On the group of purely inseparable points of an abelian variety defined over a function field of positive characteristic II

Damian RÖSSLER*

September 20, 2018

Abstract

Let $A$ be an abelian variety over the function field $K$ of a curve over a finite field. We describe several mild geometric conditions ensuring that the group $A(K^{\text{perf}})$ is finitely generated and that the $p$-primary torsion subgroup of $A(K^{\text{sep}})$ is finite. This gives partial answers to questions of Scanlon, Ghioca and Moosa, and Poonen and Voloch. We also describe a simple theory (used to prove our results) relating the Harder-Narasimhan filtration of vector bundles to the structure of finite flat group schemes of height one over projective curves over perfect fields. Finally, we use our results to give a complete proof of a conjecture of Esnault and Langer on Verschiebung divisibility of points in abelian varieties over function fields in the situation where the base field is the algebraic closure of a finite field.

1 Introduction

Let $k$ be a finite field characteristic $p > 0$ and let $S$ be a smooth, projective and geometrically connected curve over $k$. Let $K := \kappa(S)$ be its function field. Let $A$ be an abelian variety of dimension $g$ over $K$. Let $K^{\text{perf}} \subseteq \bar{K}$ be the maximal purely inseparable extension of $K$, let $K^{\text{sep}} \subseteq \bar{K}$ be the maximal separable extension of $K$ and let $K^{\text{unr}} \subseteq K^{\text{sep}}$ be the maximal separable extension of $K$, which is unramified above every place of $K$. Finally, we

*Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, United Kingdom
let \( \mathcal{A} \) be a smooth commutative group scheme over \( S \) such that \( \mathcal{A}_K = A \). We shall write 
\[
\omega_{\mathcal{A}} := \epsilon^*_{A/S}(\Omega_{A/S})
\]
for the restriction of the cotangent sheaf of \( \mathcal{A} \) over \( S \) via the zero section 
\( \epsilon_{A/S} : S \to \mathcal{A} \) of \( \mathcal{A} \).

If \( G \) is an abelian group, we shall write
\[
\text{Tor}_p(G) := \{ x \in G \mid \exists n \geq 0 : p^n \cdot x = 0 \}
\]
and
\[
\text{Tor}^p(G) := \{ x \in G \mid \exists n \geq 0 : n \cdot x = 0 \land (n, p) = 1 \}.
\]

The aim of this text is to prove the following three theorems.

**Theorem 1.1.** (a) Suppose that \( A \) is geometrically simple. If \( A(K_{\text{perf}}) \) is finitely generated and of rank \( > 0 \) then \( \text{Tor}_p(A(K_{\text{sep}})) \) is a finite group.

(b) Suppose that \( A \) is an ordinary (not necessarily simple) abelian variety. If \( \text{Tor}_p(A(K_{\text{sep}})) \) is a finite group then \( A(K_{\text{perf}}) \) is finitely generated.

**Theorem 1.2.** Suppose that \( \mathcal{A} \) is a semiabelian scheme and that \( \mathcal{A} \) is a geometrically simple abelian variety over \( K \). If \( \text{Tor}_p(A(K_{\text{sep}})) \) is infinite, then

(a) \( \mathcal{A} \) is an abelian scheme;

(b) there is \( r_{\mathcal{A}} \geq 0 \) such that \( p^{r_{\mathcal{A}}} \cdot \text{Tor}_p(A(K_{\text{sep}})) \subseteq \text{Tor}_p(A(K_{\text{unr}})) \).

Furthermore, there is

(c) an abelian scheme \( \mathcal{B} \) over \( S \);

(d) an \( S \)-isogeny \( \mathcal{A} \to \mathcal{B} \), whose degree is a power of \( p \) and such that the corresponding isogeny \( \mathcal{A}_K \to \mathcal{B}_K \) is étale;

(e) an étale \( S \)-isogeny \( \mathcal{B} \to \mathcal{B} \) whose degree is \( > 1 \) and is a power of \( p \),

and

(f) (Voloch) if \( A \) is ordinary then the Kodaira-Spencer rank of \( A \) is not maximal;

(g) if \( \text{dim}(A) \leq 2 \) then \( \text{Tr}_{\overline{\mathcal{R}}|\overline{k}}(A_{\overline{K}}) \neq 0 \);

(h) for all closed points \( s \in S \), the \( p \)-rank of \( A_s \) is \( > 0 \).
Here $\text{Tr}_{\bar{K}/k}(A_\bar{K})$ is the $\bar{K}[k]$-trace of $A_\bar{K}$. This is an abelian variety over $\bar{k}$. See after Conjecture 1.3 below for references.

Theorems 1.1 and 1.2 (b) have applications in the context of the work of Poonen and Voloch on the Brauer-Manin obstruction over function fields. In particular Theorems 1.1 and 1.2 (b) show that the conclusion of [40, Th. B] holds whenever the underlying abelian variety is geometrically simple, has semistable reduction and violates any of the conditions in Theorem 1.2, in particular if it has a point of bad reduction. Theorems 1.1 and 1.2 (b) also feed into the "full" Mordell-Lang conjecture. See [43, after Claim 4.4] and [2, Intro.] for this conjecture. In particular, in conjunction with the main result of [17] Theorems 1.1 and 1.2 (b) show that the "full" Mordell-Lang conjecture holds if the underlying abelian variety is ordinary, geometrically simple, has semistable reduction and violates any of the conditions in Theorem 1.2, in particular if it has a point of bad reduction.

Let now $L$ be a field, which is finitely generated as a field over an algebraically closed field $l_0$ of characteristic $p$. Let $C$ be an abelian variety over $L$.

**Conjecture 1.3** (Esnault-Langer). *Suppose that for all $\ell \geq 0$ we are given a point $x_\ell \in C^{(p^\ell)}(L)$ and suppose that for all $\ell \geq 1$, we have $V_{C^{(p^\ell)}}(x_\ell) = x_{\ell - 1}$. Then the image of $x_0$ in $C(L)/\text{Tr}_{L/l_0}(C)(l_0)$ is a torsion point of order prime to $p$."

See [13, Rem. 6.3 and after Lemma 6.5]. This conjecture is important in the theory of stratified bundles in positive characteristic; see [13, Question 3 in the introduction] for details.

Here $C^{(p^\ell)}$ is the base change of $C$ by the $\ell$-th power of the absolute Frobenius morphism on $\text{Spec } L$ and $V_{C^{(p^\ell)}} : C^{(p^\ell)} \to C^{(p^{\ell - 1})}$ is the Verschiebung morphism. The abelian variety $\text{Tr}_{L/l_0}(C)$ is the $L|l_0$-trace of $C$ (see [10, par. 6] for the definition). It is an abelian variety over $l_0$ and the variety $\text{Tr}_{L/l_0}(C)_L$ comes with an injective morphism to $C$. This gives in particular an injective map $\text{Tr}_{L/l_0}(C)(l_0) \to C(L)$. The Lang-Néron theorem (see [30, chap. 6, Th. 2]) asserts that $C(L)/\text{Tr}_{L/l_0}(C)(l_0)$ is a finitely generated group.

In the present text, we shall call a point $x_0 \in C(L)$ with the property described in Conjecture 1.3 an *indefinitely Verschiebung divisible point*.

We prove:

**Theorem 1.4.** *Conjecture 1.3 holds if $l_0 = \bar{F}_p$."

Using a height argument due to Raynaud, Esnault and Langer prove in [13, Th. 6.2] that the image of $P_0$ in $C(L)/\text{Tr}_{L/l_0}(C)(l_0)$ is a torsion point (but they don’t show that the order of the torsion point is prime to $p$) under the assumption that $C$ has everywhere potential
good reduction in codimension one. The height argument breaks down in the presence of bad reduction because the orders of the component groups of the special fibres of the local Néron models of the varieties $C^{(p^j)}$ might increase with $\ell$. It would be very interesting to see if our proof (which does not use heights) could point to a way to salvage a height-theoretic proof even in the presence of bad reduction. A final remark is that in the course of our proof of Theorem 1.4, we show that to prove Conjecture 1.3 in general, it is sufficient to prove it in the situation where $L$ has transcendence degree one over $l_0$ (for any algebraically closed field $l_0$ of characteristic $p > 0$). See section 9, reduction (1).

