On a conjecture of Esnault and Langer

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Set-up

$\diamond$ K the function field of a smooth and proper curve C over an algebraically closed field k of characteristic $p > 0$

$\diamond$ A an abelian variety over K. We suppose that $\mathrm{Tr}_{K|k}(A) = 0$.

$\diamond$ $\mathcal{A}$ a semiabelian model of A over C with zero section $\epsilon : C \to \mathcal{A}$

$\diamond$ $\omega = \omega_{\mathcal{A}} := \epsilon^*(\Omega_{\mathcal{A}/C})$

$\diamond$ $F_{A/K} : A \to A^{(p)}$ and $F_{\mathcal{A}/C} : \mathcal{A} \to \mathcal{A}^{(p)}$ the relative Frobenii

$\diamond$ $V_{A^{(p)}/K} : A^{(p)} \to A$ the relative Verschiebung morphism

$\diamond$ $K^{\mathrm{perf}} := K^{p^{-\infty}} = \cup_{i \geq 0} K^{p^{-i}} \subseteq \bar{K}$.

Verschiebung divisible points

Let us define

$$\mathrm{IVD}(A(K)) := \bigcap_{i\geq 0} V^{\circ i}_{A^{(p^i)}/K}(A^{(p^i)}(K))$$

Esnault and Langer conjectured that $\mathrm{IVD}(A(K))$ is a finite torsion group of order prime to p.

The purpose of this talk is to outline the proof of this conjecture.

Note that the conjecture of Esnault and Langer is apparently more general (in their formulation K is any finitely field over k, no assumption of semiabelian reduction is made) but the general case follows from the case proven here using Grothendieck's semiabelian reduction theorem and a Bertini argument.

Here is a dual, perhaps more compelling version of this conjecture.

Suppose that $\mathcal{L}$ is a line bundle on A. Suppose that there are line bundles $\mathcal{L}_i$ on $A^{(p^i)}$ $(i \geq 1)$ such that

$$F^*_{A^{(p)}/K}(\mathcal{L}_1) = \mathcal{L},\ F^*_{A^{(p^2)}/K}(\mathcal{L}_2) = \mathcal{L}_1,\ F^*_{A^{(p^3)}/K}(\mathcal{L}_3) = \mathcal{L}_2 \text{ etc.}$$

Then $\mathcal{L}$ is a torsion line bundle of order prime to p.

This has application to the theory of stratified bundles in positive characteristic (see work of Esnault and Langer).

In the case of elliptic curves and when k is a finite field, the conjecture of Esnault and Langer is a consequence of work of D. Ghioca (without the "prime-to-p" statement).

When $k = \bar{\mathbb{F}}_p$ and all the simple factors of A have p-rank > 0, it is also the consequence of recent work of E. Ambrosi, together with earlier work by him and M. d'Addezio.

Here is an application of the conjecture to the structure of perfect points on abelian varieties.

Proposition

Suppose that A is ordinary. Then the group $\cap_{j\geq 0} p^j \cdot A(K^{\mathrm{perf}})$ is a finite group of order prime to p.

To see this, note that $\mathrm{IVD}(A(K^{p^{-i}}))$ consists precisely of the points of $A(K^{p^{-i}})$, which are p^∞-divisible in $A(K^{\mathrm{perf}})$.

Two preliminary results

To study $A(K^{\mathrm{perf}})$ a basic tool is the following. Let $E \subseteq C$ be the divisor of bad reduction of $\mathcal{A}$ and $U := C \backslash E$.

Theorem (Artin-Milne)

There is a canonical injective group homomorphism

$$H^1_{\mathrm{fppf}}(K, F_{A/K}) \hookrightarrow \mathrm{Hom}_K(F_K^*(\omega_K), \Omega_{K/k}).$$

This can be refined as follows:

Theorem (R.)

There is a canonical injective group homomorphism

$$\mathrm{Sel}(K, F_{A/K}) \hookrightarrow \mathrm{Hom}_C(F_C^*(\omega), \Omega_{C/k}(E)).$$

Here $\mathrm{Sel}(K, F_{A/K})$ is the Selmer group for $F_{A/K}$.

