivative with quotients $\mathbb{C}/\Lambda$, where $\Lambda := [4K(k), 2iK(\sqrt{1 - k^2})]$ is a lattice.

• For $k = 0$ (resp. $k = 1$) the elliptic sin reduces to the trigonometric (resp. hyperbolic) sin function.

• The duplication formula proved by Fagnano and generalised by Euler is the algebraic translation of the additive group law of $\mathbb{C}/\Lambda$ on the curve $y^2 = (1 - x^2)(1 - k^2x^2)$.



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SUMTIBUS FRATRUM BORNTRÆGER

1829.

PARISIIS APUD PONTHIEU & Co. TREUTTEL & WUERZ.
LONDINI APUD TREUTTEL, WUERZ & RICHTER. H. W. KOLLER. BLACK, YOUNG & YOUNG.
AMSTELODAMI APUD MUELLER & Co. C. G. SUELPKE.
PETROPOLI APUD GRAEFF.

Figure: Jacobi's treatise on elliptic functions is still in use now. Although it is the theory of Weierstrass which is now taught in most undergraduate courses, the theory of Jacobi is better suited for numerical computations.

The evaluation of elliptic integrals

A. M. Legendre seems to have been the first one to attempt to relate the elliptic integrals $K(k)$ to other known integrals for some specific values of k.

In the treatises mentioned above, he described in particular the following computation for the length of the lemniscate:

$$\frac{1}{4}K(1/\sqrt{2}) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^4}} = \frac{1}{4}\int_0^1 \frac{\mathrm{d}u}{u^{3/4}(1-u)^{1/2}} =: \beta(1/4,1/2)$$

$$= \quad \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = \frac{\Gamma(1/4)^2}{4\sqrt{2}\pi}$$

The 2nd equality arises from the change of variable $t^4 = u$. The function

$$\beta(z_1,z_2) = \int_0^1 u^{z_1-1}(1-u)^{z_2-1}\,\mathrm{d}u$$

is Euler's *beta integral*.

The function $\beta(z_1, z_2)$ satisfies the identity (due to Euler)

$$\beta(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}$$

where $\Gamma(\cdot)$ is Euler's well-known *Gamma function*

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t.$$

The formula for the length of the lemniscate

$$\frac{1}{4}K(1/\sqrt{2}) = \frac{1}{4}\int_0^1 \frac{\mathrm{d}u}{u^{3/4}(1-u)^{1/2}} = \frac{\Gamma(1/4)^2}{4\sqrt{2}\pi}$$

can be understood as an analogue of the identity

$$4K(0) = 4\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}} = 2\pi$$

computing the length of the circle.

Legendre also provided an ingenious proof of the equalities

$$K(\sin(\frac{\pi}{12})) = K(\frac{\sqrt{2-\sqrt{3}}}{2}) = \frac{\sqrt[4]{3}\sqrt{\pi}}{6}\frac{\Gamma(1/6)}{\Gamma(2/3)} = \frac{1}{2\sqrt[4]{3}}\beta(1/3, 1/2)$$

using Cauchy's integral formula.

Until the late 19th century, the computations of Legendre were the only available evaluations of special values of $K(k)$.

Is there something specific about the values $k = 1/\sqrt{2}$ and $k = \sin(\frac{\pi}{12})$?

Complex Multiplication

We note that the lattice $\Lambda_{1/\sqrt{2}} = [1, 2i]$ associated with $k = 1/\sqrt{2}$ is contained in the lattice $\Lambda = \mathbb{Z}[i] = [1, i]$ of the Gaussian integers.

Also, the lattice $\Lambda_{\sin(\pi/12)} = [1, 2i\sqrt{3}]$ associated with $k = \sin(\pi/12)$ has the property that

$$4 \cdot [1, \frac{1}{2} + i\frac{\sqrt{3}}{2}] \subseteq [1, 2i\sqrt{3}]$$

where $[1, \frac{1}{2} + i\frac{\sqrt{3}}{2}] = [1, e^{2i\pi/3}]$ is the ring of integers of the field $\mathbb{Q}(\mu_3) = \mathbb{Q}(e^{2i\pi/3})$.

In particular, both lattices are proportional to lattices invariant under multiplication by certain complex numbers, namely i and $e^{2i\pi/3}$.

This implies that the corresponding elliptic curves $\mathbb{C}/\Lambda_\bullet$ have symmetries different from the multiplication by n maps arising from the group structure.

Elliptic curves with exotic symmetries are said to have *complex multiplication* or *CM* for short.

In Legendre's computations, the fact that the elliptic curves have CM plays a key role.