Note that Theorem 1.4 has the following consequence, which is of independent interest: if $C$ is as in Conjecture 1.3, $C$ is ordinary, $l_0 = \overline{\mathbb{F}}_p$ and $\text{Tr}_{L^{\text{perf}}|l_0}(C_{L^{\text{perf}}}) = 0$ then

$$\bigcap_{j \geq 0} p^j \cdot C(L^{\text{perf}}) = \text{Tor}^p(C(L^{\text{perf}})).$$

The structure of the article is as follows. In section 2, we state various intermediate results, from which we shall deduce Theorems 1.1 and 1.2. Theorem 2.1 in subsection 2.1 is of independent interest and is (we feel) likely to be useful for the study of the geometry of (especially ordinary) abelian varieties in general. The results in subsection 2.1 are deduced from some results in the theory of finite flat groups schemes of height one over $S$, most of which follow from the existence of a Harder-Narasimhan filtration on their Lie algebras. These results on finite flat group schemes are proven in section 4 and for the convenience of the reader, we included a section (section 3) listing the results on semistable sheaves over curves in positive characteristic that we need. To the knowledge of the author, there are very few general results on the structure of finite flat group schemes in a global situation (eg when the base is not affine) and it seems that it is the first time time that the theory of semistability of vector bundles is being used in this context. In [8] a similar idea is used in characteristic 0, where it is applied to the study of formal groups over curves (recall that all groups schemes are smooth in characteristic 0, so the Lie algebras of finite flat group schemes vanish in characteristic 0). Lemma 4.3 below (which concerns finite flat group schemes of height one) is inspired by [8, Lemma 2.9]. A prototype of Lemma 4.3 can be found in [44, Lemma 9.1.3.1] but it is not applied to the study of group schemes there. The key results here are the Lemmata 4.3 and 4.7, which will hopefully lead to further generalisations (eg in the situation when the base scheme is of dimension higher than one). The results in subsection 2.2 do not require the theory of semistable sheaves and are based on geometric class field theory, the theory of Serre-Tate canonical liftings and on the existence of moduli schemes for abelian varieties. In section 5, we prove the various claims made in subsection 2.1 and in section 6 we prove the claims made in subsection 2.2. In section 7, we prove Theorem 1.1 and in section 8 we prove Theorem 1.2. To prove Theorem 1.1
(b), we need most of the results proven in section 2 as well as the generalisation of the main result of [42] described in Appendix A. In section 9, we give a proof of Theorem 1.4. The structure of the proof is similar, but slightly more elaborate than the structure of the proof of Theorem 1.1 (b). Its first step is a reduction to the situation where the base is a curve and for this we need a geometric analog of a theorem of Néron on the specialisation of Mordell-Weil groups over function fields defined over number fields. Néron’s theorem is based on Hilbert’s irreducibility theorem but we can bypass this theorem in our situation, appealing only to Bertini’s theorem. See Appendix B for all this. We also give further explanations and references there.

In his very interesting recent preprint [47], Xinyi Yuan uses some techniques which are also used in the present paper. They were discovered independently. His text focusses on the case where the base curve is the projective line. In particular, the ”quotient process” used in step (2) of the proof of Theorem 1.4 and also in the proof of Theorem 1.2 also appears (over the projective line) in section 2.2 of [47]. Theorem 2.9 of [47] overlaps with the proof of Lemma 4.10.

The prerequisites for this article are algebraic geometry at the level of Hartshorne’s book [19], familiarity with the basic theory of finite flat group schemes, as expounded in [46] and knowledge of the basic theory of abelian schemes and varieties, as presented in [34], [37] and [36].

Acknowledgments. First, I would like to thank J.-B. Bost for his feedback, especially for pointing out the article [9], for suggesting Remark 2.4 and for providing [8, Lemma 2.9], which is at the root of the present text. I am very grateful to J.-F. Voloch for many exchanges on the material of this article and for his remarks on the text and to P. Ziegler for many discussions on and around the ”full” Mordell-Lang conjecture. Many thanks also to T. Scanlon for his interest and for interesting discussions around the group \( A(K_{\text{perf}}) \). Last but not least, many thanks to Hélène Esnault and her student Marco d’Addezio for their interest and for many enlightening discussions around Theorem 1.4. I also benefitted from A.-J. de Jong’s and F. Oort’s vast knowledge; they both very kindly took the time to answer some rather speculative messages.

Notations. If \( X \) is an integral scheme, we write \( \kappa(X) \) for the local ring at the generic point of \( X \) (which is a field). If \( X \) is a scheme of characteristic \( p \), we denote the absolute Frobenius endomorphism of \( X \) by \( F_X \). If \( f : X \to Y \) is a morphism between two schemes of characteristic \( p \) and \( \ell > 0 \), abusing language, we denote by \( X^{(p\ell)} \) the fibre product of \( f \) and \( F_Y^{\ell} \), where \( F_Y^{\ell} \) is the \( \ell \)-th power of the Frobenius endomorphism \( F_Y \) of \( Y \). If \( G \to X \)
is a group scheme, we write $\epsilon_{G/X} : X \to G$ for the zero section of $G$ and

$$\omega_{G/X} = \omega_G := \epsilon_{G/X}^*(\Omega_{G/X}).$$

If $X$ is of characteristic $p$, we shall write $F_{G/X} : G \to G^{(p)}$ for the relative Frobenius morphism and $V_{G^{(p)}/X} : G^{(p)} \to G$ for the corresponding Verschiebung morphism; we shall write $F_{G/X}^{(n)} : G \to G^{(p^n)}$ (resp. $V_{G^{(p^n)}/X}^{(j)} : G^{(p^n)} \to G^{(p^{n-j})}$) for the composition of morphisms

$$F_{G^{(p^n-1)}/X} \circ \cdots \circ F_{G/X}$$

(resp. the composition of morphisms

$$V_{G^{(p^n-j+1)}/X} \circ V_{G^{(p^n-j+2)}/X} \circ \cdots \circ V_{G^{(p^n)}/X}.$$ ) See [23, Exp. VII A, par. 4, ”Frobenius series”] for the definition of the relative Frobenius morphism and the Verschiebung. If $G$ is finite flat and commutative, we shall write $G^\vee$ for the Cartier dual of $G$.

\section{Intermediate results}

We keep the notations and terminology of the introduction.

\subsection{Consequences of infinite generation of $A(K^{\text{perf}})$}

We shall write

$$\overline{\text{rk}}_{\min}(\omega_A) := \lim_{\ell \to \infty} \text{rk}((F_S^{\text{fl},*}(\omega_A))_{\min})$$

and

$$\overline{\mu}_{\min}(\omega_A) := \lim_{\ell \to \infty} \frac{\deg((F_S^{\text{fl},*}(\omega_A))_{\min})}{\ell \cdot \text{rk}((F_S^{\text{fl},*}(\omega_A))_{\min})}.$$ Here $F^\text{fl}_S$ is the $\ell$-th power of the absolute Frobenius endomorphism of $S$ and $(F^{\text{fl},*}_S(\omega_A))_{\min}$ is the semistable quotient with minimal slope of the vector bundle $F^{\text{fl},*}_S(\omega_A)$. See section 3 for details. Our main tool will be the following theorem.

**Theorem 2.1.** There exists a (necessarily unique) multiplicative subgroup scheme $G_A \hookrightarrow \ker F_{A/S}$, with the following property: if $H$ is a finite, flat and multiplicative subgroup scheme over $S$ and $f : H \to \ker F_{A/S}$ is a morphism of group schemes, then $f$ factors through $G_A$.

If $A$ is ordinary and $\omega_A$ is not ample then the order of $G_A$ is $p^{\overline{\mu}_{\min}(\omega_A)}$.

If $\phi : A \to B$ is a morphism of smooth commutative group schemes over $S$, then the restriction of $\phi$ to $G_A$ factors through $G_B$. Furthermore, we have $\deg(\omega_A) = \deg(\omega_{A/G_A})$. 

6
Here $A/G_A$ is the "fppf quotient" of $A$ by $G$, which is also a smooth commutative group scheme over $S$. See [7, §8.2, after Prop. 10] for the exact definition.

**Remark 2.2.** Note that $\tilde{\mu}_{\text{min}}(\omega_A) > 0$ is equivalent to $\omega_A$ being ample (see [3]).

**Remark 2.3.** Theorem 2.1 holds more generally if $k$ is only supposed to be perfect (the proof does not use the fact that $k$ is finite).

**Remark 2.4.** It would be interesting to provide an explicit example of an abelian variety $A$ as in the introduction to this article, such that $A$ is ordinary, $A$ is semiabelian, $\text{Tr}_{\bar{K}/\bar{k}}(A_{\bar{K}}) = 0$ and $G_A \neq 0$. It should be possible to construct such an example by considering mod $p$ reductions of the abelian variety constructed in [9, Th. 1.3]. We hope to return to this question in a later article. The following question is also of interest: is there an ordinary abelian variety $A$ as above, such that $A$ has maximal Kodaira-Spencer rank, $A$ is semiabelian and $G_A \neq 0$?

**Proposition 2.5.** Suppose that $A$ is ordinary and that $A$ is semiabelian. Suppose that $A(K^\text{perf})$ is not finitely generated. Then $G_A$ is of order $> 1$ and $A/G_A$ is also semiabelian.

**Proposition 2.6.** Suppose that $A$ is ordinary and that $A$ is semiabelian over $S$. Suppose that $A(K^\text{perf})$ is not finitely generated. Then there is a finite flat morphism

$$\phi : A \to B$$

where $B$ is a semiabelian over $S$ and a finite flat morphism

$$\lambda : B \to B$$

such that ker($\phi$) are ker($\lambda$) are multiplicative group schemes and such that the order of ker($\lambda$) is $> 1$.

### 2.2 Consequences of infiniteness of Tor$_p(A(K^\text{sep}))$ or Tor$_p(A(K^\text{unr}))$

**Theorem 2.7.** Suppose that $\text{Tr}_{\bar{K}/\bar{k}}(A_{\bar{K}}) = 0$. Suppose that the action of Gal($K^\text{sep}|K$) on Tor$_p(A(K^\text{unr}))$ factors through Gal($K^\text{sep}|K$)$^{\text{ab}}$. Then Tor$_p(A(K^\text{unr}))$ is finite.