Recall that

$$A^{(p)}(K)/F_{A/K}(A(K)) \subseteq \mathrm{Sel}(K, F_{A/K}) \subseteq H^1_{\mathrm{fppf}}(K, F_{\mathcal{A}_K/K}).$$

This theorem can be proven by using a semistable compactification of $\mathcal{A}$ (Faltings-Chai, Mumford) and log-differentials or by a direct analysis of Raynaud uniformisations at the points of E (ongoing work of my student Zhenhua Wu).

Proposition (R.)

Let G be a finite flat group scheme of height one on C.

There is a (necessarily unique) closed subgroup scheme $G_\mu \subseteq G$, such if H is a multiplicative group scheme of height one on C and $\phi : H \to G$ a group homomorphism, then ϕ factors through G_μ.

The group scheme G_μ is compatible with Frobenius twists.

It is easy to see that the Lie algebra of G_μ is semistable of degree 0.

Note that if G is the kernel of $F_{\mathcal{A}/C} : \mathcal{A} \to \mathcal{A}^{(p)}$ then the Lie algebra of G is none other than $\omega^\vee$.

Hence, if ω is ample (in the sense of Hartshorne) then $G_\mu = 0$. So G_μ can only be non trivial if ω is not too positive.

Recall that commutative finite flat group schemes of height are in one-to-one correspondence with locally free sheaves in commutative p-Lie algebras.

The fundamental tools used to prove the proposition are the following three results:

Lemma (inspired by a result of JB Bost in char. 0)

Let V be a sheaf in commutative p-Lie algebras V over C. Suppose that the Harder-Narasimhan filtration

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{\mathrm{hn}(V)} = V$$

of V is Frobenius semistable.

Then for any V_i such that $\mu_{\min}(V_i) \geq 0$, V_i is a subsheaf in commutative p-Lie algebras V over C.

If $\mu_{\min}(V_i) > 0$ then V_i is biinfinitesimal.

Theorem (Langer)

Let V be a non zero coherent locally free sheaf on C. There is an $\ell_0 = \ell_0(V) \in \mathbb{N}$ such that the quotients of the Harder-Narasimhan filtration of $F_C^{\circ \ell_0,*}(V)$ are all Frobenius semistable.

Lemma (descent)

Let G be a finite flat commutative group scheme over C.

Let $T \to C$ be a flat, radicial and finite morphism and let $\phi : H \hookrightarrow G_T$ be a closed T-subgroup scheme, which is finite, flat and multiplicative.

Then there is a finite flat closed subgroup scheme $\phi_0 : H_0 \hookrightarrow G$, such that $\phi_{0,T} \simeq \phi$.

Here is now a proof of the proposition in the simple case.

Suppose that V the sheaf of commutative p-Lie algebras of G.

Let W be a sheaf of multiplicative p-Lie algebras, so that in particular $F_C^* W \simeq W$.

A simple calculation with Langer's theorem shows that W is Frobenius semistable of slope 0.

Let $\phi : W \to V$ be a map of p-Lie algebras.

Let $l_0 \geq 0$ such that $F_C^{\circ \ell_0,*}(V)$ has a Frobenius semistable HN filtration.

Suppose for simplicity that V is is generically multiplicative. Then a simple calculation shows that $\mu_{\max}(F_C^{\circ \ell_0,*}(V)) \leq 0$.

Then the map

$$F_C^{\circ \ell_0,*}(\phi) : F_C^{\circ \ell_0,*}(W) \to F_C^{\circ \ell_0,*}(V)$$

factors through $F_C^{\circ \ell_0,*}(V)_{\max}$ because $F_C^{\circ \ell_0,*}(W)$ is semistable of slope 0.

If $\mu((F_C^{\circ \ell_0,*}(V))_{\max}) < 0$ then $F_C^{\circ \ell_0,*}(\phi) = 0$ and hence $\phi = 0$.

Then $G_\mu = 0$.

Suppose that $\mu((F_C^{\circ\ell_0,*}(V))_{\max}) = 0$.

