A local refinement of the Adams-Riemann-Roch theorem in degree one

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Abstract

We prove that the Adams-Riemann-Roch theorem in degree one (ie at the level of the Picard group) can be lifted to an isomorphism of line bundles, compatibly with base change.

1 Introduction

The aim of this text is to provide a proof of the following theorem.

Let B be a scheme.

Let $\mathcal{S}_{\rm line,B}$ be the category whose objects are pairs (S, M), where S is a locally noetherian B-scheme and where M is a line bundle (ie a locally free sheaf of rank one) on S. An arrow $(S', M') \to (S, M)$ in $\mathcal{S}_{\rm line,B}$ is a morphism of B-schemes $\phi : S' \to S$, together with an isomorphism $\phi^*(M) \cong M'$.

Let $\mathcal{S}_{{\rm rel,line},B}$ be the category, whose objects are pairs $(Y \to S, L)$, where $Y \to S$ is a smooth and locally projective morphism of B-schemes with geometrically connected fibres and constant relative dimension, S is a locally noetherian B-scheme and L is a line bundle on Y. An arrow $(Y' \to S', L') \to (Y \to S, L)$ in $\mathcal{S}_{{\rm rel,line},B}$ is a cartesian diagram of B-schemes

$$\begin{array}{ccc} Y' & \xrightarrow{\rho} & Y \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

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together with an isomorphism $\rho^*(L) \cong L'$.

If $(Y \to S, L)$ is an object of $\mathcal{S}_{\text{rel,line},B}$, we shall write $\dim(Y/S)$ for the dimension of some (and hence any) geometric fibre of the morphism $Y \to S$.

Recall that to say that $Y \to S$ is locally projective means that every point in S has an open neighbourhood U, such that there is a factorisation of $\pi|_U$ into a closed U-immersion $Y_U \to \mathbb{P}^N_U$ followed by projection to U, for some $N \geq 0$ which depends on U.

We let $\mathcal{S}_{\text{rel,line,cf},B}$ be the full subcategory of $\mathcal{S}_{\text{rel,line},B}$, which consists of those pairs

$$(\pi : Y \to S, L),$$

where L is cohomologically flat over S. Recall that to say that L is cohomologically flat over S means that $\mathrm{R}^i\pi_*(L)$ is a locally free sheaf for all $i \geq 0$.

If $\pi : Y \to S$ is a proper and flat morphism of locally noetherian schemes and F is a vector bundle (ie a coherent locally free sheaf) on Y, we shall write $\lambda(F) := \det(\mathrm{R}^\bullet\pi_*(F))$. Here $\det(\cdot)$ is the Knudsen-Mumford determinant of a perfect complex (note that $\mathrm{R}^\bullet\pi_*(F)$ is a perfect complex by the semicontinuity theorem because π is proper and flat). We shall denote by $\mathrm{Sym}^k(F)$ the k-th symmetric power of F and we shall write $F^\vee := \underline{\mathrm{Hom}}(F, \mathcal{O}_X)$ for the dual of F. If M is a line bundle on Y and $k \in \mathbb{Z}$, we define $M^{\otimes k} := \otimes_{i=1}^k M$ if $k \geq 0$ and $M^{\otimes k} := \otimes_{i=1}^{-k} M^\vee$ if $k < 0$. As is costumary, we shall write $\Omega_{Y/S} = \Omega_\pi$ for the sheaf of differentials of π.

Note that the rule, which associates the line bundle

$$\lambda(L)^{\otimes 2^{2\dim(Y/S)+2}}$$

with the object $(Y \to S, L)$ of $\mathcal{S}_{\text{rel,line},B}$, naturally defines a functor from $\mathcal{S}_{\text{rel,line},B}$ to $\mathcal{S}_{\text{line},B}$. We shall denote this functor LRR.

Similarly, the rule, which associates the line bundle

$$\bigotimes_{j=0}^{2\dim(Y/S)} \lambda(L^{\otimes 2} \otimes \mathrm{Sym}^j(\Omega_{Y/S}))^{\otimes(-1)^j \sum_{i=0}^{2\dim(Y/S)-j} \binom{2\dim(Y/S)+1}{i}}$$

with the object $(Y \to S, L)$ of $\mathcal{S}_{\text{rel,line},B}$, naturally defines a functor from $\mathcal{S}_{\text{rel,line},B}$ to $\mathcal{S}_{\text{line},B}$. We shall denote this functor RRR.

Theorem 1.1. *Suppose that $B = \mathrm{Spec}\,\mathbb{Z}[\frac{1}{2}]$. Then the restrictions of the functors LRR and RRR to $\mathcal{S}_{\text{rel,line,cf},B}$ are isomorphic.*

In other words, it is possible to associate with any locally projective and smooth morphism of locally noetherian $\mathbb{Z}[\frac{1}{2}]$-schemes $Y \to S$ and any line bundle L on Y, which is cohomologically *flat over S, an isomorphism*

$$\lambda(L)^{\otimes 2^{2\dim(Y/S)+2}} \cong \bigotimes_{j=0}^{2\dim(Y/S)} \lambda(L^{\otimes 2} \otimes \mathrm{Sym}^j(\Omega_{Y/S}))^{\otimes(-1)^j \sum_{i=0}^{2\dim(Y/S)-j} \binom{2\dim(Y/S)+1}{i}} \tag{1}$$

compatibly with base change to any locally noetherian scheme.

Remark 1.2. (1) We conjecture that the assumption that L is cohomologically flat over S is unnecessary. In other words, we conjecture that the functors LRR and RRR are isomorphic if $B = \mathrm{Spec}\,\mathbb{Z}[\frac{1}{2}]$ (and not only their restrictions to $\mathcal{S}_{\mathrm{rel,line,cf},B}$). Proving this boils down to a problem in the linear algebra of perfect complexes. See Remark 7.4 below for details.

(2) Note if S is a scheme of characteristic 0 then the trivial line bundle $\mathcal{O}_Y$ is cohomologically flat over S by a theorem of Deligne (see [4, Th. 5.5]).

(3) It is actually plausible that LRR and RRR are isomorphic if $B = \mathrm{Spec}\,\mathbb{Z}$ (this would generalise conjecture (1) above in this remark). This is suggested by Proposition 1.3 below and Deligne's theorem [3, Th. 9.9 (3)]. See the discussion after Proposition 1.3.

(4) Our construction of the isomorphism I between the restrictions of the functors LRR and RRR to $\mathcal{S}_{\mathrm{rel,line,cf},B}$ depends on a slew of arbitrary combinatorial choices. These choices are all contained in the proof of Lemma 4.1 below. One might conjecture that, up to sign, the isomorphism I does not depend on these choices but proving this seems to be a formidable task. Presumably it is possible to show that there is only one isomorphism I, up to sign, provided it satisfies some axiomatic conditions. It would be very interesting to determine such conditions.

For example, suppose that $\dim(Y/S) = 1$. We then get an isomorphism

$$\lambda(L)^{\otimes 16} \cong \lambda(L^{\otimes 2})^{\otimes 7} \otimes \lambda(L^{\otimes 2} \otimes \Omega_{Y/S})^{\otimes(-4)} \otimes \lambda(L^{\otimes 2} \otimes \Omega_{Y/S}^{\otimes 2}) \tag{2}$$

In particular, writing $\lambda_k := \lambda(\Omega_{Y/S}^{\otimes k})$ for any $k \geq 0$, (2) gives

$$\lambda_k^{\otimes 16} \cong \lambda_{2k}^{\otimes 7} \otimes \lambda_{2k+1}^{\otimes(-4)} \otimes \lambda_{2k+2}.$$

By Grothendieck duality, there is a canonical isomorphism $\lambda_0 \cong \lambda_1$. Thus, setting $k = 0$ we obtain an isomorphism

$$\lambda_1^{\otimes 13} \cong \lambda_2. \tag{3}$$

In [26] Mumford also constructs such an isomorphism and also proves that it is invariant under base change (and he does not need the assumption that 2 is invertible on S). Our isomorphism presumably coincides with his up to a universal constant of the form $\pm 2^k$ ($k \in \mathbb{Z}$) but we did not verify this.

Suppose that $\pi : Y \to S$ is an elliptic scheme (ie an abelian scheme of relative dimension 1) over S. We then have a canonical isomorphism $\Omega_{Y/S}^{\otimes k} \cong \pi^*(\pi_*(\Omega_{Y/S}^{\otimes k}))$ for any $k \in \mathbb{Z}$. Furthermore, we have

$$\mathrm{R}^1\pi_*(\mathcal{O}_{Y/S}) \cong \pi_*(\Omega_{Y/S})^\vee$$

by Grothendieck duality. Using the projection formula, we can thus compute

$$\lambda_k = \det((\mathcal{O}_S - \mathrm{R}^1\pi_*(\mathcal{O}_{Y/S})) \otimes \pi_*(\Omega_{Y/S})^{\otimes k}) = \det(\pi_*(\Omega_{Y/S})^{\otimes k} - \pi_*(\Omega_{Y/S})^{\otimes(k-1)}) \cong \pi_*(\Omega_{Y/S})$$

for all $k \geq 0$. In particular, we are provided with an isomorphism $(\pi_*(\Omega_{Y/S}))^{\otimes 12} \cong \mathcal{O}_S$. Again, possibly up to multiplication by a term of the form $\pm 2^k$ ($k \in \mathbb{Z}$), this is presumably the classical discriminant modular form (but we did not verify this). This suggests that the isomorphism in Theorem 1.1 is in some sense optimal.

When Y is an elliptic scheme over S and L is a non trivial torsion line bundle, whose order is prime to the characteristic of all the residue fields of S, then $\mathrm{R}^\bullet\pi_*(L) = 0$. In that case, both sides of (1) are canonically isomorphic to the trivial line bundle. Thus the isomorphism (1) provides an element of $\Gamma(S, \mathcal{O}_S^*)$, in other words an elliptic unit. It seems likely that one can construct all the Siegel units in this way but to prove this, one will have probably have to wait for a metric version of Theorem 1.1. See below for a discussion.

Returning to the general situation, recall that if S is of characteristic 0, the trivial sheaf $\mathcal{O}_Y$ is cohomologically flat over S by a result of Deligne. Let us suppose that S is of characteristic 0 and $\dim(Y/S) = 2$. We then get the isomorphism

$$\lambda(\mathcal{O}_Y)^{\otimes 64} \cong \lambda(\mathcal{O}_Y)^{\otimes 31} \otimes \lambda(\Omega_{Y/S})^{\otimes(-26)} \otimes \lambda(\mathrm{Sym}^2(\Omega_{Y/S}))^{\otimes 16} \otimes \lambda(\mathrm{Sym}^3(\Omega_{Y/S}))^{\otimes(-6)} \otimes \lambda(\mathrm{Sym}^4(\Omega_{Y/S})).$$

from Theorem 1.1. This is equivalent to

$$\lambda(\mathcal{O}_Y)^{\otimes 33} \otimes \lambda(\Omega_{Y/S})^{\otimes 26} \otimes \lambda(\mathrm{Sym}^3(\Omega_{Y/S}))^{\otimes 6} \cong \lambda(\mathrm{Sym}^2(\Omega_{Y/S}))^{\otimes 16} \otimes \lambda(\mathrm{Sym}^4(\Omega_{Y/S})).$$

and there are similar identities in any relative dimension.

Here is our method of proof. We first give a proof of the geometric fixed formula for an involution, which avoids any reference to K-theory and uses only the geometric properties of quotients. This is Theorem 6.1, which is of independent interest. The idea to use quotients to prove the fixed point formula is due to Thomason (see [30]) and most probably many earlier authors but our proof relies on the crucial fact that when the fixed point scheme is a Cartier divisor then the quotient morphism is flat. This seems to be a well known fact (J. Oesterlé kindly explained the proof to me many years ago) but we could find no proof of it in the literature in the required generality and we provide one in Proposition 2.5 (1). Our proof of the geometric fixed point formula is sufficiently explicit to provide isomorphisms at every step (rather than equalities in the Picard group) but ends with an error term, which turns out to be a line bundle arising from a higher dimensional version of the Deligne pairing. This pairing was studied by Ducrot in [7] and we use his results to show that this line bundle is canonically trivial, compatibly with any base change to a locally noetherian scheme. We then apply this formula to the space $Y \times_S Y$ with the involution swapping the factors. Nori (see [27]) was apparently the first one to notice that the fixed point formula applied to this situation recovers the Adams-Riemann-Roch for the Adams operation ψ^2 and using our method we thus recover a refinement of this formula (in degree one). This is formula (1).

In [10] Eriksson gives a proof of a functorial refinement of the Adams-Riemann-Roch formula (see also [9] for an announcement), which can also be used to prove a weaker version of Theorem 1.1. It is weaker in the sense that the provided isomorphism, although invariant under base change, will include a 2^∞-torsion line bundle, which is undetermined and also because the resulting linear combination in the symmetric powers of $\Omega_{Y/S}$ will a priori depend on the dimension of the total space.

Similarly, using Franke's work in [11], it is possible to prove a weak version of Theorem 1.1, where an undetermined (not necessarily 2^∞) torsion line bundle will be included (but on the other hand the linear combination in the symmetric powers of $\Omega_{Y/S}$ should be the same as ours and should thus not depend on the dimension of the total space).

One interesting aspect of our result is thus that it removes this indeterminacy. However, the main interest of the present text is the method of proof, which is elementary (whereas Franke's and Eriksson's approaches require a vast categorical apparatus and use higher K-theory, resp. the homotopy theory of schemes). Our isomorphism is constructed very explicitly, making it in principle possible to compute its norm, when both sides are endowed with metrics (eg Quillen metrics). We hope to return to this question in a later article.