Here Gal($K^\text{sep}|K$)$^{\text{ab}}$ is the maximal abelian quotient of Gal($K^\text{sep}|K$).

**Proposition 2.8.** Suppose that dim($A$) $\leq 2$ and that $\text{Tr}_{\bar{K}/\bar{k}}(A_{\bar{K}}) = 0$. Then Tor$_p(A(K^\text{unr}))$ is finite.
Theorem 2.9. Suppose that $\text{Tor}_{p}(A(K^{\text{sep}}))$ is infinite. Then there is an étale $K$-isogeny 

$$\phi : A \rightarrow B$$

where $B$ is an abelian variety over $K$ and there is an étale $K$-isogeny 

$$\lambda : B \rightarrow B$$

such that the order of $\ker(\lambda)$ is $> 1$ and such that the orders of $\ker(\lambda)$ and $\ker(\phi)$ are powers of $p$.

Theorem 2.10. Suppose that there exists an étale $K$-isogeny $\phi : A \rightarrow A$, such that $\deg(\phi) = p^r$ for some $r > 0$. Suppose also that $A$ is a geometrically simple abelian variety and that $A$ is a semiabelian scheme.

Then $A$ is an abelian scheme and $\phi$ extends to an étale (necessarily finite) $S$-morphism $A \rightarrow A$ of group schemes.

3 Prolegomena on semistable sheaves

Let $Y$ be a scheme, which is smooth, projective and geometrically connected of relative dimension one over a field $l_0$.

Suppose to begin with that $l_0$ is algebraically closed.

If $V$ is a coherent locally free sheaf on $Y$, we write as is customary

$$\mu(V) = \deg_L(V)/\text{rk}(V)$$

where

$$\deg(V) := \int_Y c_1(V)$$

and $\text{rk}(V)$ is the rank of $V$. The quantity $\mu(V)$ is called the slope of $V$. Recall that a locally free coherent sheaf $V$ on $Y$ is called semistable if for any coherent subsheaf $W \subseteq V$, we have $\mu(W) \leq \mu(V)$. If $V/S$ is a locally free coherent sheaf on $Y$, there is a unique filtration by coherent subsheaves

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{\text{hn}(V)} = V$$

such that all the sheaves $V_i/V_{i-1}$ ($1 \leq i \leq \text{hn}(V)$) are semistable and such that the sequence $\mu(V_i/V_{i-1})$ is strictly decreasing. This filtration is called the Harder-Narasimhan filtration of $V$ (shorthand: HN filtration). One then defines

$$V_{\min} := V/V_{\text{hn}(V)-1}, \ V_{\max}(V) := V_1$$
and
\[ \mu_{\text{max}}(V) := \mu(V_1), \mu_{\text{min}}(V) := \mu(V_{\text{min}}). \]

If \( r \in \mathbb{Q} \), we shall also write
\[ V_{=r} := V_{i(r)}/V_{i(r)-1} \]
where \( i(r) \in \mathbb{N} \) is such that \( \mu(V_{i(r)}/V_{i(r)-1}) = r \). Similarly, we shall write
\[ V_{\geq r} := V_{i(r)} \]
where \( i(r) \in \mathbb{N} \) is such that \( \mu_{\text{min}}(V_{i(r)}) = r \) and
\[ V_{>r} := V_{i(r)} \]
where \( i(r) \in \mathbb{N} \) is such that \( \mu_{\text{min}}(V_{i(r)}) > r \) and \( i(r) \) is is maximal with this property.

One basic property of semistable sheaves that we shall use repeatedly is the following. If \( V \) and \( W \) are coherent locally free sheaves on \( Y \) and \( \mu_{\text{min}}(V) > \mu_{\text{max}}(W) \) then \( \text{Hom}_Y(V,W) = 0 \). This follows from the definitions.

See [5, chap. 5] (for instance) for all these notions.

If \( V \) is a coherent locally free sheaf on \( Y \) and \( l_0 \) has positive characteristic, we say that \( V \) is \textit{Frobenius semistable} if \( F_{\ell_0}^{\alpha^e,*}(V) \) is semistable for all \( \ell \in \mathbb{N} \). The terminology \textit{strongly semistable} also appears in the literature.

\textbf{Theorem 3.1.} Let \( V \) be a coherent locally free sheaf on \( S \). There is an \( \ell_0 = \ell_0(V) \in \mathbb{N} \) such that the quotients of the Harder-Narasimhan filtration of \( F_{\ell_0}^{\alpha^e,*}(V) \) are all Frobenius semistable.

\textbf{Proof.} See eg [32, Th. 2.7, p. 259]. \( \square \)

Theorem 3.1 shows in particular that the following definitions:
\[ \bar{\mu}_{\text{min}}(V) := \lim_{\ell \to \infty} \mu_{\text{min}}(F_{\ell}^{\alpha^e,*}(V))/p^{\ell}, \]
\[ \bar{\mu}_{\text{max}}(V) := \lim_{\ell \to \infty} \mu_{\text{max}}(F_{\ell}^{\alpha^e,*}(V))/p^{\ell}, \]
\[ \bar{\text{rk}}_{\text{min}}(V) := \lim_{\ell \to \infty} \text{rk}((F_{\ell}^{\alpha^e,*}(V))_{\text{min}}), \]
and
\[ \bar{\text{rk}}_{\text{max}}(V) := \lim_{\ell \to \infty} \text{rk}((F_{\ell}^{\alpha^e,*}(V))_{\text{max}}). \]
make sense.
Suppose now that $l_0$ is only perfect (not necessarily algebraically closed). If $V$ is a coherent sheaf on $Y$, then we shall write $\mu(V) := \mu(V_{\bar{l}_0})$ and we shall say that $V$ is semistable if $V_{\bar{l}_0}$ is semistable. The HN filtration of $V_{\bar{l}_0}$ is invariant under $\text{Gal}(\bar{l}_0|l_0)$ by unicity and by a simple descent argument, we see that there is a unique filtration by coherent subsheaves

$$V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{\text{hn}(V)}$$

such that

$$V_{0,l_0} \subsetneq V_{1,l_0} \subsetneq V_{2,l_0} \subsetneq \cdots \subsetneq V_{\text{hn}(V),l_0}$$

is the HN filtration of $V_{\bar{l}_0}$. We then define as before

$$\mu_{\text{max}}(V) := \mu(V_1)$$

and

$$\mu_{\text{min}}(V) := \mu(V/V_{\text{hn}(V)} - 1).$$

Notice that we have $\mu_{\text{max}}(V) = \mu_{\text{max}}(V_{\bar{l}_0})$ and $\mu_{\text{min}}(V) = \mu_{\text{min}}(V_{\bar{l}_0})$. Notice that if $V$ and $W$ are coherent locally free coherent sheaves on $Y$ and $\mu_{\text{min}}(V) > \mu_{\text{max}}(W)$ then we still have $\text{Hom}_Y(V,W) = 0$, since there is a natural inclusion

$$\text{Hom}_Y(V,W) \subseteq \text{Hom}_{\bar{l}_0}(V_{\bar{l}_0}, W_{\bar{l}_0}).$$

If $l_0$ has positive characteristic, we shall say that $V$ is Frobenius semistable if $V_{\bar{l}_0}$ is Frobenius semistable. Since Frobenius morphisms commute with all morphisms, this is equivalent to requiring that $F_S^r,V_1)$ is semistable for all $r \in \mathbb{N}$ (with our extended definition of semistability).

We can now extend the range of the terminology introduced above:

$$V_{\text{max}} := V_1, V_{\text{min}} := V/V_{\text{hn}(V)} - 1,$$

$$\bar{\mu}_{\text{min}}(V) := \lim_{\ell \to \infty} \mu_{\text{min}}(F_S^{\ell,r}(V))/p^\ell, \bar{\mu}_{\text{max}}(V) := \lim_{\ell \to \infty} \mu_{\text{max}}(F_S^{\ell,r}(V))/p^\ell,$$

$$\bar{\text{rk}}_{\text{min}}(V) := \lim_{\ell \to \infty} \text{rk}(F_S^{\ell,r}(V))_{\text{min}}, \bar{\text{rk}}_{\text{max}}(V) := \lim_{\ell \to \infty} \text{rk}(F_S^{\ell,r}(V))_{\text{max}}.$$

Note that we have $\bar{\mu}_{\text{min}}(V) = \bar{\mu}_{\text{min}}(V_{\bar{l}_0}), \bar{\mu}_{\text{max}}(V) = \bar{\mu}_{\text{max}}(V_{\bar{l}_0}), \bar{\text{rk}}_{\text{min}}(V) = \bar{\text{rk}}_{\text{min}}(V_{\bar{l}_0}), \bar{\text{rk}}_{\text{max}}(V) = \bar{\text{rk}}_{\text{max}}(V_{\bar{l}_0})$ as expected.

If $V$ is a coherent locally free coherent sheaf on $Y$ such that all the quotients of the HN filtration of $V$ are Frobenius semistable, we shall say that $V$ has a Frobenius semistable HN filtration. Note that by Theorem 3.1 above, for any coherent locally free coherent sheaf $V$ on $Y$, the sheaf $F_{S}^{r,\text{or},.}(V)$ has a Frobenius semistable HN filtration for all but finitely many $r \in \mathbb{N}$.