Also, in both computations, one first obtains a Euler beta function, before the final evaluation via Gamma functions.

This suggests the following:

• There should be a formula for $K(k)$ in terms of special values of the Gamma function, whenever k is associated with a CM elliptic curve.

• It should be possible to obtain this formula by elementary manipulations, leading to certain beta integrals first.

However, nobody seems to have suspected the existence of such a formula (or noticed the existence of the supplementary symmetries) before the end of the 19th century.

The formula of M. Lerch

In his article *Sur quelques formules relatives au nombre de classe* (1897), the Czech mathematician Matyas Lerch states the following general formula, now known as the *Chowla-Selberg formula.*

Let $d > 0$ be a square free integer and suppose for simplicity that $d - 3$ is divisible by 4. Suppose also for simplicity that the class group of $\mathbb{Q}(\sqrt{-d})$ is 1. Then we have

$$|\eta(\frac{1+i\sqrt{d}}{2})|^4 = \frac{1}{2\pi d}\prod_{j=1}^{d-1}\Gamma(\frac{j}{d})^{(\frac{-d}{j})} \quad (*)$$

where

$$\eta(z) = e^{i\pi z/12}\prod_{n=1}^{\infty}(1 - e^{2ni\pi z})$$

is the Dedekind η-function (where $\mathrm{Im}(z) > 0$).

Figure: Matyas Lerch (1860-1922) was a Czech mathematician who grew up in Susice (Bohemia), studied in Germany and became a professor in Fribourg (Switzerland) in 1896. His was mainly interested in Gauss's theory of quadratic forms. The Lerch zeta functions, which generalise the Riemann zeta function, are named after him, and (amusingly) these functions can be used as an alternative to the logarithm of the Gamma function in the derivation of the CS formula, although he was apparently not aware of this.

The link with elliptic integrals is the following.

• If the elliptic curve $y^2 = (1-x^2)(1-k^2x^2)$ is associated with a lattice proportional to $[1,(1+i\sqrt{d})/2]$ (= ring of integers of $\mathbb{Q}(\sqrt{-d})$), then k is an algebraic number and we have

$$K(k)/K(\sqrt{1-k^2}) \in \mathbb{Q}(\sqrt{-d}).$$

• In that case, we have

$$K(k) = (\text{algebraic number}) \cdot |\eta(\frac{1+i\sqrt{d}}{2})|^2.$$

and thus

$$K(k) = (\text{algebraic number}) \cdot \frac{1}{\sqrt{2\pi d}} \prod_{j=1}^{d-1} \Gamma(\frac{j}{d})^{(\frac{-d}{j})/2}$$

The proof given by Lerch of $(*)$ is based on two ingredients:

(1) The Kronecker limit formula, which shows that the derivative $L'(1)$ of the Dirichlet L-function of $\mathbb{Q}(\sqrt{-d})$ at 1 can be related to $|\eta(\frac{1+i\sqrt{d}}{2})|$.

(2) A formula of Kummer giving the Fourier development of the logarithm of the Γ-function, which can be used to relate $L'(1)$ to special values of the Gamma function.

Note that apparently *Lerch did not notice that* $|\eta(\frac{1+i\sqrt{d}}{2})|$ *could be related to an elliptic integral.*

This was first seen by E. Landau in 1902 and made explicit by S. Chowla and A. Selberg in 1949.

Lerch's version of the Chowla-Selberg formula

de sorte que la formule de Kronecker s'écrira

$$(23^*) \qquad \frac{P'(-\Delta)}{P(-\Delta)} + C - \log\Delta = \frac{2}{Cl(-\Delta)} \sum_{a,b,c} \log \frac{H(\omega_1)\,H(\omega_2)}{\sqrt{c}},$$

où

$$\omega_1 = \frac{-b + i\sqrt{\Delta}}{2c}, \qquad \omega_2 = \frac{b + i\sqrt{\Delta}}{c}.$$

Je vais prouver que la somme $P'(-\Delta)$ peut s'exprimer sous forme finie à l'aide de la transcendante gamma. Pour ce but j'emploie la formule de Kummer

$$(a) \quad \begin{cases} \log\Gamma(x) + \frac{1}{2}\log\frac{\sin x\pi}{\pi} + \left(x - \frac{1}{2}\right)(C + \log 2\pi) = \sum_{n=1}^{\infty} \frac{\log n}{n\pi} \sin 2n x\pi \\ (0 < x < 1), \end{cases}$$

et la formule également connue

$$(b) \qquad \sum_{h=1}^{\Delta-1} \left(\frac{-\Delta}{h}\right) \sin\frac{2hm\pi}{\Delta} = \left(\frac{-\Delta}{m}\right)\sqrt{\Delta}$$