Note that other constructions of the higher dimensional Deligne pairing were given in [31] and [8] but they cannot be used in our context, because they are not described in terms of determinants of cohomology and therefore cannot easily be compared with our error term. In [1], a canonical isomorphism between Ducrot's pairing and Zhang's pairing is announced (in a restricted setting), which could be used to bypass the use of

Ducrot's pairing in certain situations. However, the details of the proof of Theorem 1 of [1] have not appeared yet (thank you to one of the referees for pointing this out). In [5] Ducrot's pairing is also considered.

Finally, note that in the situation where $\dim(Y/S) = 1$, Deligne also constructed an isomorphism similar to (1) (see [3]). Deligne's work was in fact the initial motivation for the work of Franke and Eriksson. Under the assumptions of Theorem 1.1 and when $\dim(Y/S) = 1$, Deligne's theorem [3, Th. 9.9 (3)] provides in particular an isomorphism

$$\lambda(L)^{\otimes 18} \cong \lambda(\mathcal{O}_Y)^{18} \otimes \lambda(L^{\otimes 2} \otimes \Omega_{Y/S}^{\vee})^{\otimes 6} \otimes \lambda(L \otimes \Omega_{Y/S}^{\vee})^{\otimes(-6)}, \tag{4}$$

which is invariant under any base change to a locally noetherian scheme (note that Deligne's theorem is expressed in terms of the Deligne pairing; Deligne's pairing can be expressed using the determinant of cohomology - see section 4 below - and (4) is the expression one obtains when using only the determinant of cohomology). This can be seen as a variant of the isomorphism (1) when $\dim(Y/S) = 1$ and Deligne shows that it holds even if 2 is not invertible on S and L is not cohomologically flat over S.

Using Theorem 1.1 for $\dim(Y/S) = 1$, we prove

Proposition 1.3. *Under the assumptions of Theorem 1.1 and when $\dim(Y/S) = 1$, there is an isomorphism*

$$\left(\lambda(L)^{\otimes 18}\right)^{\otimes 8} \cong \left(\lambda(\mathcal{O}_Y)^{18} \otimes \lambda(L^{\otimes 2} \otimes \Omega_{Y/S}^{\vee})^{\otimes 6} \otimes \lambda(L \otimes \Omega_{Y/S}^{\vee})^{\otimes(-6)}\right)^{\otimes 8}, \tag{5}$$

which is invariant under any base change to a locally noetherian scheme.

In other words, we give a new proof of Deligne's theorem, up to a torsion line bundle of order 8 (and under the running assumption that 2 is invertible on S and that L is cohomologically flat over S). The proof of Proposition 1.3 actually also shows that one can deduce Theorem 1.1 for $\dim(Y/S) = 1$ from Deligne's theorem, up to a torsion line bundle of order 9. Thus, when $\dim(Y/S) = 1$ and under the running assumption that 2 is invertible on S and that L is cohomologically flat over S, Theorem 1.1 and Deligne's theorem are equivalent up to torsion.

The structure of the article is as follows. In section 2 we recall various facts about quotients of schemes by finite groups and we prove various supplementary properties of these in the situation where the group is isomorphic to a diagonalisable group scheme, whose order is prime and invertible in the base scheme and the fixed point scheme is a Cartier divisor. In section 4 we recall the part of Ducrot's work that is relevant to this text. In section 6, we give a proof of a local refinement of the fixed formula for an involution, in the situation where the fixed scheme is regularly immersed. In section 7, we apply this formula to the fibre product of a relative scheme by itself and we prove Theorem 1.1. In the final section 8 we give the proof of Proposition 1.3. Note that the core of the proof of Theorem 1.1 amounts to a detailed analysis of the geometry of the blow-up along the diagonal of the relative fibre product of X with itself. This is intriguing, since this particular space was believed to be relevant to a possible solution of the standard conjectures in the early days of scheme theory. It would be interesting to relate our construction to statements about algebraic cycles.

Notation. We shall say that a morphism $h : Z \to T$ of schemes is strongly projective if there is a factorisation of h into a closed T-immersion $Z \to \mathbb{P}^N_T$ followed by projection to T, for some $N \geq 0$. The notion of locally projective morphism is defined at the beginning of the introduction. If Z is a locally noetherian scheme, we write $\mathrm{Coh}(Z)$ for the category of coherent sheaves on Z. If F is an $\mathcal{O}_Z$-module on a scheme Z and $l \geq 0$, we shall write $F^{\otimes l} := \bigotimes_{k=1}^{l} F$. If Z is a scheme, we write $D(Z)$ (resp. $D^b(Z)$) for the derived category of complexes of $\mathcal{O}_Z$-modules (resp. the derived category of bounded complexes of $\mathcal{O}_Z$-modules) on Z.

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2 The geometry of quotients by finite groups

Let G be a finite group.

A scheme T together with a group homomorphism $G \to \mathrm{Aut}(T)$ will be called a G-equivariant scheme, or an equivariant scheme for short (if there is no ambiguity). A G-equivariant morphism of G-equivariant schemes is a morphism commuting with the action of G on source and target. We shall say that the action of G on the G-equivariant scheme T is trivial if the image of $G \to \mathrm{Aut}(T)$ is the identity morphism.

A G-equivariant sheaf (or equivariant sheaf for short) F on a G-equivariant scheme is a quasi-coherent sheaf F together with a morphism of sheaves $\alpha_g = \alpha_{F,g} : F \to g_*(F)$ for every $g \in G$, such that $g_*(\alpha_h) \circ \alpha_g = \alpha_{g \circ h}$ for any $g, h \in G$ and $\alpha_{\mathrm{Id}_G} = \mathrm{Id}_F$.

Suppose that T is a G-equivariant scheme with trivial action and that F is a G-equivariant sheaf on T. The G-equivariant structure on F then amounts to a homomorphism of groups $G \to \mathrm{Aut}(F)$. We then write F^G for the quasi-coherent sheaf on T such that

$$F^G(U) = F(U)^G$$

for every open set $U \subseteq T$. Here $F(U)^G$ is the subgroup of elements of $F(U)$, which are fixed under the action of G.

Suppose that $\phi : T \to Z$ is a morphism of schemes, where T is locally noetherian. Assume also that T carries G-equivariant structure and that $\phi \circ g = \phi$ for all $g \in G$. Let F be a G-equivariant sheaf. Then the sheaf $\phi_*(F)$ is also quasi-coherent. Furthermore, if Z is viewed as a G-equivariant scheme carrying the trivial G-equivariant structure, then $\phi_*(F)$ carries the G-equivariant structure given for any $g \in G$ by the composition of arrows

$$\phi_*(F) \xrightarrow{\sim} \phi_*(g_*(F)) \xrightarrow{\sim} \phi_*(F)$$

arising from the equivariant structure on F and the identity $\phi \circ g = \phi$.

Suppose that $\phi : T \to Z$ is a morphism of schemes, that T carries a G-equivariant structure and that $\phi \circ g = \phi$ for all $g \in G$. View Z as a G-equivariant scheme endowed with the trivial G-equivariant structure. Let F be a G-equivariant sheaf on Z. Then the quasi-coherent sheaf $\phi^*(F)$ carries a natural G-equivariant structure, given for any $g \in G$ by the composition of arrows

$$\phi^*(F) \overset{\phi^*(g_*)}{\xrightarrow{\sim}} \phi^*(F) \xrightarrow{\sim} g^{-1,*}(\phi^*(F)) = g_*(\phi^*(F))$$

where the first arrow comes by functoriality from the arrow $g_*(F) \to g_*(F)$, the second arrow from the identity $\phi \circ g = \phi$ and the third arrow from the identification of functors $g^{-1,*} = g_*$.

If $x \in X$, then we define $G_d(x)$ to be the stabiliser in G of x viewed as a subset of X. This group is called the decomposition group of x. The group $G_d(x)$ naturally acts on the residue field $\kappa(x)$ of x. The kernel of the homomorphism $G_d(x) \to \mathrm{Aut}(\kappa(x))$ is called the inertia group $G_i(x)$ of x.

Suppose that X is a G-equivariant scheme. A (categorical) quotient X/G of X by G (if it exists) is a G-equivariant scheme X/G together with an G-equivariant morphism $q : X \to X/G$, with the following properties:

- X/G carries the trivial action;

- if X' is a scheme with a trivial G-action and $q' : X \to X'$ is a morphism then there is a unique morphism $h : X/G \to X'$, such that $h \circ q = q'$.

These properties clearly determine X/G up to unique isomorphism.

We recall the following

Proposition 2.1. *Let X be a G-equivariant scheme. Suppose that the orbit of every point in X is contained in an affine open subscheme. Then the quotient X/G of X by G exists and*

(1) *The canonical morphism $q : X \to X/G$ is integral and surjective.*

(2) *The natural morphism of sheaves $\mathcal{O}_{X/G} \to q_*(\mathcal{O}_X)$ factors through $(q_*(\mathcal{O}_X))^G$ and induces an isomorphism $\mathcal{O}_{X/G} \to (q_*(\mathcal{O}_X))^G$.*

(3) *The underlying set of X/G is the quotient of the set X by the action of G and the topology of X/G is the quotient topology.*

(4) *if $Z \to X/G$ is a flat morphism then the natural morphism $(Z \times_{X/G} X)/G \to Z$ is an isomorphism.*

(5) Consider the X/G-morphism $\phi : G \times X \to X \times_{X/G} X$ given in set-theoretic notation by the formula $(g, x) \mapsto (g(x), x)$. Suppose that ϕ is an isomorphism. Then

- q is étale;

- if M is a G-equivariant locally free sheaf of finite rank on X then the natural morphism $q^*(q_*M)^G \to M$ is an isomorphism.

(6) If $G_i(x) = 0$ then $\mathcal{O}_{X,x}$ is étale over $\mathcal{O}_{X/G,q(x)}$.

Proof. See [15, chap. V, §1 and §2]. □

Corollary 2.2. *Suppose that there is a morphism of finite type $f : X \to S$, where S is a locally noetherian scheme. Assume that the action of G on X factors through $\mathrm{Aut}_S(X)$ and that the orbit of every point in X is contained in an affine open subscheme. Then the quotient X/G of X by G exists and the morphism $q : X \to X/G$ is finite.*

Corollary 2.2 follows from the fact that under the listed assumptions, the quotient morphism is integral and of finite type and hence finite.

Suppose that X is a G-equivariant scheme. Suppose given a morphism $X \to S$ and assume that the action of G on X factors through $\mathrm{Aut}_S(X)$. We say that X is a G-equivariant S-scheme. The fixed scheme X_G (if it exists) is a closed subscheme of X, which represents the functor on S-schemes

$$T \mapsto X(T)^G.$$

Note the following link with decomposition and inertia groups: if $x \in X$ and

$$G_d(x) = G_i(x) = G$$

then $x \in X_G$. This simply follows from the fact that the morphism $\operatorname{Spec} \kappa(x) \to X$ then lies in $X(\operatorname{Spec} \kappa(x))^G$.

Proposition 2.3. *Suppose that X is separated over S. Then X_G exists.*

Proof. For each $g \in G$, let Γ_g be the graph of g in $X \times_S X$. Let $\Delta = \Gamma_{\mathrm{Id}_X} \cong X$ be the diagonal of X over S. From the separatedness assumption, each Γ_g is a closed subscheme of $X \times_S X$. The closed subscheme $X_G = \cap_{g \in G} \Gamma_g$ is naturally a closed subscheme of Δ and can thus be viewed as a closed subscheme of X. It follows from the definitions that X_G is the fixed scheme of G. $\square$

If X_G exists, we shall write $N_{X_G/X}$ for the conormal sheaf of X_G in X. Recall that if $\mathcal{I}$ is the ideal sheaf of X_G in X, we have by definition $N_{X_G/X} = \mathcal{I}/\mathcal{I}^2$. The sheaf $\mathcal{I}/\mathcal{I}^2$ has a natural structure of $\mathcal{O}_{X_G}$-module. The conormal sheaf $N_{X_G/X}$ is thus a quasi-coherent sheaf on X_G and it carries a natural action of G.

Suppose now that G is a finite cyclic group of order n. Let us write $\widetilde{G}$ for the group scheme over $\operatorname{Spec} \mathbb{Z}$ corresponding to G. Note that we then have a canonical identification $\widetilde{G}(\operatorname{Spec} \mathbb{Z}) \cong G$. Suppose now that $\widetilde{G}_S \cong \mu_{n,S}$, where $\mu_n = \operatorname{Spec} \mathbb{Z}[t]/(1 - t^n)$ is the diagonalisable group scheme associated with the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Note that there exists an isomorphism $\widetilde{G}_S \cong \mu_{n,S}$ is iff n is invertible in S and the polynomial $x^n - 1$ splits into linear factors in $\Gamma(S, \mathcal{O}_S)$. We fix an isomorphism $G_S \cong \mu_{n,S}$.

Note the following two facts.

Suppose in this paragraph only that $X = \operatorname{Spec} R$ is affine. Then the action of G on X is given by a ring grading $R \cong \oplus_{k \in \mathbb{Z}/n\mathbb{Z}} R_k$, such that the morphism $X \to S$ factors through $\operatorname{Spec} R_0$. Furthermore, the ideal of X_G is then $R \cdot R_{\neq 0}$, where

$$R_{\neq 0} := \oplus_{k \in \mathbb{Z}/n\mathbb{Z},\, k \neq 0} R_k.$$

See [30, proof of Prop. 3.1] (this is also a good exercise for the reader).

Suppose that the action of G on X is trivial. Let F be a G-equivariant sheaf on X. The G-equivariant structure on F is then given by a $\mathbb{Z}/n\mathbb{Z}$-grading of $\mathcal{O}_X$-modules

$$F \cong \oplus_{k \in \mathbb{Z}/n\mathbb{Z}} F_k.$$

Let $g \in G$. By the above, the element g gives an element of $\widetilde{G}(\operatorname{Spec} \mathbb{Z})$ and hence after base change an element $z \in G(S)$. Applying the isomorphism $G_S \cong \mu_{n,S}$ we obtain an element $z \in \mu_n(S)$. The action of g on F is then by construction given by the formula

$$g((f_0, f_1, \ldots, f_{n-1})) = (1 \cdot f_0, z \cdot f_1, \ldots, z^{n-1} \cdot f_{n-1}),$$

where f_k is a local section of F_k. In particular, we have $F_0 = F^G$.