The following simple lemma will also prove very useful. It was suggested by J.-B. Bost.
Lemma 3.2. Let $V$ and $W$ be coherent locally free sheaves on $Y$. Suppose that $\mu(V) = \mu(W)$ and that $\text{rk}(V) = \text{rk}(W)$. Let $\phi : V \rightarrow W$ be a monomorphism of $\mathcal{O}_Y$-modules. Then $\phi$ is an isomorphism.

Proof. Let $M := \det(W) \otimes \det(V)^\vee$. The assumptions imply that $\deg(M) = 0$. Let $\det(\phi) \in H^0(Y, M)$ be the section induced by $\phi$. The zero scheme $Z(\det(\phi))$ of $\det(\phi)$ is a torsion sheaf since $\det(\phi)$ is non zero at the generic point of $Y$ and the length of $Z(\det(\phi))$ is equal to the degree of $M$ so $Z(\det(\phi))$ must be empty. In other words, $M$ is the trivial sheaf and $\det(\phi)$ is a constant non zero section of $M$. In particular, $\phi$ is an isomorphism. \qed

4 Finite flat group schemes over curves

4.1 The HN-filtration on the Lie algebra of a finite flat group scheme of height one

The terminology of this section is independent of the introduction.

Let $S$ be a smooth, projective and geometrically connected curve over a perfect field $k$. Suppose that $\text{char}(k) = p > 0$.

The following preliminary lemma will be very useful.

Lemma 4.1. Let $G$ be a finite flat commutative group scheme over $S$. Let $T \rightarrow S$ be a flat, radicial and finite morphism and let $\phi : H \hookrightarrow G_T$ be a closed subgroup scheme, which is finite, flat and multiplicative. Then there is a finite flat closed subgroup scheme $\phi_0 : H_0 \hookrightarrow G$, such that $\phi_{0,T} \simeq \phi$.

Proof. Taking Cartier duals, we get a morphism

$\phi^\vee : G_T^\vee \rightarrow H^\vee$.

Notice that $H^\vee$ is étale over $T$, since $H$ is multiplicative. By radicial invariance of étale morphisms, there is a finite flat group scheme $J_0 \rightarrow S$, such that $J_{0,T} \simeq H^\vee$. Notice also that the morphism $\phi^\vee$ is given by a section of the first projection

$G_T^\vee \times_T H^\vee \rightarrow G_T^\vee$

and since $H^\vee$ is étale over $T$, the image of this section is open and closed (see [33, Cor. 3.12]). Since the projection morphism

$G_T^\vee \times_T H^\vee \rightarrow G_T^\vee \times_S J_0$
is also radial, this open set comes from a unique open subset of $G \times_S J_0$ and this open subset defines an open and closed subscheme of $G^\vee \times_S J_0$, which is isomorphic to $G^\vee$ via the first projection. Hence the morphism $\phi^\vee$ comes from a unique morphism $G^\vee \to J_0$. Taking the Cartier dual of this morphism gives the morphism $\phi_0$. □

Recall that a commutative finite flat group scheme $\psi : G \to S$ over $S$ is said to be of height one if $F_{G/S} = \epsilon_{G/S} \circ \psi$. Recall also that a (sheaf in) commutative $p$-Lie algebras (resp. $p$-coLie) algebras $V$ over $S$ is a coherent locally free sheaf $V$ on $S$ together with a morphism of $\mathcal{O}_S$-modules $F_S^*(V) \to V$ (resp. $V \to F_S^*(V)$). A morphism of commutative $p$-Lie (resp. $p$-coLie) algebras $V \to W$ is a morphism of $\mathcal{O}_S$-modules from $V$ to $W$ satisfying an evident compatibility condition. There is a covariant functor $\text{Lie}(\cdot)$ (resp. contravariant functor $\text{coLie}(\cdot)$) from the category of commutative finite flat group schemes of height one over $S$ to the category of commutative $p$-Lie (resp. $p$-coLie) algebras, which sends a group scheme $G$ over $S$ to $\text{Lie}(G) := \epsilon_{G/S}(\Omega_{G/S})^\vee$ (resp. $\text{coLie}(G) := \epsilon_{G/S}(\Omega_{G/S})^\wedge$), together with the morphism

$$\text{Lie}(V_{G^{(p)}/S}) := (V_{G^{(p)}/S})^\vee : F_S^*(\text{Lie}(G)) = \text{Lie}(G^{(p)}) \to \text{Lie}(G)$$

(resp.

$$\text{coLie}(V_{G^{(p)}/S}) := V_{G^{(p)}/S}^* : \text{coLie}(G) \to F_S^*(\text{coLie}(G^{(p)})) = \text{coLie}(G^{(p)})$$

)

Here $(V_{G^{(p)}/S})^\vee$ (resp. $V_{G^{(p)}/S}^*$) is the dual of the pull-back morphism $V_{G^{(p)}/S}$ (resp. is the pull-back morphism) on differentials induced by the Verschiebung morphism $V_{G^{(p)}/S}$.

The category of sheaves in commutative $p$-Lie algebras is tautologically antiequivalent to the category of sheaves in commutative $p$-coLie algebras.

It can be shown that $\text{Lie}$ is an equivalence of additive categories (see [23, Exposé VIIA, rem. 7.5]). In particular, a sequence of finite flat group schemes

$$0 \to G' \to G \to G'' \to 0$$

is exact if and only if the sequence

$$0 \to \text{Lie}(G') \to \text{Lie}(G) \to \text{Lie}(G'') \to 0$$

is a sequence of commutative $p$-Lie algebras. Furthermore, we have

$$\text{order}(G) = p^{\text{rk}(\text{Lie}(G))}$$

(see [37, Proof of Th., p. 139, §14].)
Lemma 4.2. Let \( \phi : V \to W \) be a morphism of commutative \( p \)-Lie algebras. Then the image \( \text{Im}(\phi) \) (resp. the kernel \( \ker(\phi) \)) of \( \phi \) as a morphism of \( \mathcal{O}_S \)-modules is endowed with a unique structure of commutative \( p \)-Lie algebra, such that the morphism \( \text{Im}(\phi) \to W \) (resp. \( \ker(\phi) \to V \)) is a morphism of commutative \( p \)-Lie algebras.

Proof. Left to the reader. \( \square \)

If \( \phi : V \to W \) is an injective morphism of commutative \( p \)-Lie algebras, we shall say that \( \text{Im}(\phi) \) is a subsheaf in commutative \( p \)-Lie algebras. Beware that in this situation, the arrow \( \phi \) might have no cokernel in the category of commutative \( p \)-Lie algebras. So in particular, \( \text{Im}(\phi) \) might not correspond to a subgroup scheme. On the other hand, if the quotient of \( \mathcal{O}_S \)-modules \( W/\text{Im}(\phi) \) is locally free, then \( W/\text{Im}(\phi) \) can be endowed with an evident commutative \( p \)-Lie algebra structure, making it into a cokernel of \( W \) by \( \text{Im}(\phi) \) in the category of commutative \( p \)-Lie algebras. In that case, \( \text{Im}(\phi) \) corresponds to a subgroup scheme.

We inserted the following alternative proof of a special case of Lemma 4.1 to show the mechanics of \( p \)-Lie algebras at work.

Second proof of Lemma 4.1 when \( G \) is of height one and \( T \) is smooth.

We may assume that \( T \simeq S \) and that \( T \to S \) is a power \( F_{S}^{n} \) of \( F_{S} \). By induction on \( n \), we are reduced to prove the statement for \( n = 1 \).

We are given a commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & \to & F_T^*(\text{Lie}(H)) \\
\downarrow & & \downarrow F_T^*[\text{Lie}(\phi)] \\
0 & \to & F_T^*(\text{Lie}(G)_T) \\
\downarrow \text{Lie}(V_{H/T}) & & \downarrow \text{Lie}(V_{G/T/T}) \\
\text{Lie}(H) & \to & \text{Lie}(G)_T \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

With the above reductions in place, this gives a commutative diagram with exact rows and
Now consider the commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \longrightarrow F_S^*(\text{Lie}(H)) \longrightarrow F_S^{o2,*} (\text{Lie}(G)) \\
\downarrow \text{Lie}(V_{H/S}) \\
0 \longrightarrow \text{Lie}(H) \longrightarrow F_S^* (\text{Lie}(G)) \\
\downarrow \\
0
\end{array}
\]

where the diagonal arrows are defined so that the diagram becomes commutative. The labelling of the arrows shows that the upper triangle is the base change by \(F_S\) of the lower triangle. Hence the image of \(\text{Lie}(\phi)\) is the base change by \(F_S\) of the image of \(\text{Lie}(H)\) in \(\text{Lie}(G)\), since \(\text{Lie}(V_{H/S})\) is an isomorphism. So \(H_0\) can be defined as the group scheme of height one associated with the image of \(\text{Lie}(H)\) in \(\text{Lie}(G)\). \(\square\)

We shall say that a finite flat commutative group scheme \(G\) of height one (or its associated commutative \(p\)-Lie algebra) is biinfinitesimal if \(F_S^*(\text{Lie}(G)) \to \text{Lie}(G)\) is nilpotent. To say that \(F_S^*(\text{Lie}(G)) \to \text{Lie}(G)\) is nilpotent means that for some \(n \geq 1\), the composition

\[F_S^{o_n,*} (\text{Lie}(G)) \to F_S^{o(n-1),*} (\text{Lie}(G)) \to \cdots \to F_S^*(\text{Lie}(G)) \to \text{Lie}(G) \to 0\]

vanishes. We notice without proof that if

\[0 \to G' \to G \to G'' \to \]

is an exact sequence of commutative finite flat group schemes, then \(G'\) and \(G''\) are biinfinitesimal if and only if \(G\) is biinfinitesimal.