qui a lieu pour les discriminants fondamentaux et pour un entier positif m quelconque. En posant dans la formule (a) $x = \frac{h}{\Delta}$, multipliant par $\left(\frac{-\Delta}{h}\right)$ et faisant la somme pour $h = 1, 2, \ldots, \Delta - 1$, on a d'abord

$$\sum_{h=1}^{\Delta-1}\left(\frac{-\Delta}{h}\right)\log\Gamma\left(\frac{h}{\Delta}\right) + \frac{1}{2}\sum_{h=1}^{\Delta-1}\left(\frac{-\Delta}{h}\right)\log\frac{\sin\frac{h\pi}{\Delta}}{\pi}$$
$$+ (C + \log 2\pi)\sum_{h=1}^{\Delta-1}\left(\frac{-\Delta}{h}\right)\left(\frac{h}{\Delta} - \frac{1}{2}\right) = \sum_{n=1}^{\infty}\frac{\log n}{n\pi}\sum_{h=1}^{\Delta-1}\left(\frac{-\Delta}{h}\right)\sin\frac{2nh\pi}{\Delta}.$$

En faisant usage des équations identiques

$$\sum_h \left(\frac{-\Delta}{h}\right) = 0, \qquad \sum_h \left(\frac{-\Delta}{h}\right)\log\sin\frac{h\pi}{\Delta} = 0,$$

le premier membre devient

$$\sum_{h=1}^{\Delta-1}\left(\frac{-\Delta}{h}\right)\log\Gamma\left(\frac{h}{\Delta}\right) + (C + \log 2\pi)\sum_{h=1}^{\Delta-1}\left(\frac{-\Delta}{h}\right)\frac{h}{\Delta},$$

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et le second se transforme, à l'aide de la formule (b), en série

$$\sqrt{\Delta}\sum_{n=1}^{\infty}\left(\frac{-\Delta}{n}\right)\frac{\log n}{n\pi} \quad \text{ou bien} \quad \frac{\sqrt{\Delta}}{\pi}P'(-\Delta).$$

En faisant usage de la formule (1) on a, par conséquent,

$$\frac{\sqrt{\Delta}}{\pi}P'(-\Delta)=\sum_{h=1}^{\Delta-1}\left(\frac{-\Delta}{h}\right)\log\Gamma\left(\frac{h}{\Delta}\right)-(C+\log 2\pi)\frac{2}{\tau}Cl(-\Delta)$$

ou bien

$$(24) \quad P'(-\Delta)=\frac{\pi}{\sqrt{\Delta}}\sum_{h=1}^{\Delta-1}\left(\frac{-\Delta}{h}\right)\log\Gamma\left(\frac{h}{\Delta}\right)-\frac{2\pi}{\tau\sqrt{\Delta}}(C+\log 2\pi)Cl(-\Delta).$$

L'équation (14)

$$P(-\Delta)=\frac{2\pi}{\tau\sqrt{\Delta}}Cl(-\Delta)$$

permet donc d'écrire

$$(25) \quad \frac{P'(-\Delta)}{P(-\Delta)}+C+\log 2\pi=\frac{\tau}{2\,Cl(-\Delta)}\sum_{h=1}^{\Delta-1}\left(\frac{-\Delta}{h}\right)\log\Gamma\left(\frac{h}{\Delta}\right).$$

Grâce à cette relation, la formule (23*) de Kronecker prend une forme plus simple

$$(26) \quad \frac{\tau}{2}\sum_{h=1}^{\Delta-1}\left(\frac{-\Delta}{h}\right)\log\Gamma\left(\frac{h}{\Delta}\right)=2\sum_{a,b,c}\log\left[\sqrt{\frac{2\Delta\pi}{c}}H(\omega_1)H(\omega_2)\right]$$

$$\left(\omega_1=\frac{-b+i\sqrt{\Delta}}{2c}, \qquad \omega_2=\frac{b+i\sqrt{\Delta}}{2c}\right),$$

que nos méthodes permettent d'ailleurs d'obtenir plus directement.

NOTE.

La formule (6) nous fournit une nouvelle propriété du signe de Legendre. Pour l'obtenir, je multiplie les deux membres par $e^{-ux}\,dx$ et j'intègre de zéro à l'infini; il vient

$$\frac{2m}{\tau}Cl(-\Delta)=-u\sum_{\alpha=1}^{\Delta-1}\left(\frac{-\Delta}{\alpha}\right)\int_0^{\infty}e^{-ux}E^*\left(x+\frac{\alpha m}{\Delta}\right)dx.$$

Figure: The CS formula is stated at the end of Lerch's paper. Lerch refers to an article in the *Mémoires de l'académie impériale de Prague* for the proof. I could not find this article, which was probably written in Czech.