We record the following

Lemma 2.4. *Let X be an G-equivariant S-scheme. Suppose that the orbit of every point in X is contained in an affine open subscheme. Assume that G is a finite cyclic group of order n and that $G_S \cong \mu_{n,S}$. If $Z \to X/G$ is a morphism then the natural morphism $(Z \times_{X/G} X)/G \to Z$ is an isomorphism.*

In other words, when $G_S \cong \mu_{n,S}$, the quotient construction commutes with any base change on X/G (not only flat base changes as in Proposition 2.1 (4)).

Proof. By Proposition 2.1 (4), we may assume that Z and X are affine, say $Z = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$. In this case, we have to prove that the morphism of A_0-modules

$$B \to (B \otimes_{A_0} A)_0$$

given by the formula $b \mapsto b \otimes 1$ is an isomorphism. We have

$$B \otimes_{A_0} A = B \otimes_{A_0} \bigoplus_{k \in \mathbb{Z}/n\mathbb{Z}} A_k = \bigoplus_{k \in \mathbb{Z}/n\mathbb{Z}} B \otimes_{A_0} A_k$$

so that $(B \otimes_{A_0} A)_0 = B \otimes_{A_0} A_0 = B$, proving the assertion. $\square$

The next proposition collects the main results of this section.

Proposition 2.5. *Suppose that X is a G-equivariant S-scheme such that S is locally noetherian and the morphism $X \to S$ is separated and of finite type. Assume that the orbit of every point in X is contained in an affine open subscheme. Finally, suppose that G is a finite cyclic group of order n and that $G_S \cong \mu_{n,S}$. Let $\iota : X_G \to X$ be the fixed point scheme of X. Then:*

(1) *Suppose that n is prime and that X_G is a (possibly empty) Cartier divisor. Then q is flat.*

(2) *Suppose that X_G is a Cartier divisor. Then $(N_{X_G/X})_0 = 0$.*

(3) *The morphism $q \circ \iota : X_G \to X/G$ is a closed immersion and we have the set-theoretic equality $q^{-1}(q(X_G)) = X_G$. Thus we have a natural isomorphism $(X/G) \backslash q(X_G) \cong (X \backslash X_G)/G$.*

(4) *Let $U = X\backslash X_G$ (so that $U/G = (X/G)\backslash q(X_G)$ by (3)). Consider the U/G-morphism*

$$\phi : G \times U \to U \times_{U/G} U$$

given in set-theoretic notation by the formula $(g, u) \mapsto (g(u), u)$. If n is prime then ϕ is an isomorphism.

(5) *Let M be a G-equivariant locally free sheaf of finite rank on X. Suppose that $\iota^* M$ carries the trivial action, that q is flat and that n is prime. Then the natural morphism $q^*(q_* M)_0 \to M$ is an isomorphism.*

(6) *If $X \to S$ is smooth and $X_G \to S$ is flat then $X_G \to S$ is smooth.*

(7) *If $X \to S$ is smooth, X_G is a Cartier divisor in X and $X_G \to S$ is flat then $X/G \to S$ is also smooth.*

Remark 2.6. A variant (for algebraic varieties) of (5) is proven in [6, Th. 2.3]. See also [20, Lemma 4.8] and [21, Proposition (3.3.4.i)], where most of the above proposition is proven in the restricted context of algebraic varieties.

Proof. We begin with (1). We may suppose that $X = \mathrm{Spec}(R)$ is affine. Then $X/G = \mathrm{Spec}(R_0)$ by Proposition 2.1 (2). To show that R is flat over R_0, it is sufficient to show that for all $\mathfrak{p} \in \mathrm{Spec}(R)$, the ring $R_\mathfrak{p}$ is flat over the ring $R_{0,\mathfrak{p}\cap R_0}$. If $\mathfrak{p} \not\supseteq R \cdot R_{\neq 0}$, then $\mathfrak{p} \notin X_G$ by the previous discussion. Thus $G_i(x) \neq G$ and thus $G_i(x) = 0$ since n is prime; thus $R_\mathfrak{p}$ is flat over the ring $R_{0,\mathfrak{p}\cap R_0}$ by Proposition 2.1 (6). Thus we may assume that $\mathfrak{p} \supseteq R \cdot R_{\neq 0}$. The prime ideal $\mathfrak{p}$ is then graded by construction (if $r \in \mathfrak{p}$, write $r = r_0 + \cdots + r_{n-1}$, where the r_i are homogenous for the grading; by assumption $r_1, \ldots, r_{n-1} \in \mathfrak{p}$; thus $r_0 \in \mathfrak{p}$ as well). The ring $R_\mathfrak{p}$ is thus naturally a $\mathbb{Z}/n\mathbb{Z}$-graded local ring. Now notice that we have a natural identification

$$R_{0,\mathfrak{p}\cap R_0} = (R_\mathfrak{p})_0$$

(use the fact that $R\backslash\mathfrak{p} \subseteq R_0$). Also by construction the ideal generated by the image of the ideal $R \cdot R_{\neq 0}$ in $R_\mathfrak{p}$ is $R_\mathfrak{p} \cdot R_{\mathfrak{p},\neq 0}$. Thus the assumption that $R \cdot R_{\neq 0}$ is a Cartier divisor implies that there exists $t \in R_\mathfrak{p}$, which is not a zero divisor, such that $(t) = R_\mathfrak{p} \cdot R_{\mathfrak{p},\neq 0}$.

Thus we may assume without restriction of generality that R is a local ring and that $R \cdot R_{\neq 0}$ is generated by an element t, which is not a zero divisor.

We claim that t can be taken to be homogenous of degree $\neq 0 \pmod{n}$. To verify the claim, let

$$R \cdot R_{\neq 0} = (a_1, \ldots, a_k)$$

where the $a_i \in R_{\neq 0}$ are homogenous and of degree $\neq 0 \pmod{n}$ (recall that R is noetherian). We take k minimal. We may assume that $k > 1$, otherwise there is nothing to prove. Then for some family of $x_i \neq 0$, we have

$$x_1 a_1 + \cdots + x_k a_k = t$$

Let $b_1 \in R$ be such that $a_1 = t \cdot b_1$. If b_1 is a unit then $R \cdot R_{\neq 0} = (a_1)$ contradicting the assumption that $k > 1$. Thus b_1 is not a unit and thus $1 - x_1 b_1$ is a unit since R is local. We compute

$$t = \frac{x_2}{1 - x_1 b_1} a_2 + \cdots + \frac{x_k}{1 - x_1 b_1} a_k$$

contradicting minimality again. Thus $k = 1$ and the claim is verified.

So we may suppose that $(t) = R \cdot R_{\neq 0}$ where t is homogenous of degree $\neq 0 \pmod{n}$.

I am grateful to one of the referees for suggesting the argument below.

sub-lemma 2.7. *For any $i \in \mathbb{Z}/n\mathbb{Z}$, we have $t^i \cdot R_0 = R_{i \deg(t)}$.*

Proof. (of the sublemma) The proof is by induction on i, where i is viewed as an element of the ordered set $\{0, \ldots n-1\}$. The identity of course holds if $i = 0$. We suppose that $t^j \cdot R_0 = R_{j \deg(t)}$ for all $j < i$. Note first that we certainly have $t^i \cdot R_0 \subseteq R_{i \cdot \deg(t)}$. To conclude the proof, we need to show that $R_{i \cdot \deg(t)} \subseteq t^i \cdot R_0$. To show this, let $e \in R_{i \cdot \deg(t)}$. By assumption e can be written in the form $e = t \cdot r$, with $r \in R$. For $k \in \{0, \ldots, n-1\}$, let r_k be the homogenous component of degree k of r. We have

$$e = t \cdot r = t \cdot r_0 + \cdots + t \cdot r_{n-1}$$

so that $t \cdot r_{(i-1) \cdot \deg(t)} = t \cdot r = e$. By induction, we have $r_{(i-1) \cdot \deg(t)} \in t^{i-1} R_0$ so that $e \in t \cdot (t^{i-1} R_0) = t^i \cdot R_0$, as required. $\square$

Now since n is prime and $\deg(t) \neq 0 \pmod{n}$, every element of $\mathbb{Z}/n\mathbb{Z}$ is a multiple of $\deg(t)$. We can thus conclude from the sublemma that R is a direct sum of copies of R_0 so in particular R is flat over R_0.

To prove (2), localising at points of X_G, we may still assume that $X = \mathrm{Spec}(R)$, where R is a local ring and $R \cdot R_{\neq 0}$ is generated by a single element t, which is not a zero divisor. In the proof of (1), it was shown that we may suppose that t is homogenous of degree $\neq 0$. The sheaf $N_{X_G/X}$ corresponds to the R-module $(t)/(t^2)$ and thus $(N_{X_G/X})_0 = 0$, since t is of degree $\neq 0 \pmod{n}$.

Proof of (3). We may suppose that $X = \mathrm{Spec}\, R$ is affine. The first statement now corresponds to the statement that $R_0 \to R/(R \cdot R_{\neq 0})$ is surjective. This follows from the definitions. The fact that $q^{-1}(q(X_G)) = X_G$ follows from Proposition 2.1 (3). The third assertion follows from Proposition 2.1 (4).

Proof of (4). Note that for all $x \in X \backslash X_G$, we have $G_i(x) \neq G$ and thus $G_i(x) = 0$, since n is prime. By Proposition 2.1 (6) this implies that q is étale, in particular flat. Hence the morphism $U \to U/G$ is finite and flat.

We first compute its degree. For this, let $u_0 \in U/G$ and let H be the spectrum of the strict henselisation of $\mathcal{O}_{U/G,u_0}$. Then $H \cong (U \times_{U/G} H)/G$ by Proposition 2.1 (4) and the fact that H is flat over $\mathcal{O}_{U/G,u_0}$ (see [12, I, §1, 1.20] for this). We only have to compute the degree of $U \times_{U/G} H$ over H. Now note that $U \times_{U/G} H$ is a disjoint union $\coprod_{i \in I} H_i$ of copies of H, since H is strictly henselian and $U \times_{U/G} H \to H$ is étale. Furthermore, the group G permutes the H_i and also the closed points of the H_i. Hence the degree is the cardinality of the orbit of a closed point $P \in H_{i_0}$ (i_0 arbitrary). Since $G_i(P) = G_d(P)$, we must have $G_d(P) = 0$, since n is prime and $(U \times_{U/G} H)_G$ is empty. Hence the orbit of P has n elements and thus the degree of $U \to U/G$ is n.

Now consider the morphism $\phi : G \times U \to U \times_{U/G} U$. Let T be a connected scheme. The map $G(T) \times U(T) \to U(T) \times_{(U/G)(T)} U(T)$ is injective. To see this note that otherwise there is $e \in U(T)$ and $g \in G(T)$ such that $g \neq 0$ and $g(e) = e$; since $G(T)$ is of prime order this means that $e \in U(T)^G$ and thus $e \in U_G(T)$, which is not possible, since U_G is empty. Since T was arbitrary, the morphism ϕ is a monomorphism of schemes. Since it is also proper (because $G \times U$ and $U \times_{U/G} U$ are proper over U/G), it is a closed immersion (see [14, IV.3, 8.11.5] for this). Since both $G \times U$ and $U \times_{U/G} U$ are flat and finite of the same rank over U by the previous paragraph, this implies that ϕ is an isomorphism.

Proof of (5). Consider the natural morphism

$$\alpha : q^*(q_*M)_0 \to M$$

The restriction $\alpha_{X \backslash X_G}$ is an isomorphism by (4) and Proposition 2.1 (5). Now both sides are locally free of finite rank by (1). Thus, by Nakayama's lemma, it is sufficient to show that $\alpha_{\kappa(x)}$ is surjective for $x \in X_G$. In particular, it is sufficient to show that the restriction $\iota^*(\alpha)$ of α to X_G is an isomorphism. Now note that since q is an affine morphism, the natural adjunction morphism $\alpha : q^*(q_*M) \to M$ is a surjection and thus we have a surjection

$$\iota^*(q^*(q_*M)) \to \iota^*(M)$$

restricting α. Hence we have a surjection

$$\iota^*(q^*((q_*M)_0)) \to \iota^*(M)_0$$

and since $\iota^*(M)_0 = \iota^*(M)$ by assumption we get a surjection

$$\iota^*(q^*((q_*M)_0)) \to \iota^*(M)$$

which must be an isomorphism, since both sides are locally free of the same rank.

Proof of (6). We need to check that the geometric fibres $X/G \to S$ are regular. So let $\operatorname{Spec} k \to S$ be a geometric point. By assumption, X_k is regular and by [30, Prop. 3.1] , $(X_k)_G = (X_G)_k$ is then also regular.