**Lemma 4.3.** Let \(V\) be a sheaf in commutative \(p\)-Lie algebras \(V\) over \(S\). Suppose that the HN filtration

\[0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{\text{hn}(V)} = V\]

is an exact sequence of commutative finite flat group schemes, then \(G'\) and \(G''\) are biinfinitesimal if and only if \(G\) is biinfinitesimal.
of $V$ is Frobenius semistable. Then for any $V_i$ such that $\mu_{\min}(V_i) \geq 0$, $V_i$ is a subsheaf in commutative $p$-Lie algebras $V$ over $S$. If $\mu_{\min}(V_i) > 0$ then $V_i$ is biinfinitesimal.

**Proof.** For the first statement, consider the morphism $\phi : F_S^*(V_i) \to V$ given by the composition of the inclusion $F_S^*(V_i) \to F_S^*(V)$ with the morphism $F_S^*(V) \to V$ given by the commutative $p$-Lie algebra structure. We have to check that the image of $\phi$ lies in $V_i$. The composition of $\phi$ with the quotient morphism $V \to V/V_i$ gives a morphism $F_S^*(V_i) \to V/V_i$ and it is equivalent to check that this morphism vanishes. Now compute

$$\mu_{\min}(F_S^*(V_i)) = p \cdot \mu(V_i/V_{i-1})$$

and

$$\mu_{\max}(V/V_i) = \mu(V_{i+1}/V_i) < \mu(V_i/V_{i-1})$$

and thus $\mu_{\min}(F_S^*(V_i)) > \mu_{\max}(V/V_i)$. We conclude that $\text{Hom}_S(F_S^*(V_i), V/V_i) = 0$ (see the discussion after Theorem 3.1) which concludes the proof of the first statement. To prove the second statement, it is sufficient by the remarks preceding the lemma to show that $V_i/V_{i-1}$ is biinfinitesimal for all indices $i$ such that $\mu(V_i/V_{i-1}) > 0$. By the above computation, we have

$$\mu_{\min}(F_S^*(V_i/V_{i-1})) = \mu(F_S^*(V_i/V_{i-1})) = p \cdot \mu(V_i/V_{i-1})$$

and thus $\mu_{\min}(F_S^*(V_i/V_{i-1})) > \mu(V_i/V_{i-1})$. Again, this implies that $\text{Hom}_S(F_S^*(V_i/V_{i-1}), V_i/V_{i-1}) = 0$, showing that $V_i/V_{i-1}$ is biinfinitesimal. \square

**Remark 4.4.** As explained in the introduction, a characteristic 0 analog of Lemma 4.3 can be found in [8, Lemma 2.9]. See also [44, Lemma 9.1.3.1], where a variant of a special case of Lemma 4.3 is proven under the assumption that $p$ is sufficiently large.

**Lemma 4.5.** Let $G$ be a commutative finite flat group scheme of height one over $S$ and suppose given an exact sequence

$$0 \to G_{\text{binf}} \to G \to G_\mu \to 0$$

of finite flat group schemes such that $G_\mu$ is multiplicative and $G_{\text{binf}}$ is biinfinitesimal. Then the sequence splits canonically.

**Proof.** Consider the commutative diagram with exact rows and columns

\[
\begin{array}{ccccccc}
0 & \to & \ker(\text{Lie}(V_G^{(n)}_{\text{binf}(p^n)/S})) & \to & \ker(\text{Lie}(V_G^{(n)}_{G(p^n)/S})) & \to & 0 \\
& & \downarrow_{\cong} & & \downarrow & & \\
0 & \to & F_S^{*\text{nm}}(\text{Lie}(G_{\text{binf}})) & \to & F_S^{*\text{nm}}(\text{Lie}(G)) & \to & F_S^{*\text{nm}}(\text{Lie}(G_\mu)) & \to & 0 \\
& & \downarrow_{=0} & & \downarrow & & \downarrow_{\cong} & & \\
0 & \to & \text{Lie}(G_{\text{binf}}) & \to & \text{Lie}(G) & \to & \text{Lie}(G_\mu) & \to & 0
\end{array}
\]
where \( n \geq 0 \) is chosen so that \( V_{G_{\text{binf}}(p^n)/S}^{(n),*} = 0 \). Then the image of the arrow

\[
F_{S}^{\text{on,*}}(\text{Lie}(G)) \to \text{Lie}(G)
\]

splits the bottom sequence. \( \square \)

**Lemma 4.6.** Let \( G \) be a commutative finite flat group scheme of height one over \( S \). Suppose that \( \text{Lie}(G) \) is Frobenius semistable of slope 0. Let \( n \geq 0 \) be such that \( \text{rk}(\ker(V_{G(p^n)/S}^{(n),*})) \) is maximal. Then there is a canonical decomposition

\[
G(p^n) \cong H_{\text{binf}} \times_S H_{\mu}
\]

where \( H_{\text{binf}} \) (resp. \( H_{\mu} \)) is a biinfinitesimal (resp. multiplicative) finite flat group scheme over \( S \).

**Proof.** Consider the commutative diagram

\[
\begin{array}{c}
0 \to F_{S}^{\text{on,*}}(\ker(\text{Lie}(V_{G(p^n)/S}^{(n)}))) \to F_{S}^{\text{on,*}}(\ker(\text{Lie}(V_{G(p^n)/S}^{(n)}))) \to 0 \\
\downarrow \sim \\
0 \to F_{S}^{\text{on,*}}(\ker(\text{Lie}(V_{G(p^n)/S}^{(n)}))) \to F_{S}^{\text{o}(2n),*}(\text{Lie}(G)) \to F_{S}^{\text{on,*}}(W) \to 0 \\
\downarrow =0 \\
0 \to \ker(\text{Lie}(V_{G(p^n)/S}^{(n)})) \to F_{S}^{\text{on,*}}(\text{Lie}(G)) \to W \to 0
\end{array}
\]

where \( n \geq 0 \) is such that \( \text{rk}(\ker(\text{Lie}(V_{G(p^n)/S}^{(n)}))) \) is maximal and \( W \) is the image of \( \text{Lie}(V_{G(p^n)/S}^{(n)}) \). The two bottom rows and the two leftmost columns in this diagram are exact by construction. Furthermore the map \( F_{S}^{(n),*}W \to W \) is a monomorphism for otherwise \( \text{rk}(\ker(\text{Lie}(V_{G(p^n)/S}^{(n)}))) \) is not maximal. The diagram thus has exact rows and columns. Since the second row gives a surjection

\[
F_{S}^{\text{o}(2n),*}(\text{Lie}(G)) \to F_{S}^{\text{on,*}}(W)
\]

we have \( \mu_{\text{min}}(F_{S}^{\text{on,*}}(W)) \geq 0 \). Also, since the second column gives an injection

\[
F_{S}^{\text{on,*}}(W) \hookrightarrow F_{S}^{(n),*}(\text{Lie}(G))
\]

we have \( \mu_{\text{max}}(F_{S}^{\text{on,*}}(W)) \leq 0 \). Thus \( F_{S}^{\text{on,*}}(W) \) is of slope 0. Thus \( W \) is also of slope 0. Hence by Lemma 3.2, the monomorphism

\[
F_{S}^{\text{on,*}}(W) \to W
\]

is an isomorphism. Now we see that the image of the morphism \( F_{S}^{\text{o}(2n),*}(\text{Lie}(G)) \to F_{S}^{\text{on,*}}(\text{Lie}(G)) \) splits the bottom sequence. \( \square \)
Lemma 4.7. Let $G$ be a finite flat commutative group scheme of height one over $S$. There exists a (necessarily unique) multiplicative subgroup scheme $G_\mu \hookrightarrow G$, such that if $H$ is a multiplicative subgroup scheme of height one over $S$ and $f : H \to G$ is a morphism of group schemes, then $f$ factors through $G_\mu$. Furthermore, for any $n \geq 0$, we have $(G_\mu)^{(p^n)} = (G^{(p^n)})_\mu$. If $G$ is multiplicative over an open subset of $S$ and $\text{Lie}(G)$ has Frobenius semistable HN filtration then $\text{Lie}(G) = \text{Lie}(G)_{\leq 0}$ and $G_\mu$ corresponds to the subgroup scheme associated with $\text{Lie}(G)=0$.

