The conjecture of Beauville and Catanese over function fields

Y a projective variety over $\mathbb{C}$;

$\mathrm{Pic}^0(Y)$ the Picard variety of Y;

$S_m^{i,j}(Y) :=$

$\{\mathcal{L} \in \mathrm{Pic}^0(Y) | \dim_{\mathbb{C}} H^i_{\bar{\partial}}(Y, \Omega^j \otimes \mathcal{L}) \geq m\}$;

Conjecture of Beauville and Catanese [Thm.; Simpson; Green-Lazarsfeld]:

$S_m^{i,j}(Y)$ is a finite union of translates of abelian subvarieties of $\mathrm{Pic}^0(Y)$ by points of finite order (= "torsion subvariety").

The conjecture of Mordell-Lang

A an abelian variety over $\mathbb{C}$;

$G \subseteq A$ a subgroup such that $G \otimes_{\mathbb{Z}} \mathbb{Q}$ has a finite number of generators;

$S \subseteq G$ a subset;

$\overline{S} := \bigcap_{(X \supseteq S,\ X \text{ closed analytic})} X$

Mordell-Lang conjecture [Thm. Faltings et al.]:

$\overline{S}$ is a finite union of translates of abelian subvarieties of A (= "linear").

Beauville-Catanese $\sim$ Mordell-Lang ?

The conjecture of Beauville and Catanese as a Hilbert-Samuel theorem

Proposition. Fix $i \geq 0$. There exists $k = k(i) \geq 1$ such that $\dim_{\mathbb{C}} H^i(Y, \mathcal{L}^{\otimes(1+t \cdot k)}) \geq \dim_{\mathbb{C}}(Y, \mathcal{L})$ for all $\mathcal{L} \in \mathrm{Pic}^0(Y)$ and all $t \geq 1$.

(follows from the conjecture of B. and C.)

Proposition. Fix $i \geq 0$ and $\mathcal{L} \in \mathrm{Pic}^0(Y)$. There exists $k = k(\mathcal{L}, i), r = r(\mathcal{L}, i) \geq 1$, such that $\dim_{\mathbb{C}}(H^i(Y, \mathcal{L}^{\otimes(r+t \cdot k)})) \geq \dim_{\mathbb{C}}(Y, \mathcal{L}^{\otimes r})$ for all $t \geq 1$.

(follows from the conjecture of Mordell-Lang)

Review of the proofs of the conjecture of Beauville and Catanese

Green-Lazarsfeld [1991] prove the conjecture without "of finite order". They deduce from an analysis of the relative Dolbeault complex of the universal fibration on $\mathrm{Pic}^0(Y) \times Y$ that $S_m^{i,j}(Y)$ is totally geodesic.

Simpson [1993] shows that $\mathrm{Pic}^0(Y)$ carries a *Betti* and *de Rham* algebraic structure. Any closed irreducible real analytic subset of $\mathrm{Pic}^0(Y)$ which is algebraic for both *Betti* and *de Rham* is linear.

Pink-R. [2003] give a proof based on the properties of some specific l-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$. In the special case of function fields, they give a purely algebraic proof.

Ingredients of the proof of the conjecture of Beauville and Catanese over function fields I

Deligne-Illusie [1987] - absolute version.

Y a projective variety smooth over $\mathbb{Q}$;

p a prime number $\geq \dim(Y)$;

Y has good reduction $Y_p/\mathbb{F}_p$ at p;

$\mathcal{L}/Y_p$ a line bundle;

then for any $i, j, k \geq 0$

$$\sum_{i+j=k} \dim_{\mathbb{F}_p} H^i(Y_p, \Omega^j \otimes \mathcal{L})$$

$$\leq$$

$$\sum_{i+j=k} \dim_{\mathbb{F}_p} H^i(Y_p, \Omega^j \otimes \mathcal{L}^{\otimes p})$$

Ingredients. . . II

Deligne-Illusie - function fields.