Proof of (7). Since q is faithfully flat, we see that $X/G \to S$ is also flat. To see that $X/G \to S$ is smooth, we need to check that the geometric fibres $X/G \to S$ are regular. Now since X_G is flat over S and a Cartier divisor, we see that for any base change $T \to S$, $(X_T)_G \to T$ is also flat and a Cartier divisor. Furthermore, by Lemma 2.4, for any base change $T \to S$, we have $(X/G)_T \cong (X_T)/G$. So let $\operatorname{Spec} k \to S$ be a geometric point. By assumption X_k is regular and since $(X_k)_G$ is a Cartier divisor, we see that $(X_k)/G = (X/G)_k$ is regular, since q_k is faithfully flat by (1) and Proposition 2.1 (1). $\square$

3 Free algebras

This section is mainly here to fix some notation that will be needed in section 4. Let I be a finite set. We shall write $\langle I \rangle$ for the free monoid generated by the set I. See for instance [24, chap. I] for this. Recall that the set $\langle I \rangle$ consists of all the finite words written in the alphabet I. A finite word is a map $\{1, \ldots, n\} \to I$, where n is a positive integer. The integer n is called the length of the word. If

$$w_1 : \{1, \ldots, n_1\} \to I$$

and

$$w_2 : \{1, \ldots, n_2\} \to I$$

are two finite words, their concatenation w_1w_2 is by definition the map

$$w_1w_2 : \{1, \ldots, n_1 + n_2\} \to I,$$

such that $w_1w_2(k) = w_1(k)$ if $k \leq n_1$ and $w_1w_2(k) = w_2(k - n_1)$ if $k > n_1$. The monoid structure of $\langle I \rangle$ is given by the concatenation of finite words. We shall write $\mathbb{Z}\langle I \rangle$ for the free $\mathbb{Z}$-module with basis the elements of $\langle I \rangle$. If $I = \{X_1, \ldots, X_n\}$ then we shall use the shorthand

$$\mathbb{Z}\{X_1, \ldots X_n\} := \mathbb{Z}\langle \{X_1, \ldots, X_n\} \rangle$$

The set $\mathbb{Z}\langle I\rangle$ has the structure of a unital ring, where the addition is given by the addition on $\mathbb{Z}\langle I\rangle$ provided by its structure of $\mathbb{Z}$-module and the multiplication $\cdot$ is given by the formula

$$(\sum_{w\in\langle I\rangle} n_w\cdot w)\cdot(\sum_{v\in\langle I\rangle} m_v\cdot v):=\sum_{h\in\langle I\rangle}(\sum_{w,v\in\langle I\rangle,\, wv=h} n_w\cdot m_v)\cdot h.$$

Note that there is an equivalence relation $\sim_{\rm ro}$ on $\langle I\rangle$, defined as follows. If w_1 and w_2 are two finite words as above, then $w_1\sim_{\rm ro} w_2$ iff $n_1 = n_2$ and there is a bijection $\sigma:\{1,\dots,n_1\}\to\{1,\dots,n_1\}$ such that $w_1=w_2\circ\sigma$. We shall write $[w]_{\rm ro}$ for the equivalence class of a finite word w in $\langle I\rangle$.

We shall write $\mathbb{Z}[I]$ for the polynomial ring over the set I (ie the polynomial ring with coefficients in $\mathbb{Z}$ where each element of I is a variable). This is by definition the free $\mathbb{Z}$-module with basis the free commutative monoid generated by I.

Note that there is an obvious surjective map of rings

$$\mathbb{Z}\langle I\rangle\to\mathbb{Z}[I]$$

Lemma 3.1. *An element $\sum_{w\in\langle I\rangle} n_w\cdot w$ is in the kernel of $\mathbb{Z}\langle I\rangle\to\mathbb{Z}[I]$ iff for all $v\in\langle I\rangle$, we have*

$$\sum_{w\in[v]_{\rm ro}} n_w=0$$

Proof. Left to the reader. $\square$

Abusing language, we shall say that $\mathbb{Z}\langle I\rangle$ is the ring of non commutative polynomials with variables I and with coefficients in $\mathbb{Z}$. In particular $\mathbb{Z}\{X_1,\dots X_n\}$ is the ring of non commutative polynomials in the variables $X_1,\dots,X_n$ and coefficients in $\mathbb{Z}$.

If $P\in\mathbb{Z}\langle I\rangle$, then for each $w\in\langle I\rangle$, the integer P_w is defined by the equality

$$P=\sum_{w\in\langle I\rangle} P_w\cdot w$$

4 The determinant of cohomology and Ducrot's generalisation of the Deligne pairing

Let $f:X\to S$ be a flat and strongly projective morphism. Suppose that S is locally noetherian. Let I be a finite set. Let $\{F_i\}_{i\in I}$ be a collection of vector bundles on X indexed by I. If $w = i_1 i_2 \dots i_k$ is a non empty word in the alphabet I, then we shall write

$$\lambda(w) := \lambda(\bigotimes_{t=1}^{k} F_{i_t})$$

where $\lambda(F_{i_t}) := \det(\mathrm{R}^\bullet f_*(F_{i_t}))$ is the determinant of cohomology of the vector bundle F_{i_t}, relatively to f (see beginning of the introduction). If w is the empty word then by convention $\lambda(w) := \lambda(\mathcal{O}_X)$.

If we are given a non commutative polynomial $P = P((F_i)_{i\in I})$ with variables in I and integral coefficients (see section 3), we shall write

$$\lambda(P) := \bigotimes_{w\in\langle I\rangle} \lambda(w)^{\otimes P_w}$$

Note that with this definition, if P and Q are two non commutative polynomials with variables in I and integral coefficients, then in view of the distributivity of the tensor product, there is a canonical isomorphism

$$\lambda(P+Q) \cong \lambda(P) \otimes \lambda(Q).$$

Abusing language, we shall mostly write non commutative polynomials $P = P((F_i)_{i\in I})$ with variables in I using the F_i as variable symbols instead of the elements of the index set I. Also we shall mostly use the tensor product symbol $\otimes$ instead of the symbol $\cdot$. So eg if $I = \{X_1, X_2\}$ we would write

$$F_1 \otimes F_2 + F_2 \otimes F_2$$

instead of $X_1 \cdot X_2 + X_2 \cdot X_2$.

Lemma 4.1. *Let $\{F_i\}_{i\in I}$ be a finite collection of vector bundles on X. Let $P = P((F_i)_{i\in I}) \in \mathbb{Z}\langle I\rangle$ be a non commutative polynomial with integral coefficients in the F_i and suppose that P lies in the kernel of the natural map of rings $\mathbb{Z}\langle I\rangle \to \mathbb{Z}[I]$. Then there is an isomorphism $\lambda(P) \cong \mathcal{O}_S$, which can be chosen compatibly with any base change to a locally noetherian scheme.*

Proof. Let us write O for the set of equivalence classes of the relation $\sim_{\mathrm{ro}}$ in $\langle I\rangle$ (see section 3 for the definition). Choose a representative $w'(o) \in o$ (arbitrary but fixed) for each $o \in O$. Furthermore, for each $o \in O$ and each $w \in o$, choose an automorphism σ_w of $\{1, \dots, \mathrm{length}(w'(o))\}$ such that $w'(o) = \sigma_w \circ w$. Finally, choose an isomorphism $\alpha_o : o \cong \{1, \dots, \#o\}$ for each $o \in O$.

According to Lemma 3.1, if we write

$$P = \sum_{o\in O} \sum_{w\in o} P_w \cdot w$$

then $\sum_{w\in o} P_w = 0$ for each $o \in O$. On the other hand, by the definition of $\lambda(P)$ and the distributivity of the tensor product, we have a canonical isomorphism

$$\lambda(P) \cong \bigotimes_{o\in O} \bigotimes_{w\in o} \lambda(w)^{P_w}$$

and by the commutativity of the tensor product, there is a canonical isomorphism

$$\lambda(w) \cong \lambda(w'(o))$$

for each $w \in o$, which depends of the choice of the automorphism σ_w. Hence there is a canonical isomorphism

$$\lambda(P) \cong \bigotimes_{o\in O} \lambda(w'(o))^{\sum_{w\in o} P_w},$$

which depends on the isomorphism α_o. The conclusion follows. Note that this isomorphism depends a priori on the choices of the representatives $w'(o) \in o$ and of the automorphisms σ_w and α_o. One might conjecture that different choices of representatives and automorphisms will lead to the same isomorphism $\lambda(P) \cong \mathcal{O}_S$, up to sign (but this is irrelevant to the conclusion of the lemma, which contains no unicity statement). $\square$

Let $L_1, \dots L_{d+1}$ be line bundles on X. Suppose that X is of constant relative dimension d over S. We shall write

$$I_{X/S}(L_1, \dots, L_{d+1}) := \lambda((\mathcal{O}_X - L_1) \otimes (\mathcal{O}_X - L_2) \otimes \cdots \otimes (\mathcal{O}_X - L_{d+1}))^{\otimes(-1)^d}$$

where the expression defining $I_{X/S}(L_1, \dots, L_{d+1})$ is to be read with the above notational conventions in mind. In particular, the expression $(\mathcal{O}_X - L_1) \otimes (\mathcal{O}_X - L_2) \otimes \cdots \otimes (\mathcal{O}_X - L_{d+1})$ should be understood as a non commutative polynomial in the line bundles $\mathcal{O}_X, L_1, \dots, L_{d+1}$ and $\lambda((\mathcal{O}_X - L_1) \otimes (\mathcal{O}_X - L_2) \otimes \cdots \otimes (\mathcal{O}_X - L_{d+1}))$ is to be computed according to the conventions described above.

So for example, if $d = 1$,

$$\begin{aligned} & I_{X/S}(L_1, L_2)^\vee = \lambda((\mathcal{O}_X - L_1) \otimes (\mathcal{O}_X - L_2)) \\ = \ & \lambda(\mathcal{O}_X \otimes \mathcal{O}_X - \mathcal{O}_X \otimes L_2 - L_1 \otimes \mathcal{O}_X + L_1 \otimes L_2) \\ = \ & \lambda(\mathcal{O}_X \otimes \mathcal{O}_X) \otimes \lambda(\mathcal{O}_X \otimes L_2)^\vee \otimes \lambda(L_1 \otimes \mathcal{O}_X)^\vee \otimes \lambda(L_1 \otimes L_2) \\ \cong \ & \lambda(\mathcal{O}_X) \otimes \lambda(L_2)^\vee \otimes \lambda(L_1)^\vee \otimes \lambda(L_1 \otimes L_2). \end{aligned} \tag{6}$$

Ducrot showed in [7, §5] that the line bundle $I_{X/S}(L_1, \dots, L_{d+1})$ is multiadditive in the line bundles $L_1, \dots L_{d+1}$. In particular, he shows that if Q is a line bundle on X, then there is a canonical isomorphism

$$I_{X/S}(L_1 \otimes Q, \dots, L_{d+1}) \cong I_{X/S}(L_1, \dots, L_{d+1}) \otimes I_{X/S}(Q, \dots, L_{d+1}) \tag{7}$$

The canonical isomorphism (7) is compatible with any base change to a locally noetherian scheme. See [7, Th. 4.2 (BC)].

We may thus compute

$$
\begin{aligned}
& \lambda((\mathcal{O}_X - Q) \otimes (\mathcal{O}_X - L_1) \otimes (\mathcal{O}_X - L_2) \otimes \cdots \otimes (\mathcal{O}_X - L_{d+1})) \\
\overset{(1)}{\cong} \; & \lambda((\mathcal{O}_X - L_1) \otimes (\mathcal{O}_X - L_2) \otimes \cdots \otimes (\mathcal{O}_X - L_{d+1})) \\
\otimes \; & \lambda((Q - Q \otimes L_1) \otimes (\mathcal{O}_X - L_2) \otimes (\mathcal{O}_X - L_3) \otimes \cdots \otimes (\mathcal{O}_X - L_{d+1}))^\vee \\
\overset{(2)}{\cong} \; & \lambda((\mathcal{O}_X - L_1) \otimes (\mathcal{O}_X - L_2) \otimes \cdots \otimes (\mathcal{O}_X - L_{d+1})) \\
\otimes \; & \lambda(\big(\mathcal{O}_X - L_1 \otimes Q - (\mathcal{O}_X - Q)\big) \otimes (\mathcal{O}_X - L_2) \otimes (\mathcal{O}_X - L_3) \otimes \cdots \otimes (\mathcal{O}_X - L_{d+1}))^\vee \\
\overset{(3)}{\cong} \; & I_{X/S}(L_1, \ldots, L_{d+1})^{\otimes(-1)^d} \\
\otimes \; & I_{X/S}(L_1 \otimes Q, L_2, \ldots, L_{d+1})^{\otimes(-1)^{d+1}} \otimes I_{X/S}(Q, L_2, \ldots, L_{d+1})^{\otimes(-1)^d} \\
\overset{(4)}{\cong} \; & I_{X/S}(L_1, L_2, \ldots, L_{d+1})^{\otimes(-1)^d} \otimes I_{X/S}(L_1, L_2, \ldots, L_{d+1})^{\otimes(-1)^{d+1}} \\
\otimes \; & I_{X/S}(Q, L_2, \ldots, L_{d+1})^{\otimes(-1)^{d+1}} \otimes I_{X/S}(Q, L_2, \ldots, L_{d+1})^{\otimes(-1)^d} \overset{(5)}{\cong} \mathcal{O}_X
\end{aligned}
$$

and this trivialisation is invariant under any base change to a locally noetherian scheme. The isomorphisms $(1), (2), (3)$ are formal consequences of Lemma 4.1 and of the polynomial equalities

$$
\begin{aligned}
& (1-y)(1-x_1)\cdots(1-x_{d+1}) \\
= \; & (1-x_1)\cdots(1-x_{d+1}) - (y - yx_1)(1-x_2)\ldots(1-x_{d+1}) \\
= \; & (1-x_1)\cdots(1-x_{d+1}) - ((1-x_1y) - (1-y))(1-x_2)\ldots(1-x_{d+1}) \\
= \; & (1-x_1)\cdots(1-x_{d+1}) - (1-x_1y)(1-x_2)\ldots(1-x_{d+1}) + (1-y)(1-x_2)\ldots(1-x_{d+1})
\end{aligned}
$$

(in the same order). The isomorphism (4) comes from the multiadditivity of the symbol $I_{X/S}$ described above. Isomorphism (5) is just a cancellation.

The following theorem summarises the discussion and it is one of the main consequences of the theory developed in [7].

Theorem 4.2. *Suppose that $X \to S$ is flat, strongly projective and of relative dimension d. Suppose that S is locally noetherian. Let $L_1, \ldots, L_{d+2}$ be line bundles on X. Then the line bundle*

$$\lambda((\mathcal{O}_X - L_1) \otimes (\mathcal{O}_X - L_2) \otimes \cdots \otimes (\mathcal{O}_X - L_{d+2}))$$

is canonically trivial and the trivialisation is invariant under base change to any locally noetherian scheme.