**Proof.** In view of Lemma 4.1, we may replace $G$ by $G^{(p^n)}$ for any $n \geq 0$ and in particular suppose that $\text{Lie}(G)$ has a Frobenius semistable HN filtration. Let $f : H \to G$ be a morphism of group schemes and consider the corresponding map

$$\text{Lie}(f) : \text{Lie}(H) \to \text{Lie}(G).$$

Since $H$ is multiplicative, $\text{Lie}(H)$ is Frobenius semistable of slope 0 (this is a consequence of Theorem 3.1). Thus the image of $\text{Lie}(f)$ lies in $\text{Lie}(G)_{\geq 0}$. According to Lemma 4.3 there is an exact sequence of $p$-Lie algebras

$$0 \to \text{Lie}(G)_{>0} \to \text{Lie}(G)_{\geq 0} \xrightarrow{\pi} \text{Lie}(G)_{=0} \to 0$$

and we may assume according to Lemma 4.6 that there is a splitting

$$\text{Lie}(G)_{=0} \simeq \text{Lie}(G)_{=0,\text{binf}} \oplus \text{Lie}(G)_{=0,\mu}$$

of $\text{Lie}(G)_{=0}$ into multiplicative and biinfinitesimal part (we might have to twist $G$ some more for this). The inverse image of $\text{Lie}(G)_{=0,\mu}$ by $\pi$ gives a $p$-Lie subalgebra $\pi^*(\text{Lie}(G)_{=0,\mu})$ of $\text{Lie}(G)_{\geq 0}$. This gives an exact sequence

$$0 \to \pi^*(\text{Lie}(G)_{=0,\mu}) \to \text{Lie}(G)_{\geq 0} \to \text{Lie}(G)_{=0,\text{binf}} \to 0$$

Since $\text{Lie}(H)$ is multiplicative, the image of $\text{Lie}(H)$ in $\text{Lie}(G)_{=0,\text{binf}}$ vanishes and thus the image of $\text{Lie}(H)$ lies in $\pi^*(\text{Lie}(G)_{=0,\mu})$. On the other hand by Lemma 4.5 and Lemma 4.3, we have again a canonical decomposition

$$\pi^*(\text{Lie}(G)_{=0,\mu})_{\mu} \oplus \pi^*(\text{Lie}(G)_{=0,\mu})_{\text{binf}}$$

into multiplicative and biinfinitesimal part and thus the image of $\text{Lie}(f)$ lies in $\pi^*(\text{Lie}(G)_{=0,\mu})_{\mu}$. Now $\pi^*(\text{Lie}(G)_{=0,\mu})_{\mu}$ is a multiplicative $p$-Lie subalgebra of $\text{Lie}(G)$ and it defines the required subgroup scheme.

If $G$ is multiplicative over an open subset of $S$ then we have an injection

$$F_S^\text{en,*}(\text{Lie}(G)) \hookrightarrow \text{Lie}(G)$$
(obtained by composition) for any \( n \geq 0 \) and thus if \( \text{Lie}(G) \) has Frobenius semistable HN filtration then we must have \( \text{Lie}(G) = \text{Lie}(G)_{\leq 0} \). Secondly the morphism \( F_{S}^{*}(\text{Lie}(G)) \hookrightarrow \text{Lie}(G) \) then induces an injection

\[
F_{S}^{*}(\text{Lie}(G)_{=0}) \hookrightarrow \text{Lie}(G)_{=0}
\]

and since both source and target in this map have the same rank and the same slope, we deduce from Lemma 3.2 that this map must be an isomorphism. Thus \( \text{Lie}(G)_{=0} \) is multiplicative and by the explicit construction above, it is associated with \( G_{\mu} \).

**Remark 4.8.** Note that the ”connected étale” decomposition of \( G_{K}^{/} \) (see the beginning of [46]) gives a canonical exact sequence of group schemes

\[
0 \rightarrow (G_{K}^{/})_{\text{inf}} \rightarrow G_{K}^{/} \rightarrow (G_{K}^{/})_{\text{et}} \rightarrow 0
\]

over \( K \), where \((G_{K}^{/})_{\text{inf}}\) is an infinitesimal group scheme and \((G_{K}^{/})_{\text{et}}\) is an étale group scheme over \( K \). The group scheme \((G_{K}^{/})_{\text{et}}\) corresponds to a representation of \( \text{Gal}(K^{\text{sep}}|K) \) into a finite \( p \)-group \( E \) and one might be tempted to think that \( G_{\mu} \) is the Cartier dual of the group scheme corresponding to the largest unramified quotient of \( E \), ie the largest quotient of \( E \), such that the action of \( \text{Gal}(K^{\text{sep}}|K) \) factors through the fundamental group \( \pi_{1}(S) \). This not so, however. Indeed, consider a finite flat commutative group scheme \( G \) of height one, which is such that \( \bar{\mu}_{\max}(\text{Lie}(G)) < 0 \). Then \( G_{\mu} = 0 \) and for any finite flat base change \( S' \rightarrow S \), we also have \((G_{S'})_{\mu} = 0 \). On the other hand \((G_{K})_{\text{et}}\) will become constant (and hence entirely unramified) after a finite separable field extension \( K'|K \).

### 4.2 Quotients of semiabelian schemes by finite flat multiplicative group schemes

**Lemma 4.9.** Let \( A \rightarrow S \) be a semiabelian scheme. Suppose that there is an open dense subset \( U \subseteq S \), such that \( A_{U} \rightarrow U \) is an abelian scheme. Suppose that \( G \hookrightarrow A \) is a finite, flat, closed subgroup scheme, which is multiplicative. Then the quotient scheme \( A/G \) is also a semiabelian scheme and \( (A/G)_{U} \rightarrow U \) is an abelian scheme.

**Proof.** See [6, Cor. 5.4.6].

**Lemma 4.10.** Let \( G \rightarrow S \) be a finite flat group scheme of multiplicative type. Then there is a finite étale morphism \( T \rightarrow S \) such that \( G_{T} \) is a diagonalisable group scheme.

**Proof.** See [24, Exp. IX, Intro.].
Lemma 4.11. Let $\mathcal{A} \to S$ be a smooth commutative group scheme. Suppose that $G \hookrightarrow \mathcal{A}$ is a finite, flat, closed subgroup scheme, which is multiplicative. Then

$$\deg(\omega_{\mathcal{A}}) = \deg(\omega_{\mathcal{A}/G})$$

Proof. By Lemma 4.10, we may assume that $G$ is diagonalisable. In particular, we may assume that there is a finite group scheme $G_0 \to \text{Spec}(k)$ such that $G_{0,S} \simeq G$. Let $\mathcal{B} := \mathcal{A}/G$. Let $f : \mathcal{A} \to S$ and $g : \mathcal{B} \to S$ be the structural morphisms and let $\pi : \mathcal{A} \to \mathcal{B}$ be the quotient morphism. The triangle of cotangent complexes associated with the morphisms $\pi$, $g$ and $f$ gives an exact sequence

$$0 \to \mathcal{H}_1(\text{CT}(\pi)) \to \pi^*(\Omega_g) \to \Omega_f \to \Omega_\pi \to 0 \quad (1)$$

where $\text{CT}(\pi)$ is the cotangent complex of $\pi$ and $\mathcal{H}_1(\text{CT}(\pi))$ is its first homology sheaf. Now $\pi$ makes $\mathcal{A}$ into a torsor over $\mathcal{B}$ and under $G_\mathcal{B}$. Hence there is a faithfully flat morphism $T \to \mathcal{B}$ (for instance, we may take $T = \mathcal{A}$), such that $\mathcal{A}_T \simeq (G_\mathcal{B}) \times_\mathcal{B} T$. In particular we have

$$\Omega_{\pi_T} \simeq \Omega_{G_0/k,T}$$

and

$$\mathcal{H}_1(\text{CT}(\pi_T)) \simeq \mathcal{H}_1(\text{CT}(G_0/k))_T$$

because the homology sheaves of the cotangent complex of $G_0$ over $k$ are flat (since they are $k$-vector spaces).

On the other hand, since $T \to \mathcal{B}$ is flat, we have

$$\Omega_{\pi_T} \simeq \Omega_{\pi,T}$$

and

$$\mathcal{H}_1(\text{CT}(\pi_T)) \simeq \mathcal{H}_1(\text{CT}(\pi))_T$$

Finally, notice that $\Omega_{G_0/k,T}$ and $\mathcal{H}_1(\text{CT}(G_0/k))_T$ are flat and thus by flat descent, the sheaves $\mathcal{H}_1(\text{CT}(\pi))$ and $\Omega_\pi$ are flat (in other words: locally free). Hence the sequence

$$0 \to \epsilon_{A/S}^*(\mathcal{H}_1(\text{CT}(\pi))) \to \epsilon_{B/S}^*(\Omega_g) \to \epsilon_{A/S}^*(\Omega_f) \to \epsilon_{A/S}^*(\Omega_\pi) \to 0 \quad (2)$$

is also exact. Furthermore, we then have

$$\epsilon_{A/S}^*(\mathcal{H}_1(\text{CT}(\pi))) \simeq \mathcal{H}_1(\text{CT}(G_0/k))_S$$
and
\[ \epsilon_{A/S}^*(\Omega_{\pi}) \simeq \Omega_{G_0/k,S} \]
and thus the sheaves \( \epsilon_{A/S}^*(\mathcal{H}_1(CT(\pi))) \) and \( \epsilon_{A/S}^*(\Omega_{\pi}) \) are trivial sheaves. In particular, we have \( \deg(\epsilon_{A/S}^*(\mathcal{H}_1(CT(\pi)))) = \deg(\epsilon_{A/S}^*(\Omega_{\pi})) = 0 \) and by the additivity of \( \deg(\cdot) \), we deduce from the existence of the sequence (2) that \( \deg(\omega_A) = \deg(\omega_{A/G}) \).