C a smooth affine curve over $\mathbb{Q}$;

$\mathcal{Y}/C$ a smooth projective fibration;

p a prime number $\geq \dim(\mathcal{Y}/C)$;

C has good reduction C_p at p; $\mathcal{Y}$ has good reduction $\mathcal{Y}_p$ over C_p;

set $Y := \mathcal{Y}_{\mathbb{Q}(C)}$; $Y_p := \mathcal{Y}_{p,\mathbb{F}_p(C_p)}$;

$\mathcal{L}/Y_p$ a line bundle;

then for any $i, j, k \geq 0$

$$\sum_{i+j=k} \dim_{\mathbb{F}_p(C_p)} H^i(Y_p, \Omega^j \otimes \mathcal{L})$$

$$\leq$$

$$\sum_{i+j=k} \dim_{\mathbb{F}_p(C_p)} H^i(Y_p, \Omega^j \otimes \mathcal{L}^{\otimes p})$$

Application of the results of Deligne-Illusie to the conjecture of Beauville and Catanese for $Y/\mathbb{Q}(C)$

Notation.

$$S_m^k(Y) :=$$

$$\{\mathcal{L} \in \mathrm{Pic}^0(Y) | \sum_{i+j=k} \dim_{\overline{\mathbb{Q}(C)}} H^i(Y, \Omega^j \otimes \mathcal{L}) \geq m\};$$

Deligne-Illusie (function fields) + existence of the Picard scheme implies that for almost all primes (f.a.a.) p :

$$p \cdot S_m^k(Y)_p \subseteq S_m^k(Y)_p \ (*).$$

Problem. Study the varieties with the property (*).

Ingredients. . . III

Hrushovski (1) [1998] (or Pink-R.) Suppose that for some prime p, $\mathrm{Pic}^0(Y/\mathbb{Q}(C))$ has a good reduction $\mathrm{Pic}^0(Y/\mathbb{Q}(C))_p$ with no isotrivial factors. Then any closed subvariety Z of $\mathrm{Pic}^0(Y/\mathbb{Q}(C))_p$ such that $p^k \cdot Z = Z$ is linear $(k \geq 1)$.

Hrushovski (2). [1998] If $\mathrm{Pic}^0(Y/\mathbb{Q}(C))$ has no isotrivial factors then $\mathrm{Pic}^0(Y/\mathbb{Q}(C))_p$ has no isotrivial factors f.a.a. primes p.

Application of the results of Hrushovski and Deligne-Illusie to the conjecture of Beauville and Catanese

Deligne-Illusie + Hrushovski (1), (2) imply:

The irreducible components of maximal dimension of $S^k_m(Y) \subseteq \mathrm{Pic}^0(Y/\mathbb{Q}(C))$ are linear for any $k, m \geq 0$.

Proof: let Z be the union of these components. By Deligne-Illusie (function fields), $p \cdot Z \subseteq Z$, f.a.a. p. By Hrushovski (1), (2), Z_p is thus linear, f.a.a. p. Thus Z is linear.

Complements I

Y a smooth projective variety over $\mathbb{Q}$ (or $\mathbb{C}$);

$\mathcal{L}/Y$ a line bundle such that $\mathcal{L}^{\otimes n}$ is trivial ($n \geq 1$);

Proposition. If $(l, n) = 1$, then

$$\sum_{i+j=k} \dim_{\mathbb{C}} H^i(Y, \Omega^j \otimes \mathcal{L})$$

$$=$$

$$\sum_{i+j=k} \dim_{\mathbb{C}} H^i(Y, \Omega^j \otimes \mathcal{L}^{\otimes l}) \quad (**)$$

Proof: Fix $n, l \geq 1$. Deligne-Illusie (absolute version) implies that (**) holds for Y_p if $p = l \ (mod\ n)$. But by Dirichlet's theorem on arithmetic progressions, there are infinitely many primes p such that $p = l \pmod{n}$.

Complements II

Question. Is the equation (**) true in char. $p > 0$, for any l, n with $(l, n) = 1$?.

The Weil conjectures together with a positive answer to the question would imply the conjecture of Beauville and Catanese in general.