See also [2, Th. A.21, Appendix], where it is verified that some noetherian assumptions in Theorem 4.2 can be removed (we do not exploit this because noetherian assumptions are needed elsewhere in this text).

Corollary 4.3. *Let $\mathcal{F} := n_1 M_1 + \cdots + n_k M_k$, where M_i is a line bundle on X (resp. $n_i \in \mathbb{Z}$) for all $i \in \{1, \ldots, k\}$. Let $L_1, \ldots, L_{d+1}$ be line bundles on X. Suppose that $\sum_i n_i = 0$. Then the line bundle*

$$\lambda(\mathcal{F} \otimes (\mathcal{O}_X - L_1) \otimes (\mathcal{O}_X - L_2) \otimes \cdots \otimes (\mathcal{O}_X - L_{d+1}))$$

is canonically trivial and the trivialisation is invariant under base change to any noetherian scheme.

Proof. (of Corollary 4.3). By Theorem 4.2, there is a canonical isomorphism

$$\lambda((n_i M_i) \otimes (\mathcal{O}_X - L_1) \otimes (\mathcal{O}_X - L_2) \otimes \cdots \otimes (\mathcal{O}_X - L_{d+1})) \cong \lambda((\mathcal{O}_X - L_1) \otimes (\mathcal{O}_X - L_2) \otimes \cdots \otimes (\mathcal{O}_X - L_{d+1}))^{\otimes n_i}$$

for any n_i. The Corollary follows from this. $\square$

5 Equivariant derived functors

We first recall the definition of a perfect complex. Let Z be a locally noetherian scheme. We shall as usual write $D^b(Z)$ for the derived category of bounded complexes of $\mathcal{O}_Z$-modules. A complex $J^\bullet$ of $\mathcal{O}_Z$-modules is said to be of finite tor-dimension if there are integers $a < b$ such that for all $\mathcal{O}_Z$-modules M, we have $\underline{\mathrm{Tor}}^k(J^\bullet, M) = 0$ if $k < a$ or $k > b$. A *bounded* complex $J^\bullet$ is said to be perfect if

- the homology sheaves $\mathcal{H}^k(J^\bullet)$ are coherent for all $k \in \mathbb{Z}$;

- there is a covering (U_i) of Z by open subschemes, such that $J^\bullet|_{U_i}$ is of finite tor-dimension.

In view of this definition, we see that the property of being perfect depends only on the image of $J^\bullet$ in the category $D^b(Z)$.

Let G be a finite group. If Z is a G-equivariant locally noetherian scheme, we let $\mathrm{Coh}^{\mathrm{eq}}(Z)$ be the category of coherent G-equivariant sheaves on Z. Recall also that $\mathrm{Coh}(Z)$ refers to the category of coherent sheaves on Z. Note that the category $\mathrm{Coh}^{\mathrm{eq}}(Z)$ has a natural structure of abelian category. We shall write $D^b(\mathrm{Coh}^{\mathrm{eq}}(Z))$ for the derived category of bounded complexes in $\mathrm{Coh}^{\mathrm{eq}}(Z)$. Note that there is a natural forgetful functor from $D^b(\mathrm{Coh}^{\mathrm{eq}}(Z))$ to $D^b(\mathrm{Coh}(Z))$ and thus also to $D^b(Z)$ via the forgetful functor $D^b(\mathrm{Coh}(Z)) \to D^b(Z)$.

Lemma 5.1. *Suppose that $f : X \to Y$ is a strongly projective morphism of G-equivariant noetherian schemes. Let $J^\bullet$ be a bounded complex of G-equivariant coherent sheaves on X. Then there is a bounded complex $H^\bullet$ of G-equivariant f-acyclic coherent sheaves on X and a G-equivariant quasi-isomorphism $J^\bullet \to H^\bullet$.*

Recall that if F is a quasi-coherent sheaf on X, one says that F is f-acyclic if $\mathrm{R}^k f_*(F) = 0$ when $k > 0$.

Proof. When the action of G on X and Y is trivial, this is standard. The proof in the equivariant situation is completely similar and we skip it. □

In view of Lemma 5.1 and [18, Th. I.5.1], in the situation of Lemma 5.1 the functor f_*^{eq} has a right derived functor

$$\mathrm{R}^\bullet f_*^{\mathrm{eq}} : D^b(\mathrm{Coh}^{\mathrm{eq}}(X)) \to D^b(\mathrm{Coh}^{\mathrm{eq}}(S)).$$

The functor $\mathrm{R}^\bullet f_*^{\mathrm{eq}}$ is compatible with the usual right derived functor

$$\mathrm{R}^\bullet f_* : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(Y))$$

via the forgetful functors $D^b(\mathrm{Coh}^{\mathrm{eq}}(X)) \to D^b(\mathrm{Coh}(X))$ and $D^b(\mathrm{Coh}^{\mathrm{eq}}(Y)) \to D^b(Y)$. If $f : X \to Y$ and $h : X \to Y$ are strongly projective morphism of G-equivariant noetherian schemes then we have a natural isomorphism of functors $\mathrm{R}^\bullet(h \circ f)_*^{\mathrm{eq}} \cong \mathrm{R}^\bullet h_*^{\mathrm{eq}} \circ \mathrm{R}^\bullet f_*^{\mathrm{eq}}$. This follows from [18, Prop. 5.4 and following remark]. We leave the details to the reader. The point is that for any bounded complex of G-equivariant coherent sheaves $J^\bullet$ on X, there is a bounded complex $H^\bullet$ of G-equivariant coherent sheaves on X, which is both $f-$ and $h \circ f$-acyclic, and a G-equivariant quasi-isomorphism $J^\bullet \to H^\bullet$.

If F is a G-equivariant locally free sheaf on a G-equivariant locally noetherian scheme Z, we have a functor $F \otimes (\cdot) : D^b(\mathrm{Coh}^{\mathrm{eq}}(Z)) \to D^b(\mathrm{Coh}^{\mathrm{eq}}(Z))$ (resp. a functor $(\cdot) \otimes F : D^b(\mathrm{Coh}^{\mathrm{eq}}(Z)) \to D^b(\mathrm{Coh}^{\mathrm{eq}}(Z))$. This functor simply sends a complex $J^\bullet \in D^b(\mathrm{Coh}(Z))$ on the complex $J^\bullet \otimes F$ (resp. the complex $F \otimes J^\bullet$).

We have a projection formula:

Proposition 5.2. *Suppose that $f : X \to Y$ is a strongly projective morphism of G-equivariant noetherian schemes. Let F be a G-equivariant locally free sheaf on Y. Then there is a natural isomorphism of functors*

$$\mathrm{R} f_*^{\mathrm{eq}}(f^*(F) \otimes (\cdot)) \cong \mathrm{R} f_*^{\mathrm{eq}}(\cdot) \otimes F$$

Proof. Left to the reader. Apply the usual projection formula to the definition of $\mathrm{R}f_*^{\mathrm{eq}}(\cdot)$. $\square$

Recall that a morphism $h : T \to S$ of locally noetherian schemes is called lci (local complete intersection), if locally on S, there is a factorisation of h into a regular closed immersion $T \to T_1$ followed by a smooth morphism $T_1 \to S$. We recall the

Proposition 5.3. *If a morphism $h : T \to S$ of noetherian schemes is lci and strongly projective and $F^\bullet$ is an object of $D^b(\mathrm{Coh}(T))$, which is a perfect complex then $\mathrm{R}^\bullet f_*(F^\bullet)$ is also a perfect complex.*

Proof. See [17, Cor. 4.8.1, Exp. III]. $\square$

Suppose now that Z is a locally noetherian scheme, that $G = \mathbb{Z}/2\mathbb{Z}$ and that 2 is invertible on Z. Suppose also that the scheme Z is endowed with a trivial G-equivariant structure. If F is an equivariant coherent sheaf on Z, we shall write

$$F_+ := F_0$$

and

$$F_- := F_1.$$

The functors

$$(\cdot)_- = (\cdot)_1 : \mathrm{Coh}^{\mathrm{eq}}(Z) \to \mathrm{Coh}(Z)$$

and

$$(\cdot)_+ = (\cdot)_0 : \mathrm{Coh}^{\mathrm{eq}}(Z) \to \mathrm{Coh}(Z)$$

are exact functors and so they uniquely extend to functors from $D^b(\mathrm{Coh}^{\mathrm{eq}}(Z))$ to $D^b(\mathrm{Coh}(Z))$, which are their right and left derived functors simultaneously. We shall also call these extensions $(\cdot)_+$ and $(\cdot)_-$. If $F^\bullet$ is an object in $D^b(\mathrm{Coh}^{\mathrm{eq}}(Z))$ then we have by construction a canonical direct sum decomposition $F^\bullet \cong (F^\bullet)_+ \oplus (F^\bullet)_-$ in $D^b(\mathrm{Coh}(Z))$. In particular, if the image of $F^\bullet$ in $D^b(\mathrm{Coh}(Z))$ is a perfect complex, so are $(F^\bullet)_+$ and $(F^\bullet)_-$. We shall say that an object $F^\bullet$ of $D^b(\mathrm{Coh}^{\mathrm{eq}}(Z))$ is a perfect complex if its image in $D^b(\mathrm{Coh}(Z))$ (or $D^b(Z)$) is a perfect complex. If $F^\bullet$ is an object of $D^b(\mathrm{Coh}^{\mathrm{eq}}(Z))$, which is a perfect complex, we can thus write

$$\det{}^{\mathrm{eq}}(F^\bullet) := \det((F^\bullet)_+) \otimes \det((F^\bullet)_-)^\vee.$$

If

$$F^\bullet \to H^\bullet \to J^\bullet \to F^\bullet[1]$$

is a triangle of perfect complexes in $D^b(\mathrm{Coh}^{\mathrm{eq}}(Z))$, we then have a canonical isomorphism

$$\det{}^{\mathrm{eq}}(F^\bullet) \otimes \det{}^{\mathrm{eq}}(J^\bullet) \cong \det{}^{\mathrm{eq}}(H^\bullet)$$

by the standard properties of determinants (see [22]) and the fact that the functors

$$(\cdot)_\pm : D^b(\mathrm{Coh}^{\mathrm{eq}}(Z)) \to D^b(\mathrm{Coh}(Z))$$

respect triangulations (because they are derived functors).

Let $f : X \to Y$ be a locally projective and lci morphism of G-equivariant locally noetherian schemes, where the G-action on Y is trivial. Suppose that $G = \mathbb{Z}/2\mathbb{Z}$ and that 2 is invertible on Y (and thus on X). Let $F^\bullet$ be an object of $D^b(\mathrm{Coh}^{\mathrm{eq}}(X))$, which is a perfect complex. Let $U \subseteq Y$ be an open subset, such that $f|_U : f^{-1}(U) \to U$ is strongly projective. By Corollary 5.3 and the above discussion, we may define

$$\lambda^{\mathrm{eq}}(F^\bullet|_{f^{-1}(U)}) := \det((\mathrm{R}^\bullet(f^{\mathrm{eq}}|_U)_*(F^\bullet|_{f^{-1}(U)}))_+) \otimes \det((\mathrm{R}^\bullet(f^{\mathrm{eq}}|_U)_*(F^\bullet|_{f^{-1}(U)}))_-)^\vee$$

which is a line bundle on U. Since this line bundle is defined locally on Y, by varying U, we obtain a line bundle on all of Y, which we denote by $\lambda^{\mathrm{eq}}(F^\bullet)$. If the equivariant structure on X and F is trivial, then we of course have a canonical identification

$$\lambda^{\mathrm{eq}}(F^\bullet) \cong \lambda(F^\bullet).$$

Note that if

$$F^\bullet \to H^\bullet \to J^\bullet \to F^\bullet[1]$$

is a triangle of perfect complexes in $D^b(\mathrm{Coh}^{\mathrm{eq}}(X))$, then we have canonically

$$\lambda^{\mathrm{eq}}(F^\bullet) \otimes \lambda^{\mathrm{eq}}(J^\bullet) \cong \lambda^{\mathrm{eq}}(H^\bullet) \tag{8}$$

(because $\mathrm{R}^\bullet f_*^{\mathrm{eq}}(\cdot)$ respects triangles, locally in Y).

Let I be a finite set. Let $\{F_i\}_{i\in I}$ be a collection of equivariant vector bundles on X indexed by I. For any non commutative polynomial $P = ((F_i)_{i\in I})$ with integral coefficients and variables in I, we may now define $\lambda^{\mathrm{eq}}(P)$ in a manner entirely similar to the non equivariant case (see beginning of section 4). The evident equivariant analog of Lemma 4.1 then also holds.

Finally, we shall write $\{-1\}$ for the trivial sheaf $\mathcal{O}_X$, endowed with the G-equivariant structure such that for any $g \in G$ the isomorphism $\alpha_{g,\{-1\}} : \{-1\} \to g_*(\{-1\})$ composed with the canonical non equivariant identification $g_*(\{-1\}) \cong \{-1\}$ is given by multiplication by $(-1)^g$. If F is a G-equivariant sheaf on X, we shall write $F\{-1\}$ for $F \otimes \{-1\}$. Note that if F is an equivariant coherent locally free sheaf on X and $l \in \mathbb{Z}$, we have canonical isomorphisms

$$\lambda^{\mathrm{eq}}((F\{-1\})^{\otimes l}) \cong \lambda^{\mathrm{eq}}(F^{\otimes l})^{\otimes(-1)^l} \cong \lambda^{\mathrm{eq}}((-F)^{\otimes l}). \tag{9}$$

6 Local refinement of the fixed point formula for an involution

Let S be a locally noetherian scheme and let $f : X \to S$ be a separated morphism of finite type. Suppose that 2 is invertible in S. Let $G = \mathbb{Z}/2$, so that we have a canonical isomorphism $G_S \cong \mu_{2S}$. Suppose that we have a G-equivariant structure on X over S. Suppose finally that the orbit of every point in X is contained in an open affine subscheme. Let $\iota : X_G \hookrightarrow X$ be the fixed scheme of X and let $q : X \to X/G$ be the quotient morphism. These morphisms exist by Proposition 2.3 and Theorem 2.1. Note that if q is flat then it is faithfully flat (since it is surjective) and thus if q and f are flat the natural morphism $X/G \to S$ is also flat. Similarly, if f is locally projective then so is the natural morphism $X/G \to S$.