By the the first lemma $F_C^{\circ\ell_0,*}(V)_{\max}$ is a p-Lie subalgebra of $F_C^{\circ\ell_0,*}(V)$ and it must be multiplicative.

By the descent lemma, its descends to a p-Lie subalgebra W_0 of W and ϕ then factors through W_0.

W_0 is the p-Lie algebra of G_μ.

Sketch of the proof of the conjecture of Esnault and Langer

Let now $G_{\mathcal{A}}$ be the kernel of $F_{\mathcal{A}/C} : \mathcal{A} \to \mathcal{A}^{(p)}$. This is a finite flat group scheme of height one over C.

The coLie algebra of $G_{\mathcal{A}}$ is canonically isomorphic to ω.

Proposition (R.)

Let $n >> 1$. If $x \in \mathrm{IVD}(A^{(p^n)}(K)) \cap A(K)$, then the map

$$F_C^{\circ n,*}(\omega) \to \Omega_{C/k}(E)$$

corresponding to x factors through

$$F_C^{\circ n,*}(\mathrm{coLie\ algebra}(G_{\mathcal{A},\mu})).$$

The proof is based on the theory of semistable sheaves together with the existence of the diagram

$$\begin{array}{ccc} A^{(p)}(L) & \longrightarrow & \mathrm{Hom}_C(\omega_{\mathcal{A}^{(p)}}, \Omega_{C/k}(E)) \\ \uparrow{\scriptstyle V_{A^{(p^2)}/L}} & & \uparrow{\scriptstyle V^*_{\mathcal{A}^{(p^2)}/C}} \\ A^{(p^2)}(L) & \longrightarrow & \mathrm{Hom}_C(\omega_{\mathcal{A}^{(p^2)}}, \Omega_{C/k}(E)) \\ \uparrow & & \uparrow \\ \vdots & \longrightarrow & \vdots \end{array} \quad (1)$$

which heavily constrains the map $F_C^{\circ n,*}(\omega) \to \Omega_{C/k}(E)$.

Corollary

Let $\phi : A \to A/G_{\mathcal{A},\mu,K} := A_1$*. Then*

$$\phi(\mathrm{IVD}(A(K))) \subseteq p \cdot \mathrm{IVD}(A_1(K)).$$

Now consider the sequence of isogenies

$$A \to A_1 \to A_2 \to \ldots \quad (*)$$

where $A_1 = A/G_{\mathcal{A},\mu,K}$, $A_2 = A_1/G_{\mathcal{A}_1,\mu,K}$ etc.

The image of $\mathrm{IVD}(A(K))$ in $A_i(K)$ is thus divisible by p^i.

Proposition (R.)

The elements of the sequence $(*)$ *are members of a bounded family.*

This follows from a computation which shows that the (Faltings) heights of the members of (*) are bounded.

Proposition (R.)

The order of the image of $A_i(K)$ in the group of connected components of the Néron model of A_i over C is bounded independently of i for infinitely many i.

For the proof, we consider a semiabelian scheme $\mathcal{B}$ over $C \times_k S$, where S is of finite type over k, such that the A_i correspond to certain elements of $S(k)$.

Possibly after finite base change, $\mathcal{B}$ can be compactified into a family $\bar{\mathcal{B}}$, which is proper over $C \times_k U$ and smooth over U, for some an open subset U of S (work of Chai-Faltings and Künnemann).

Then we derive a bound from the fact that the fibres of $\bar{\mathcal{B}}$ over $C \times_k U$ have a bounded number of irreducible components.

Corollary

The denominators of the heights (for suitable polarisations) of the elements of $A_i(K)$ are bounded for infinitely many i.

Let now $x \in \mathrm{IVD}(A(K))$. The image x_i of x in $A_i(K)$ is divisible by p^i by the above and this implies that the height of x_i is divisible by p^{2i}.

Using the corollary, we see that the height of x_i must eventually be 0 and hence a torsion point, by the Lang-Néron theorem.

Hence x is also a torsion point, which is what we wanted to prove.

The fact that x_i is of order prime to p follows from the fact that the torsion of the $A_i(K)$ is bounded for infinitely many i (this is based on the fact that the points of vanishing height are represented by a scheme of finite type).