Remark 4.12. The computation of the cotangent complex made in the proof of Lemma 4.10 is in essence also contained in [12, Prop. 1.1] (but the assumptions made there are not quite the right ones for us).

5 Proofs of the claims made in subsection 2.1

Recall that we now use the terminology of the introduction. So let \( k \) be a finite field of characteristic \( p > 0 \) and let \( S \) be a smooth, projective and geometrically connected curve over \( k \). Let \( K := \kappa(S) \) be its function field. Let \( A \) be an abelian variety of dimension \( g \) over \( K \). Let \( K_{\text{perf}} \subseteq \bar{K} \) be the maximal purely inseparable extension of \( K \) and let \( K_{\text{unr}} \subseteq K_{\text{sep}} \) be the maximal separable extension of \( K \), which is unramified above every place of \( K \). Finally, we let \( A \) be a smooth commutative group scheme over \( S \) such that \( A_K = A \).

Proof of Theorem 2.1. Recall the statement: there exists a (necessarily unique) multiplicative subgroup scheme \( G_A \hookrightarrow \ker F_{A/S} \), with the following property: if \( H \) is a multiplicative subgroup scheme over \( S \) and \( f : H \to \ker F_{A/S} \) is a morphism of group schemes, then \( f \) factors through \( G_A \). If \( A \) is ordinary and \( \omega_A \) is not ample then the order of \( G_A \) is \( p^{\bar{\mu}_{\min}(\omega_A)} \). If \( \phi : A \to B \) is a morphism of smooth commutative group schemes over \( S \), then the restriction of \( \phi \) to \( G_A \) factors through \( G_B \). Furthermore, we have \( \deg(\omega_A) = \deg(\omega_{A/G_A}) \).

In spite of its lengthy statement, the proof Theorem 2.1 readily follows from Lemma 4.7 and Lemma 4.11.

Proof of Proposition 2.5. Recall the assumptions of Proposition 2.5: \( A \) is ordinary, \( A \) is semiabelian and \( A(K_{\text{perf}}) \) is not finitely generated. We have to prove that \( G_A \) is of order \( > 1 \) and that \( A/G_A \) is also semiabelian.

We know that \( \bar{\mu}_{\min}(\omega_{A/S}) \geq 0 \) by Lemma 4.7 and since \( A(K_{\text{perf}}) \) is not finitely generated, we know by Theorem A.1 that \( \bar{\mu}_{\min}(\omega_{A/S}) = 0 \). Proposition 2.5 now follows from Theorem 2.1 and Lemma 4.9.

Proof of Proposition 2.6. Recall the assumptions of Proposition 2.6: \( A \) is ordinary, \( A \) is semiabelian over \( S \) and \( A(K_{\text{perf}}) \) is not finitely generated. We have to prove that there a
finite flat morphism
\[ \phi : \mathcal{A} \rightarrow \mathcal{B} \]
where \( \mathcal{B} \) is a semiabelian over \( S \) and a finite flat morphism
\[ \lambda : \mathcal{B} \rightarrow \mathcal{B} \]
such that \( \ker(\phi) \) are \( \ker(\lambda) \) are multiplicative group schemes and such that the order of \( \ker(\lambda) \) is \( > 1 \).

Consider now \( \mathcal{A}_1 := \mathcal{A}/G_A \). By Lemma 4.9, the group scheme \( \mathcal{A}_1 \) is also semiabelian and of course \( A_1 := A_{1,K} \) is also an ordinary abelian variety. We also have that \( A_1(K_{\text{perf}}) \) is not finitely generated, since the natural map \( A(K_{\text{perf}}) \rightarrow A_1(K_{\text{perf}}) \) has finite kernel. Finally, the quotient morphism is \( \mathcal{A} \rightarrow \mathcal{A}_1 \) is finite, flat, with multiplicative kernel and \( G_A \) is non trivial by Proposition 2.5.

Repeating the above procedure for \( \mathcal{A}_1 \) in place of \( \mathcal{A} \) and continuing this way, we obtain an infinite sequence of semiabelian schemes over \( S \)
\[ \mathcal{A} \rightarrow \mathcal{A}_1 \rightarrow \mathcal{A}_2 \rightarrow \ldots \] (3)
where all the connecting morphisms are finite, flat, of degree \( > 1 \) and with multiplicative kernel. Applying Lemma 4.11, we see that
\[ \deg(\omega_{\mathcal{A}}) = \deg(\omega_{\mathcal{A}_1}) = \deg(\omega_{\mathcal{A}_2}) = \ldots \]

Let now \( K' \) be a finite separable extension of \( K \) such that \( A(K)[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2 \dim(A)} \) for some \( n \geq 3 \) such that \( (p, n) = 1 \). Let \( S' \) be the normalisation of \( S \) in \( K' \). After base-change, we obtain an infinite sequence of semiabelian schemes over \( S' \)
\[ \mathcal{A}_{S'} \rightarrow \mathcal{A}_{1,S'} \rightarrow \mathcal{A}_{2,S'} \rightarrow \ldots \] (4)
and applying a theorem of Zarhin (see [41, Th. 3.1] for a statement, explanations and further references), we conclude that in the sequence (4), there are only finitely many isomorphism classes of semiabelian schemes over \( S' \). On the other hand, applying a basic finiteness result in Galois cohomology proven by Borel and Serre (see [39, §3, p. 69]), we can now conclude that in the sequence (3), there are also only finitely many isomorphism classes of semiabelian schemes over \( S \).

Hence there are integers \( j > i \geq 0 \) and an isomorphism
\[ I : \mathcal{A}_i \simeq \mathcal{A}_j \]
over \( S \). Letting \( \phi : \mathcal{A} \rightarrow \mathcal{A}_i \) be the constructed morphism and letting \( \lambda \) be the constructed morphism \( \mathcal{A}_i \rightarrow \mathcal{A}_j \) composed with \( I^{-1} \), we can now conclude the proof of Proposition 2.6.
6 Proofs of the claims made in subsection 2.2

We start with the proof of Theorem 2.7. We recall the statement:

Suppose that \( \text{Tr}_{\bar{K}/\bar{k}}(A_{\bar{K}}) = 0 \). Suppose that the action of \( \text{Gal}(K_{\text{sep}}|K) \) on \( \text{Tor}_p(A(K_{\text{unr}})) \) factors through \( \text{Gal}(K_{\text{sep}}|K)^{ab} \). Then \( \text{Tor}_p(A(K_{\text{unr}})) \) is finite.

For the proof, let \( L|K \) be the maximal subextension of \( K_{\text{unr}}|K \), which is Galois with abelian Galois group. Since \( S \) is geometrically integral, \( K \otimes_k \bar{k} \) is a field and \( L \) contains a subfield isomorphic to \( K \otimes_k \bar{k} \) (note that \( \bar{k} = k_{\text{sep}} \) and that \( \text{Gal}(\bar{k}|k) \simeq \hat{\mathbb{Z}} \), which is an abelian group). Furthermore, geometric class field theory (see eg [45, Cor. 1.3]) tells us that \( \text{Gal}(L|K \otimes_k \bar{k}) \) is a finite group. In particular, the field \( L \) is finitely generated (as a field) over \( \bar{k} \), since \( K \otimes_k \bar{k} \) is finitely generated over \( \bar{k} \).

Now suppose to obtain a contradiction that \( \text{Tor}_p(A(K_{\text{unr}})) \) were infinite. By assumption, we have

\[
\text{Tor}_p(A(K_{\text{unr}})) \subseteq \text{Tor}_p(A(L))
\]

Thus \( \text{Tor}_p(A(L)) \) is infinite as well. By the Lang-Néron theorem, this implies that

\[
\text{Tr}_{L|\bar{k}}(A_L) \neq 0,
\]

contradicting the first assumption.

We now turn to the proof of Proposition 2.8. We recall the statement:

Suppose that \( \dim(A) \leq 2 \) and that \( \text{Tr}_{\bar{K}/\bar{k}}(A_{\bar{K}}) = 0 \). Then \( \text{Tor}_p(A(K_{\text{unr}})) \) is finite.

For the proof, notice that if \( \text{Tor}_p(A(K_{\text{unr}})) \) is infinite then we have

\[
\bigcap_{\ell \geq 0} p^\ell \cdot (\text{Tor}_p(A(K_{\text{unr}}))) \neq 0
\]

This follows from the fact that for each \( n \geq 0 \), the set

\[
\{ x \in \text{Tor}_p(A(K_{\text{unr}})) \mid p^n \cdot x = 0 \}
\]

is finite (the details are left to the reader). Let \( G \subseteq \bigcap_{\ell \geq 0} p^\ell \cdot (\text{Tor}_p(A(K_{\text{unr}}))) \) be the subgroup of elements annihilated by the multiplication by \( p \) map.