The period 1950-present time

S. Chowla and A. Selberg announced their formula (identical to Lerch's) in 1949 and finally published a proof in 1967, in their paper *On Epstein's zeta-function.*

This formula is difficult to understand conceptually, because although it computes a large class of elliptic integrals, the computation does not involve any elementary manipulations (= variable change, integration by parts, fundamental theorem of calculus) and in particular does not lead to β-integrals.

A. Weil was the first one to attempt to approach the formula via elementary manipulations. He provided a few pointers about this (but no general proof) in his 1976 paper *Sur les périodes des intégrales abéliennes.*

In particular, he gave a proof of the CS formula for an elliptic curve with CM by $\mathbb{Q}(\sqrt{-7})$ by relating it to the so-called Klein curve, where β-integrals can be obtained.

However, after a discussion with A. Weil, the Harvard mathematician B. Gross, devised a completely new proof of the CS formula, which involved a deformation argument.

His proof worked roughly as follows:

(1) construct a family of algebraic varieties, where one element of the family is a self-product of a CM elliptic curve, and another element is a piece of the Jacobian of a Fermat curve $x^n + y^n = z^z$; that this should be possible follows from the Kronecker-Weber theorem;

(2) show that a type of elliptic integral exists for all the elements of the family, and that these integrals are essentially constant in the family;

(3) compute the integral corresponding to the piece of the Jacobian of the Fermat curve; this turns out be a β-integral and can be identified with the right-hand side of the CS formula.

Figure: Benedict Gross, an American mathematician who has been a professor at Harvard since 1985, is mainly famous for his work with D. Zagier on the Birch and Swinnerton-Dyer conjecture.

B. Gross's proof is much closer to what A. Weil had in mind, but it still did not provide a completely elementary approach to the CS formula.

What was still missing? After some work by P. Deligne in the early 1980s, which was inspired by B. Gross's work, it became clear that this new proof suggested (but did not provide) the following solution.

(1) Any CM elliptic curve E has internal symmetries which look like some of the internal symmetries of some Fermat curve $x^n + y^n = z^n$, again by the Kronecker-Weber theorem.

(2) Replacing the Fermat curve by its Jacobian F_n, there should be subvarieties in the product $E \times F_n$, which are compatible with the symmetries on both sides. This would follow from a central conjecture in algebraic geometry, the so-called *Hodge conjecture.*

(3) These subvarieties encode elementary operations relating the elliptic integral on the first factor to the β-integrals on the second factor. So if the Hodge conjecture holds in this situation, we would finally have an elementary approach to the CS formula.

Epilogue

However, to this day, nobody knows how to prove the Hodge conjecture in this situation.

In fact, A. Weil proposed this as a challenge in the 1980s, and in a very recent article, B. Gross worked out precisely what subvarieties in $E \times F_n$ have to be exhibited.

B. Gross's proof also suggests that if *any* algebraic variety has a finite group of symmetries, then some integrals associated with it (the so-called *period integrals*) should be given by β-integrals. This was formalised in the *Gross-Deligne conjecture*.

A weak form of this statement was proven in 2004 by V. Maillot and the speaker, using more advanced deformation-theoretic methods.

I will end by explaining Deligne's construction of a motive of rank 2 and weight n with complex multiplication by an imaginary quadratic field k from an abelian variety of dimension n with endomorphisms by k (which is an abstraction of the geometric argument in my Chowla-Selberg paper). I will work this construction out for the abelian variety $B = B(p) = \mathrm{Res}_{H/k}\, A(p)$ of dimension $h(-p)$ as well as for a factor C of the Jacobian of the Fermat curve of exponent p, which has dimension $(p-1)/2$. The comparison of these two motives (they differ only by a Tate twist) yields a Hodge class in the middle cohomology of the product variety $B \times C$. In the simplest non-trivial case, when $p = 7$, B is the elliptic

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curve $A(7) = X_0(49)$ with affine equation

$$y^2 + xy = x^3 - x^2 - 2x - 1$$

and C is the Jacobian of the hyperelliptic curve of genus 3 with affine equation

$$z^2 - z = t^7.$$

In this case, the abelian variety $B \times C$ has dimension 4 and has a Hodge class of type $(2, 2)$. Is there a codimension 2 cycle on $B \times C$ which is defined over $\mathbb{Q}$ and has this class in cohomology?

Figure: Weil's challenge in a specific situation. Extract of B. Gross's article *On the periods of abelian varieties* (2018).

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