In this section, we shall prove a version of the relative geometric fixed point formula for the G-action of G on X, which avoids K-theory entirely, replacing all the equalities in a Grothendieck group or a Picard group by explicit isomorphisms. This is the following Theorem.

Theorem 6.1. *Suppose in addition that f is smooth, locally projective and that f has constant relative dimension d. Suppose also that the morphism $X_G \to S$ is flat. Then $X_G \to S$ is smooth and thus X_G is regularly immersed in X. Let $N = N_{X_G/X}$ be the conormal bundle of $\iota : X_G \hookrightarrow X$, endowed with its canonical G-equivariant structure. Let M be a G-equivariant line bundle on X. We have a canonical isomorphism*

$$\lambda^{\mathrm{eq}}(M)^{\otimes 2^{d+1}} \cong \bigotimes_{j=0}^{d} \lambda^{\mathrm{eq}}(\iota^*(M) \otimes \mathrm{Sym}^j(N))^{\otimes \sum_{i=0}^{d-j} \binom{d+1}{i}}$$

which is compatible with any base change $h : S' \to S$ such that S' is locally noetherian.

For the proof, we shall need the following

Lemma 6.2. *Let $Z \to T$ be a morphism of locally noetherian schemes. Let $C \hookrightarrow Z$ be a regular closed immersion. Suppose that C and Z are flat over T. Let $h : T' \to T$ be a morphism of schemes, where T' is locally noetherian. Then*

(a) *the natural morphism $\mathrm{Bl}_{C_{T'}}(Z_{T'}) \to \mathrm{Bl}_C(Z)_{T'}$ is an isomorphism;*

(b) *$\mathrm{Bl}_C(Z)$ is flat over T.*

Proof. Let I be the sheaf of ideals of C in Z. By definition, we have

$$\mathrm{Bl}_C(Z) := \mathrm{Proj}(\bigoplus_{i \geq 0} I^i)$$

so that

$$\mathrm{Bl}_C(Z)_{T'} := \mathrm{Proj}(\bigoplus_{i\geq 0} h_Z^*(I^i))$$

where $h_Z : Z_{T'} \to Z$ is the base change of h to Z and $h_Z^*(I^i)$ is the pull-back to $Z_{T'}$ of I^i as a coherent sheaf on Z. On the other hand, we have again by definition

$$\mathrm{Bl}_{C_{T'}}(Z_{T'}) := \mathrm{Proj}(\bigoplus_{i\geq 0} h_Z^{-1}(I)^i) = \mathrm{Proj}(\bigoplus_{i\geq 0} h_Z^{-1}(I^i))$$

where $h_Z^{-1}(I^i)$ is the ideal sheaf on $Z_{T'}$, which is the image of $h_Z^*(I^i)$ in $\mathcal{O}_{Z_{T'}}$. The surjection of sheaves $h_Z^*(I^i) \to h_Z^{-1}(I^i)$ provide a natural $Z_{T'}$-morphism from $\mathrm{Bl}_{C_{T'}}(Z_{T'})$ to $\mathrm{Bl}_C(Z)_{T'}$, which is the natural map mentioned in the lemma. To prove (a), we need to show that this morphism is an isomorphism. For this, it is sufficient to show that the surjection $h_Z^*(I^i) \to h_Z^{-1}(I^i)$ is an isomorphism for all $i \geq 0$. We will show that the closed subscheme of Z defined by I^i is flat over T, from which this immediately follows. Now note that because C is regularly immersed in Z we have $I^k/I^{k+1} \cong \mathrm{Sym}^k(N_{C/Z})$ for all $k \geq 0$. Here $N_{C/Z}$ is the conormal sheaf of C in Z. See eg [13, IV, par. 2, Cor. 2.4] for this. Since $N_{C/Z}$ is locally free over C and C is flat over T, we see that I^k/I^{k+1} is flat over T for all $k \geq 0$. Since $\mathcal{O}_Z/I^i$ has a natural filtration, whose quotients are of the form I^k/I^{k+1}, we conclude that $\mathcal{O}_Z/I^i$ is also flat over T. In other words, the closed subscheme of Z defined by I^i is flat over T. This concludes the proof of (a). For (b), note that since Z is flat over T and $\mathcal{O}_Z/I^i$ is flat over T (see the proof of (a)), the sheaf I^i is also flat over T (for all $i \geq 0$). Thus the graded $\mathcal{O}_Z$-algebra $\bigoplus_{i\geq 0} I^i$ is flat over T, which implies that $\mathrm{Bl}_C(Z)$ is flat over T. $\square$

Proof. (of Theorem 6.1). First note that since the advertised isomorphism of line bundles is local on S, we may assume that S is affine. In particular, we may assume that f is a strongly projective morphism.

We start with an identity in $\mathbb{Z}[t]$. Define

$$P_k(t) := 2^k + 2^{k-1}(2-t) + 2^{k-2}(2-t)^2 + \cdots + (2-t)^k \in \mathbb{Z}[t].$$

Setting $q := 1 - \frac{t}{2}$, we have

$$tP_k(t) = 2(1-q)2^k(1+q+\cdots+q^k) = -2^{k+1}(q^{k+1}-1) = 2^{k+1} - (2q)^{k+1} = 2^{k+1} - (2-t)^{k+1}.$$

(I am grateful to one of the referees for providing a simplification of earlier calculations).

Now suppose first that X_G is a Cartier divisor. Let $L := \mathcal{O}(-X_G)$.

We have an exact sequence

$$0 \to L \otimes M \to M \to \iota_*(\iota^*(M)) \to 0 \tag{10}$$

The existence of this sequence, unspectacular as it may seem, is the linchpin of the proof.

Note that by the adjunction formula (or by definition, according to taste) we have a canonical equivariant isomorphism $\iota^*(L) \cong N$. Note also that by Proposition 2.5 (2), G acts by -1 on N. Let $J := q_*(L\{-1\})_0$. Proposition 2.5 (5) implies that this is a line bundle on X/G such that $q^*(J) = L\{-1\}$.

Now we compute

$$
\begin{aligned}
& \lambda^{\mathrm{eq}}(\iota^*(M) \otimes P_k(\mathcal{O}_{X_G} - N))) \\
\overset{(b)}{\cong}\ & \lambda^{\mathrm{eq}}(\iota^*(M) \otimes P_k(\mathcal{O}_{X_G} - \iota^*(L))) \overset{(c)}{\cong} \lambda^{\mathrm{eq}}(M \otimes (\mathcal{O}_X - L) \otimes P_k(\mathcal{O}_X - L)) \\
\overset{(d)}{\cong}\ & \lambda^{\mathrm{eq}}(M \otimes (\mathcal{O}_X^{\oplus 2^{k+1}} - (\mathcal{O}_X^{\oplus 2} - (\mathcal{O}_X - L))^{\otimes(k+1)})) \\
\overset{(e)}{\cong}\ & \lambda^{\mathrm{eq}}(M \otimes (\mathcal{O}_X^{\oplus 2^{k+1}} - (\mathcal{O}_X^{\oplus 2} - (\mathcal{O}_X + L\{-1\}))^{\otimes(k+1)})) \\
\overset{(f)}{\cong}\ & \lambda^{\mathrm{eq}}(M \otimes (\mathcal{O}_X^{\oplus 2^{k+1}} - (\mathcal{O}_X - L\{-1\})^{\otimes(k+1)})) \\
\overset{(g)}{\cong}\ & \lambda^{\mathrm{eq}}(M)^{\otimes 2^{k+1}} \otimes \lambda^{\mathrm{eq}}(M \otimes (\mathcal{O}_X - L\{-1\})^{\otimes(k+1)})^{\vee} \\
\overset{(h)}{\cong}\ & \lambda^{\mathrm{eq}}(M)^{\otimes 2^{k+1}} \otimes \lambda^{\mathrm{eq}}(q_*(M) \otimes (\mathcal{O}_{X/G} - J)^{\otimes(k+1)})^{\vee} \\
\overset{(i)}{\cong}\ & \lambda^{\mathrm{eq}}(M)^{\otimes 2^{k+1}} \otimes \lambda((q_*(M)_+ - q_*(M)_-) \otimes (\mathcal{O}_{X/G} - J)^{\otimes(k+1)})^{\vee} \\
\overset{(j)}{\cong}\ & \lambda^{\mathrm{eq}}(M)^{\otimes 2^{k+1}} \otimes \lambda(((\mathcal{O}_{X/G} - q_*(M)_-) - (\mathcal{O}_{X/G} - q_*(M)_+)) \otimes (\mathcal{O}_{X/G} - J)^{\otimes(k+1)})^{\vee} \\
\overset{(k)}{\cong}\ & \lambda^{\mathrm{eq}}(M)^{\otimes 2^{k+1}} \otimes \lambda((\mathcal{O}_{X/G} - q_*(M)_-) \otimes (\mathcal{O}_{X/G} - J)^{\otimes(k+1)})^{\vee} \\
\otimes\ & \lambda((\mathcal{O}_{X/G} - q_*(M)_+) \otimes (\mathcal{O}_{X/G} - J)^{\otimes(k+1)})
\end{aligned}
$$

Equality (b) is justified by the adjunction formula. Equality (c) follows from the existence of the exact sequence (10) and the compatibility of $\lambda^{\mathrm{eq}}(\cdot)$ with triangles. Equality (d) follows from the equality $t \cdot P_k(t) = 2^{k+1} - (2-t)^{k+1}$ and the equivariant analogue of Lemma 4.1. Equality (e) follows from (9). Equality (f) is a simple cancellation and so is equality (g). Equality (h) follows from the projection formula 5.2, the compatibility of equivariant derived functors with compositions of morphisms (see before Proposition 5.2) and the fact that we have $q^*(J) \cong L\{-1\}$. Equality (i) follows from the definition of $\lambda^{\mathrm{eq}}(\cdot)$. Equality (j) is a simple cancellation and so is equality (k).

Now if we let $k = d$, we obtain by Theorem 4.2 canonical trivialisations

$$\lambda^{\mathrm{eq}}((\mathcal{O}_{X/G} - q_*(M)_-) \otimes (\mathcal{O}_{X/G} - J)^{\otimes(k+1)}) \cong \lambda((\mathcal{O}_{X/G} - q_*(M)_-) \otimes (\mathcal{O}_{X/G} - J)^{\otimes(k+1)}) \cong \mathcal{O}_S$$

and

$$\lambda^{\mathrm{eq}}((\mathcal{O}_{X/G} - q_*(M)_+) \otimes (\mathcal{O}_{X/G} - J)^{\otimes(k+1)}) \cong \lambda((\mathcal{O}_{X/G} - q_*(M)_+) \otimes (\mathcal{O}_{X/G} - J)^{\otimes(k+1)}) \cong \mathcal{O}_S$$

and thus a canonical isomorphism

$$\lambda^{\mathrm{eq}}(\iota^*(M)\otimes P_d(\mathcal{O}_{X_G}-N))\cong\lambda^{\mathrm{eq}}(M)^{\otimes 2^{d+1}}. \tag{11}$$

Note that all the isomorphisms (b),..., (k) are compatible with any base change to a locally noetherian scheme. This follows from that fact that $X\to S$ and $X_G\to S$ are flat, from Lemma 2.4 and from Theorem 4.2.

We repeat the calculation for $M=\mathcal{O}_X$ and $d=1$ (ie when $X\to S$ is a fibration in curves) to make the calculation completely explicit in a simple situation. In the case $d=1$, we may choose $k=d=1$ (see above). We then have $P_k(t)=P_1(t)=4-t$. We shall write $F:=q_*(\mathcal{O}_X)_-$. We compute

$$\begin{aligned}
&\lambda^{\mathrm{eq}}(\mathcal{O}_{X_G})^{\otimes 3}\otimes\lambda^{\mathrm{eq}}(N)\\
\overset{\alpha}{\cong}\ &\lambda^{\mathrm{eq}}(\mathcal{O}_X)^{\otimes 3}\otimes\lambda^{\mathrm{eq}}(L)^{\otimes(-3)}\otimes\lambda^{\mathrm{eq}}(L)\otimes\lambda^{\mathrm{eq}}(L^{\otimes 2})^{\otimes(-1)}\\
\overset{\beta}{\cong}\ &\lambda^{\mathrm{eq}}(\mathcal{O}_X)^{\otimes 3}\otimes\lambda^{\mathrm{eq}}(L\{-1\})^{\otimes 2}\otimes\lambda^{\mathrm{eq}}(L\{-1\}^{\otimes 2})^{\otimes(-1)}\\
\overset{\gamma}{\cong}\ &\lambda^{\mathrm{eq}}(\mathcal{O}_X)^{\otimes 3}\otimes\lambda(J)^{\otimes 2}\otimes\lambda(J\otimes F)^{\otimes(-2)}\otimes\lambda(J^{\otimes 2})^{\otimes(-1)}\otimes\lambda(J^{\otimes 2}\otimes F)\\
\overset{\delta}{\cong}\ &\lambda^{\mathrm{eq}}(\mathcal{O}_X)^{\otimes 4}\otimes\lambda(\mathcal{O}_{X/G})^{\otimes(-1)}\otimes\lambda(F)\otimes\lambda(J)^{\otimes 2}\otimes\lambda(J\otimes F)^{\otimes(-2)}\otimes\lambda(J^{\otimes 2})^{\otimes(-1)}\otimes\lambda(J^2\otimes F)\\
\overset{\epsilon}{\cong}\ &\lambda^{\mathrm{eq}}(\mathcal{O}_X)^{\otimes 4}\otimes\lambda((1-F)\otimes(1-J)\otimes(1-J))^{\otimes(-1)}\\
\overset{\zeta}{\cong}\ &\lambda^{\mathrm{eq}}(\mathcal{O}_X)^{\otimes 4}
\end{aligned}$$

The isomorphism α comes from the adjunction formula, the exact sequence (10) and the identity (8). The isomorphism β is a consequence of the identities (9). The isomorphisms γ and δ come from the equivariant projection formula (Proposition 5.2) and the fact that equivariant derived functors are compatible with compositions of morphisms (see before Proposition 5.2). Isomorphism ϵ is just a reshuffling of terms, taking into account the commutativity of the tensor product. Isomorphism ζ comes from Theorem 4.2.

We now go back to the general situation. If X_G is not a Cartier divisor let $\widetilde{X}$ be the blow-up of X along X_G and let $b:\widetilde{X}\to X$ be the canonical morphism. Note that since S is affine, the scheme X carries an ample line bundle. In particular the morphism b is strongly projective. Also, the scheme $\widetilde{X}$ is flat over S by Lemma 6.2 (b) and it has geometrically regular fibres over S by Lemma 6.2 (a) and the fact that $X_G\to S$ is smooth. Thus $\widetilde{X}$ is smooth over S and this implies that b is lci. The scheme $\widetilde{X}$ is canonically G-equivariant since the sheaf of ideals of X_G is equivariant. The exceptional divisor E of $\widetilde{X}$ is isomorphic to the projectivised bundle $\mathbb{P}(N)$. Since G acts by multiplication by -1 on

N, we see that the action of G is trivial on E. Hence $E = \widetilde{X}_G$ and $\widetilde{X}_G$ is a Cartier divisor, which is clearly smooth over S.

Let $\mu : \widetilde{X}_G \hookrightarrow \widetilde{X}$ and $p : \widetilde{X}_G \to X_G$ be the canonical morphisms. From equality (11), we obtain

$$
\begin{aligned}
& \lambda^{\mathrm{eq}}(M)^{\otimes 2^{d+1}} \overset{(p)}{\cong} \lambda^{\mathrm{eq}}(b^*(M))^{\otimes 2^{d+1}} \\
\overset{(o)}{\cong} \quad & \lambda^{\mathrm{eq}}(\mu^*(b^*(M)) \otimes P_d(\mathcal{O}_{\widetilde{X}_G} - N_{\widetilde{X}_G/\widetilde{X}})) \\
\overset{(l)}{\cong} \quad & \lambda^{\mathrm{eq}}(\iota^*(M) \\
\otimes \quad & \mathrm{R}^\bullet p_*^{\mathrm{eq}}\Big(\mathcal{O}_{\widetilde{X}_G}^{\oplus 2^d} + 2^{d-1}(\mathcal{O}_{\widetilde{X}_G}^{\oplus 2} - (\mathcal{O}_{\widetilde{X}_G} - N_{\widetilde{X}_G/X})) + 2^{d-2}(\mathcal{O}_{\widetilde{X}_G}^{\oplus 2} - (\mathcal{O}_{\widetilde{X}_G} - N_{\widetilde{X}_G/\widetilde{X}}))^{\otimes 2} + \dots \\
+ \quad & (\mathcal{O}_{\widetilde{X}_G}^{\oplus 2} - (\mathcal{O}_{\widetilde{X}_G} - N_{\widetilde{X}_G/\widetilde{X}}))^{\otimes d}\Big)) \\
\overset{(m)}{\cong} \quad & \lambda^{\mathrm{eq}}(\iota^*(M) \\
\otimes \quad & \mathrm{R}^\bullet p_*^{\mathrm{eq}}\Big(\mathcal{O}_{\widetilde{X}_G}^{\oplus 2^d} + 2^{d-1}(\mathcal{O}_{\widetilde{X}_G} + N_{\widetilde{X}_G/\widetilde{X}}) + 2^{d-2}(\mathcal{O}_{\widetilde{X}_G} + N_{\widetilde{X}_G/\widetilde{X}})^{\otimes 2} + \dots \\
+ \quad & (\mathcal{O}_{\widetilde{X}_G} + N_{\widetilde{X}_G/\widetilde{X}})^{\otimes d}\Big)) \\
\overset{(n)}{\cong} \quad & \lambda^{\mathrm{eq}}(\iota^*(M) \otimes \mathrm{R}^\bullet p_*^{\mathrm{eq}}\Big(\sum_{i=0}^{d} \sum_{j=0}^{i} 2^{d-i} \binom{i}{j} (N_{\widetilde{X}_G/\widetilde{X}})^{\otimes j}\Big))
\end{aligned}
$$

For equality (l), use the projection formula (Proposition 5.2) and the fact that the functors $\mathrm{R}f_*^{\mathrm{eq}} \cdot \mathrm{R}b_*^{\mathrm{eq}}$ and $\mathrm{R}(f \circ b)_*^{\mathrm{eq}}$ are naturally isomorphic (see discussion after Lemma 5.1). Equality (m) is a simple cancellation. Equality (n) follows from the equivariant analogue of Lemma 4.1 and from the polynomial identity $P_d(1-t) = \sum_{i=0}^{d} \sum_{j=0}^{i} 2^{d-i} \binom{i}{j} t^j$, which itself follows from the binomial formula. Equality (o) follows from (11). Equality (p) follows from the projection formula and the fact that $\mathrm{R}^\bullet b_*(\mathcal{O}_{\widetilde{X}}) = \mathcal{O}_X$ (see [13, VI, §4, proof of Prop. 4.1] for lack of a better reference).

Now since $\widetilde{X}_G = \mathbb{P}(N)$ we have

$$
\mathrm{R}^\bullet p_*(N_{\widetilde{X}_G/\widetilde{X}}^{\otimes j}) \cong \mathrm{Sym}^j(N)
$$

(see [19, Lemma 3.1]) and we obtain

$$
\lambda^{\mathrm{eq}}(M)^{\otimes 2^{d+1}} \cong \lambda^{\mathrm{eq}}\Big(\iota^*(M) \otimes \sum_{i=0}^{d} \sum_{j=0}^{i} 2^{d-i} \binom{i}{j} \mathrm{Sym}^j(N)\Big).
$$

Now note that we have the formal equality

$$
\sum_{i=0}^{d} \sum_{j=0}^{i} 2^{d-i} \binom{i}{j} \mathrm{Sym}^j(N) = \sum_{j=0}^{d} \Big[\sum_{i=0}^{d-j} 2^{d-j-i} \binom{i+j}{j}\Big] \mathrm{Sym}^j(N).
$$

To simplify this expression, we shall make use of the following combinatorial lemma, that was kindly communicated to us by E. Gomezllata Marmolejo.

Lemma 6.3 (E. Gomezllata Marmolejo). *For $0 \leq j \leq d$, we have*

$$\sum_{i=0}^{d-j} 2^{d-j-i}\binom{i+j}{j} = \sum_{i=0}^{d-j}\binom{d+1}{i}.$$

Proof. (of Lemma 6.3). The equality clearly holds if $d = j$. We prove it by induction on d, starting at $d = j$:

$$\begin{aligned}
\sum_{i=0}^{d-j}\binom{d+1}{i} &= \binom{d}{0} + \sum_{i=1}^{d-j}[\binom{d}{i} + \binom{d}{i-1}] = 2[\sum_{i=0}^{d-j-1}\binom{d}{i}] + \binom{d}{d-j} \\
&= 2[\sum_{i=0}^{(d-1)-j}\binom{(d-1)+1}{i}] + \binom{d}{j} = 2[\sum_{i=0}^{(d-1)-j} 2^{(d-1)-j-i}\binom{i+j}{j}] + \binom{d}{j} \\
&= \sum_{i=0}^{d-j} 2^{d-j-i}\binom{i+j}{j}
\end{aligned} \tag{12}$$

The first and third equality in (12) follow from standard properties of binomial coefficients, the second and last one are just simplifications and the fourth one relies on the inductive hypothesis. $\square$

Using Lemma 6.3, we finally get the advertised canonical isomorphism

$$\begin{aligned}
\lambda^{\mathrm{eq}}(M)^{\otimes 2^{d+1}} &\cong \lambda^{\mathrm{eq}}\Big(\iota^*(M) \otimes \sum_{j=0}^{d}\Big[\sum_{i=0}^{d-j}\binom{d+1}{i}\Big]\mathrm{Sym}^j(N)\Big) \\
&\cong \bigotimes_{j=0}^{d} \lambda^{\mathrm{eq}}(\iota^*(M) \otimes \mathrm{Sym}^j(N))^{\otimes \sum_{i=0}^{d-j}\binom{d+1}{i}}
\end{aligned}$$

Note again that this isomorphism is invariant under any base change to a locally noetherian scheme by Lemma 6.2 and by the fact that it is invariant under any base change to a locally noetherian scheme when X_G is a Cartier divisor. $\square$

7 Local refinement of the Adams-Riemann-Roch formula

We shall now prove Theorem 1.1. We recall the terminology. We let $\pi : Y \to S$ be a smooth and locally projective morphism of locally noetherian schemes. We suppose that the fibres of π are geometrically connected and that π has constant relative dimension $d > 0$. We suppose that 2 is invertible on S. We want to prove that there is a canonical isomorphism

$$\lambda(L)^{\otimes 2^{2d+2}} \cong \bigotimes_{j=0}^{2d} \lambda(L^{\otimes 2} \otimes \mathrm{Sym}^j(\Omega_{Y/S}))^{\otimes (-1)^j \sum_{i=0}^{2d-j} \binom{2d+1}{i}}$$

(this is (1) in Theorem 1.1) which is invariant under any base change to a locally noetherian scheme.

We shall write

$$X := Y \times_S Y$$

and we shall write $\pi_1 : X \to Y$ and $\pi_2 : X \to Y$ for the two projections. The group scheme $G = \mathbb{Z}/2\mathbb{Z}$ acts on X by swapping the coordinates, with fixed point scheme the relative diagonal Δ. The diagonal Δ is then regularly immersed.

Note that we used the fact that the fibres of π are smooth and geometrically connected here. If π is only supposed to be smooth, the diagonal Δ might not be regularly immersed. This can be seen on the example of a finite and étale morphism. In that case, the immersion of the diagonal is open and closed and thus Δ is not a Cartier divisor.

Let L be a line bundle on Y and suppose that L is cohomologically flat over S (see the beginning of the introduction for the definition of cohomological flatness). The line bundle $M = \pi_1^*(L) \otimes \pi_2^*(L)$ is naturally G-equivariant and $M|_\Delta \cong L^{\otimes 2}$ carries the trivial action. Furthermore $N_{\Delta/X} \cong \Omega_{Y/S}$ by definition.

Proposition 7.1. *We have a canonical isomorphism*

$$\lambda^{\mathrm{eq}}(M) \cong \lambda(L)^{\otimes 2}$$

where $\lambda^{\mathrm{eq}}(M)$ is computed using the above equivariant structure on M. This isomorphism is invariant under any base change to a locally noetherian scheme.

Lemma 7.2. *Let W be a vector bundle on a locally noetherian scheme T. Suppose that 2 is invertible on T. Endow $W \otimes W$ with the G-action which swaps the factors. There is a canonical isomorphism*

$$\det{}^{\mathrm{eq}}(W \otimes W) := \det((W \otimes W)_+) \otimes \det((W \otimes W)_-)^\vee \cong \det(W)^{\otimes 2} \tag{13}$$

which is compatible with any base change to a locally noetherian scheme.

Proof. (of Lemma 7.2) Note that we have by definition

$$\mathrm{Sym}^2(W) := (W \otimes W)_+$$

and

$$\Lambda^2(W) := (W \otimes W)_-.$$

The identity (13) can be proven "by pure thought". We sketch the argument, leaving some of the details to the reader. Let $r := \mathrm{rk}(W)$. Recall that there is an additive and exact functor A from the additive category of the GL_r-comodules (ie representations of the group scheme GL_r), which are finitely generated and free $\mathbb{Z}$-modules, to the additive category of vector bundles over T. This functor can be described as follows. Choose an open covering (U_i) of S, such that $W|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$ for all indices i. This leads to transition functions $\tau_{ij} : U_i \cap U_j \to \mathrm{GL}_r(U_i \cap U_j)$. Now let $h \geq 0$ and choose a GL_r-comodule structure on $\mathbb{Z}^h$. This corresponds to a homomorphism of group schemes $\rho : \mathrm{GL}_r \to \mathrm{GL}_h$. We then define the vector bundle $A(W)$ as the vector bundle described by the transition functions $\rho(\tau_{ij})$. See [17, Exp. VI, after Th. 3.3] for more details on this. The functor A is compatible by construction with all the usual tensor constructions (tensor powers, exterior powers, etc.) and the construction of A is naturally compatible with any base change of W. Let now V be the standard representation of GL_r (so that $V = \mathbb{Z}^r$ as a $\mathbb{Z}$-module). Consider the two GL_r-comodules $\det(\Lambda^2(V))$ and $\det(V)$. These are both one-dimensional GL_r-representations. Since the one dimensional GL_r-comodules are all of the form $(\det(V))^{\otimes n}$ for some $n \in \mathbb{Z}$ (see eg [29, par. 3.8] for this), we see that there exists a uniquely determined integer m and an isomorphism of comodules

$$\det(\Lambda^2(V)) \cong \det(V)^{\otimes m}.$$

We fix one such isomorphism (it is actually fixed up to sign, since $\det(W)$ is a one dimensional $\mathbb{Z}$-module). In view of the definition of the functor $A(\cdot)$, we see that this isomorphism induces an isomorphism of vector bundles

$$\det(\Lambda^2(W)) \cong \det(W)^{\otimes m}$$

To compute m, it is sufficient to find a locally noetherian scheme Z and a vector bundle J of rank r on Z, such that $\det(\Lambda^2(J))$ is isomorphic to at most one tensor power of $\det(J)$. The scheme $\mathbb{P}^1$ has this property, since $\mathrm{Pic}(\mathbb{P}^1)) = \mathbb{Z}$, provided $\det(J) \not\cong \mathcal{O}_{\mathbb{P}^1}$. So supposing that $Z = \mathbb{P}^1$ and $J = \mathcal{O}(1)^{\oplus r}$, we compute

$$\det(\Lambda^2(J)) \cong \det(\bigoplus_{1 \leq i < j \leq r} \mathcal{O}(1) \otimes \mathcal{O}(1)) \cong \mathcal{O}(2 \sum_{1 \leq i < j \leq r} 1) = \mathcal{O}(2\binom{r}{2})$$

We can repeat this reasoning for $\mathrm{Sym}^2(\cdot)$ in place of $\Lambda^2(\cdot)$ and we obtain

$$\det(\mathrm{Sym}^2(J)) = \mathcal{O}(2\binom{r+1}{r-1}).$$

We conclude that for any T and W, we have

$$\det(\Lambda^2(W)) \cong \det(W)^{\otimes\frac{(r-1)!}{(r-2)!}} \cong \det(W)^{\otimes(r-1)}$$

and

$$\det(\mathrm{Sym}^2(W)) \cong \det(W)^{\otimes\frac{(r+1)!}{r(r-1)!}} \cong \det(W)^{\otimes(r+1)}$$

and the lemma follows from these two equations. □

Lemma 7.3. *Let W be a vector bundle on a locally noetherian scheme T. Suppose that 2 is invertible on T. Let G be the action of G on $W \oplus W$, which swaps the summands. Then there is a canonical isomorphism*

$$\det{}^{\mathrm{eq}}(W \oplus W) := \det((W \oplus W)_+) \otimes \det((W \oplus W)_-)^\vee \cong \mathcal{O}_T, \tag{14}$$

which is compatible with any base change to a locally noetherian scheme.

Proof. (of Lemma 7.3). Note that the diagonal morphism of sheaves $W \to W \oplus W$ identifies $(W \oplus W)_+$ with W. Similarly, the antidiagonal morphism $W \to W \oplus W$ (given by the formula $w \mapsto (w, -w)$) identifies $(W \oplus W)_-$ with W. The lemma follows from this. □

Proof. (of Proposition 7.1) Let $f : X \to S$ be the canonical morphism. By the Künneth formula (see [14, III, par. 6, Th. 6.7.3]), we have a canonical isomorphism

$$\mathrm{R}^i f_*(M) \cong \bigoplus_t \mathrm{R}^t\pi_*(L) \otimes \mathrm{R}^{i-t}\pi_*(L). \tag{15}$$

Note that we used the fact that L is cohomologically flat here. The vector bundle

$$\bigoplus_t \mathrm{R}^t\pi_*(L) \otimes \mathrm{R}^{i-t}\pi_*(L)$$

carries a natural G-action by permutation, namely the action such that the non trivial element of G sends $\bigoplus_t w_t \otimes w_{i-t}$ to $\bigoplus_t (-1)^{t(i-t)} w_{i-t} \otimes w_t$. By the Koszul rule of signs, the isomorphism (15) becomes G-equivariant with this choice of G-action on the righthand side. Let $\mathrm{sgn} : G \to \{0, 1\}$ be the non trivial character of G. Let us first suppose that i is odd. We compute

$$\det{}^{\mathrm{eq}}(\mathrm{R}^i f_*(M)) \cong \bigotimes_{0 \le t \le \lfloor i/2 \rfloor} \det{}^{\mathrm{eq}}(\mathrm{R}^t\pi_*(L) \otimes \mathrm{R}^{i-t}\pi_*(L) \oplus \mathrm{R}^{i-t}\pi_*(L) \otimes \mathrm{R}^t\pi_*(L)). \tag{16}$$

In the righthand side of the isomorphism (16), the terms

$$\mathrm{R}^t\pi_*(L) \otimes \mathrm{R}^{i-t}\pi_*(L) \oplus \mathrm{R}^{i-t}\pi_*(L) \otimes \mathrm{R}^t\pi_*(L)$$

carry a G-equivariant structure of the form considered in Lemma 7.3. We thus see that we have a canonical isomorphism

$$\det^{\mathrm{eq}}(\mathrm{R}^i f_*(M)) \cong \mathcal{O}_S$$

Now suppose that i is even. We then have

$$\begin{aligned} & \det^{\mathrm{eq}}(\mathrm{R}^i f_*(M)) \\ \cong\ & \det^{\mathrm{eq}}(\mathrm{R}^{i/2}\pi_*(L) \otimes \mathrm{R}^{i/2}\pi_*(L)) \otimes \bigotimes_{0 \le t < i/2} \det^{\mathrm{eq}}(\mathrm{R}^t\pi_*(L) \otimes \mathrm{R}^{i-t}\pi_*(L) \oplus \mathrm{R}^{i-t}\pi_*(L) \otimes \mathrm{R}^t\pi_*(L)). \end{aligned}$$

Here the summands $\mathrm{R}^t\pi_*(L) \otimes \mathrm{R}^{i-t}\pi_*(L) \oplus \mathrm{R}^{i-t}\pi_*(L) \otimes \mathrm{R}^t\pi_*(L)$ carry a G-equivariant structure of the type considered in Lemma 7.3 multiplied by sgn^t and the summand $\mathrm{R}^{i/2}\pi_*(L) \otimes \mathrm{R}^{i/2}\pi_*(L)$ carries the equivariant structure considered in Lemma 7.2 multiplied by $\mathrm{sgn}^{i/2}$. As before, we conclude that

$$\det^{\mathrm{eq}}(\mathrm{R}^i f_*(M)) \cong \det^{\mathrm{eq}}(\mathrm{R}^{i/2}\pi_*(L) \otimes \mathrm{R}^{i/2}\pi_*(L)).$$

On the other hand, by Lemma 7.2, we have

$$\det^{\mathrm{eq}}(\mathrm{R}^{i/2}\pi_*(L) \otimes \mathrm{R}^{i/2}\pi_*(L)) \cong \det(\mathrm{R}^{i/2}\pi_*(L))^{\otimes 2(-1)^{i/2}}$$

Summarising, we have

$$\det^{\mathrm{eq}}(\mathrm{R}^i f_*(M)) \cong \det(\mathrm{R}^{i/2}\pi_*(L))^{\otimes 2(-1)^{i/2}}$$

if i is even and

$$\det^{\mathrm{eq}}(\mathrm{R}^i f_*(M)) \cong \mathcal{O}_S$$

if i is odd. We conclude that we have

$$\lambda^{\mathrm{eq}}(M) = \bigotimes_{i \ge 0} \det^{\mathrm{eq}}(\mathrm{R}^i f_*(M))^{\otimes(-1)^i} \cong \bigotimes_{i \ge 0,\, i \text{ even}} \det^{\mathrm{eq}}(\mathrm{R}^i f_*(M)) \cong \bigotimes_{j \ge 0} \det(\mathrm{R}^j\pi_*(L))^{\otimes 2(-1)^j} = \lambda(L)^{\otimes 2}$$

which is what we wanted to prove. $\square$

Remark 7.4. Lemma 7.1 is the only place in the proof of Theorem 1.1 where we use the assumption that L is cohomologically flat over S. We conjecture that Lemma 7.1 holds without that assumption. If this is true then Theorem 1.1 holds without the assumption that L is cohomologically flat over S. If one tries to prove Lemma 7.1 without the assumption of cohomological flatness, one is faced with a difficult problem in the linear algebra of perfect complexes that to date we have not been able to solve. See also [28] about this.

Finally, combining Proposition 7.1 and Theorem 6.1 we get an isomorphism

$$\lambda(L)^{\otimes 2^{2d+2}} \cong \bigotimes_{j=0}^{2d} \lambda(L^{\otimes 2} \otimes \mathrm{Sym}^j(\Omega_{Y/S}))^{\otimes(-1)^j \sum_{i=0}^{2d-j} \binom{2d+1}{i}}. \tag{17}$$

and this completes the proof of Theorem 1.1.

8 Proof of Proposition 1.3

We work with the assumptions and terminology of Theorem 1.1 and we suppose that $d = 1$. Consider the formal linear combinations of line bundles

$$\mathrm{MT}(L) := 7L^{\otimes 2} - 4\Omega_{Y/S} \otimes L^{\otimes 2} + L^{\otimes 2} \otimes \Omega_{Y/S}^{\otimes 2}$$

and

$$\mathrm{DT}(L) := 18 + 6L^{\otimes 2} \otimes \Omega_{Y/S}^{\vee} - 6L \otimes \Omega_{Y/S}^{\vee}$$

Theorem 1.1 for $\dim(Y/S) = 1$ says that we have a canonical isomorphism

$$\lambda(\mathrm{MT}(L)) \cong \lambda(L)^{\otimes 16}.$$

Similarly, Deligne's theorem (4) implies that there is a canonical isomorphism

$$\lambda(\mathrm{DT}(L)) \cong \lambda(L)^{\otimes 18}.$$

We shall prove that the line bundle $\lambda\Big(9\mathrm{MT}(L) - 8\mathrm{DT}(L)\Big)$ is canonically trivial, even without the assumption that L is cohomologically flat over S. Assuming Theorem 1.1 for $\dim(Y/S) = 1$, this will prove that $\lambda(8\mathrm{DT}(L))$ is canonically trivial, which is the conclusion of Proposition 1.3 (note that $9 \cdot 16 = 8 \cdot 18 = 144$).

Now since L is arbitrary, it is sufficient to prove that

$$\lambda\Big(9\mathrm{MT}(L \otimes \Omega_{Y/S}) - 8\mathrm{DT}(L \otimes \Omega_{Y/S})\Big)$$

is canonically trivial.

We first compute

$$9\mathrm{MT}(L\otimes\Omega_{Y/S}) - 8\mathrm{DT}(L\otimes\Omega_{Y/S}) = (9\Omega_{Y/S}^{\otimes 4} - 36\Omega_{Y/S}^{\otimes 3} + 63\Omega_{Y/S}^{\otimes 2} - 48\Omega_{Y/S}) \otimes L^{\otimes 2} + 48L - 144.$$

Let

$$P(x, y) := (9y^4 - 36y^3 + 63y^2 - 48y)x^2 + 48x - 144 \in \mathbb{Z}[x, y].$$

We compute

$$
\begin{aligned}
& P(x,y) = P(1-(1-x), 1-(1-y)) \\
= \;& (9(1-y)^4 + 9(1-y)^2 - 6(1-y) - 12)(1-x)^2 \\
+ \;& (-18(1-y)^4 - 18(1-y)^2 + 12(1-y) - 24)(1-x) + (9(1-y)^4 + 9(1-y)^2 - 6(1-y) - 108) \\
= \;& -12(1-x)^2 + (12(1-y) - 24)(1-x) + (9(1-y)^2 - 6(1-y) - 108) \\
& \operatorname{mod}\,((1-y)^3, (1-y)(1-x)^2, (1-y)^2(1-x), (1-x)^3)
\end{aligned}
$$

(where $((1-y)^3, (1-y)(1-x)^2, (1-y)^2(1-x), (1-x)^3)$ refers to the ideal of $\mathbb{Z}[x,y]$ generated by $(1-y)^3$, $(1-y)(1-x)^2$, $(1-y)^2(1-x)$ and $(1-x)^3$).

We deduce from this identity, Lemma 4.1 and Corollary 4.3 that we have a canonical isomorphism

$$
\begin{aligned}
& \lambda\Big(9\mathrm{MT}(L\otimes\Omega_{Y/S}) - 8\mathrm{DT}(L\otimes\Omega_{Y/S})\Big) \\
\cong \;& \lambda\Big(-12(1-L)^2 + (12(1-\Omega_{Y/S}) - 24)(1-L) + (9(1-\Omega_{Y/S})^2 - 6(1-\Omega_{Y/S}) - 108)\Big) \\
\cong \;& \lambda\Big(-12L^{\otimes 2} + (12\Omega_{Y/S} + 36)\otimes L + (9\Omega_{Y/S}^{\otimes 2} - 24\Omega_{Y/S} - 129)\Big)
\end{aligned}
$$

Note that by Grothendieck duality, we have a canonical isomorphism

$$\lambda(\Omega_{Y/S}\otimes L) \cong \lambda(L^\vee).$$

We deduce that we have

$$\lambda\Big(9\mathrm{MT}(L\otimes\Omega_{Y/S}) - 8\mathrm{DT}(L\otimes\Omega_{Y/S})\Big) \cong \lambda\Big(-12L^{\otimes 2} + 12L^\vee + 36L + 9\Omega_{Y/S}^{\otimes 2} - 24\Omega_{Y/S} - 129\Big). \tag{18}$$

Now by Corollay 4.3, we have a canonical trivialisation

$$\lambda(L^\vee \otimes (1-L)^{\otimes 3}) \cong \mathcal{O}_S$$

or in other words a canonical isomorphism

$$\lambda(L^\vee) \cong \lambda(L^{\otimes 2} - 3L + 3).$$

Merging this with (18), we obtain a canonical isomorphism

$$\lambda\Big(9\mathrm{MT}(L) - 8\mathrm{DT}(L)\Big) \cong \lambda(9\Omega_{Y/S}^{\otimes 2} - 117) \cong (\lambda(\Omega_{Y/S}^{\otimes 2}) \otimes \lambda(\mathcal{O}_X)^{\otimes -13})^{\otimes 9}. \tag{19}$$

To conclude, notice that by (3), we have canonically

$$\lambda(\Omega_{Y/S}^{\otimes 2}) \cong \lambda(\mathcal{O}_X)^{\otimes 13}.$$

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