If \( G = 0 \) then there the conclusion holds, because then \( \bigcap_{\ell \geq 0} p^\ell \cdot (\text{Tor}_p(A(K_{\text{unr}}))) = 0 \) and thus \( \text{Tor}_p(A(K_{\text{unr}})) \) is finite by the above remark.

Suppose now that \( \#G = p \). Then \( \bigcap_{\ell \geq 0} p^\ell \cdot (\text{Tor}_p(A(K_{\text{unr}}))) \) is infinite and the action of \( \text{Gal}(K_{\text{sep}}|K) \) on \( \bigcap_{\ell \geq 0} p^\ell \cdot (\text{Tor}_p(A(K_{\text{unr}}))) \) factors through \( \text{Gal}(K_{\text{sep}}|K)^{ab} \) (details left to the reader). But this contradicts Theorem 2.7 and thus we must have \( \#G > p \). If \( \#G > p \)
then by the assumption that \( \dim(A) \leq 2 \), we see that we must have \( \#G = p^2 \) and thus the inclusions

\[
\text{Tor}_p(A(K^{unr})) \subseteq \text{Tor}_p(A(K^{sep})) \subseteq \text{Tor}_p(A(\bar{\mathbb{K}}))
\]

are both equalities. In particular, \( A \) is an ordinary abelian surface. Let now \( s \in S \) be a closed point such that \( A_s \) is an ordinary abelian variety over \( s \). Let \( W := \text{Spec}(\mathcal{O}^{\text{sh}}_{S,s}) \) be the spectrum of the completion of the strict henselisation of the local ring at \( s \) and write \( \bar{K}^{\text{sh}}_s \) for the fraction field of \( \mathcal{O}^{\text{sh}}_{S,s} \). The abelian scheme \( A_W \to W \) gives rise to an element \( e \) of

\[
\text{Hom}_{\mathbb{Z}_p}(T_p(A_s(\bar{s})), \mathcal{O}^{\text{sh}}_{S,s}^*)
\]

Here \( T_p(A_s(\bar{s})) \) and \( T_p(A_s^\vee(\bar{s})) \) are the \( p \)-adic Tate modules of \( A_s \) and \( A_s^\vee \) respectively and \( \mathcal{O}^{\text{sh}}_{S,s}^* \) is the group of multiplicative units of \( \mathcal{O}^{\text{sh}}_{S,s} \). The element \( e \) is called the Serre-Tate pairing associated with \( A_W \). See [28] for the construction of this pairing. We have \( e = 0 \) if and only if \( A_W \simeq A_s \times_{\bar{s}} W \). Furthermore, the fact that

\[
\text{Tor}_p(A(W)) = \text{Tor}_p(A(\bar{K}^{\text{sh}})) = \text{Tor}_p(A(K^{unr})) = \text{Tor}_p(A(\bar{K}^{\text{sh}}))
\]

in our situation shows that \( e = 0 \) (this follows directly from the definition of the Serre-Tate pairing). Thus we have \( A_W \simeq A_s \times_{\bar{s}} W \) and in particular \( \text{Tr}_{\bar{K}/\bar{k}}(A_{\bar{K}}) \neq 0 \). This contradicts one of our assumptions. We conclude that \( G = 0 \), so that the conclusion must hold.

We shall now prove Theorem 2.9. Recall the statement:

**Suppose that** \( \text{Tor}_p(A(K^{sep})) \) **is infinite. Then there is an étale** \( K \)-isogeny

\[
\phi : A \to B
\]

where \( B \) **is an abelian variety over** \( K \) **and there is an étale** \( K \)-isogeny

\[
\lambda : B \to B
\]

**such that the order of** \( \ker(\lambda) \) **is** \( > 1 \) **and such that the orders of** \( \ker(\lambda) \) **and** \( \ker(\phi) \) **are powers of** \( p \).

For the proof, note that in [41, Th. 1.4], this statement is proven under the supplementary assumption that there exist \( n \in \mathbb{Z} \), such that \( (n,p) = 1 \) and \( n > 3 \) and such that \( A[n](\bar{K}) \simeq (\mathbb{Z}/n\mathbb{Z})^{2\dim(A)} \). Using [39, §3, p. 69] in the proof, it can be seen that this assumption is not necessary. A completely parallel argument is described in the proof of Proposition 2.6. We leave the details to the reader.

We now turn to the proof of Theorem 2.10. Recall the statement:
Suppose that there exists an étale $K$-isogeny $\phi: A \to A$, such that $\deg(\phi)$ is strictly larger than 1 and that $\deg(\phi) = p^r$ for some $r > 0$. Suppose also that $A$ is a geometrically simple abelian variety and that $A$ is a semiabelian scheme.

Then $A$ is an abelian scheme and $\phi$ extends to an étale $S$-morphism $A \to A$ of group schemes.

For the proof, notice first that by a result of Raynaud (see [14, Prop. 2.7]), the morphism $\phi$ extends uniquely to an $S$-morphism $\bar{\phi}: A \to A$ of group schemes. Since $\bar{\phi}$ is étale over $K$, we have an exact sequence of coherent sheaves

$$0 \to \bar{\phi}^*(\Omega_A/S) \to \Omega_A$$

on $A$. Let $\sigma \in H^0(A, \det(\bar{\phi}^*(\Omega_A/S)) \otimes \det(\Omega_A/S))$ be the corresponding section. Since $\sigma_K \in H^0(A, \det(\phi^*(\Omega_{A/K})) \otimes \det(\Omega_{A/K}))$ has an empty zero-scheme, the zero scheme $Z(\sigma)$ is supported on a finite number of closed fibres of $A$. Hence there exists a finite number $P_1, \ldots, P_n$ of closed point of $S$, such that $Z(\sigma) = \bigcup_{i=1}^n n_i A_{P_i}$ (as Weil divisors) for some $n_i \geq 0$. On the other hand, the Weil divisor $\bigcup_{i=1}^n n_i P_i$ is rationally equivalent to 0 on $S$, which implies that $n_i = 0$ for all $i = 1, \ldots, n$. In other words, we have $Z(\sigma) = \emptyset$ and thus the morphism $\bar{\phi}^*(\Omega_A/S) \to \Omega_A$ is an isomorphism. By [19, III, Prop. 10.4], this implies that $\bar{\phi}$ is étale.

Let now $s \in S$ be a closed point such that $A_s$ has a presentation

$$0 \to G \to A_s \to A^0_s \to 0$$

where $G$ is a torus over $s$ of dimension $d > 0$ and $A^0_s$ is an abelian variety over $s$. The morphism $\bar{\phi}_s|_G: G \to A_s$ factors through $G$, since there is no non-constant $s$-morphism $G \to A^0_s$. Call $\gamma: G \to G$ the resulting morphism. The morphism $\gamma$ is étale. Indeed, we have a commutative diagram

$$\begin{array}{ccc}
\gamma^*(t^*(\Omega_{A_s/s})) & \longrightarrow & \gamma^*(\Omega_{G/s}) \\
\sim \downarrow & & \downarrow=
\tilde{t}^*(\bar{\phi}_s^*(\Omega_{A_s/s})) & \longrightarrow & \tilde{t}^*(\Omega_{A_s/s}) \longrightarrow \Omega_{G/s}.
\end{array}$$

and in the lower row of this diagram all the arrows are surjective. Thus the arrow

$$\gamma^*(\Omega_{G/s}) \to \Omega_{G/s}$$
must also be surjective and hence an isomorphism. Since $G$ is smooth over $\kappa(s)$, we conclude that $\gamma$ is smooth by [19, III, Prop. 10.4]. In particular $\gamma$ is faithfully flat, because it is a morphism of group schemes and $G$ is connected (see eg [23, SGA 3.1, Exp. IV-B, Cor. 1.3.2]).

Now recall that there is a $K$-morphism $\psi : A \to A$ such that $\psi \circ \phi = [p^{\deg(\phi)}]_A$ (because finite commutative group schemes over $K$ are annihilated by their order; see [38, Theorem (Deligne), p. 4]). The morphism $\psi$ extends uniquely to $\overline{\psi} : A \to A$ and thus by unicity, we have $\overline{\psi} \circ \overline{\phi} = [p^{\deg(\phi)}]_A$. In particular, $\ker(\gamma)$ is a closed subscheme of $\ker([p^{\deg(\phi)}]_A)$. Since $\ker([p^{\deg(\phi)}]_A)$ is an infinitesimal group scheme and $\gamma$ is étale, we see that $\ker(\gamma) = 0$ (since $\ker(\gamma)$ is étale over $s$). Thus $\gamma$ is an isomorphism.

Now choose a $\bar{s}$-isomorphism $G_{\bar{s}} \simeq \mathbb{G}_m^d$ (here $\bar{s}$ if the spectrum of the algebraic closure of $\kappa(s)$). The morphism $\gamma_{\bar{s}}$ is described by a matrix $M \in \text{GL}_d(\mathbb{Z})$ (because the group scheme dual to $G_{\bar{s}}$ is the constant group scheme over $\bar{s}$ associated with $\mathbb{Z}^d$). Hence there exists a monic polynomial $P(x) \in \mathbb{Z}[x]$, such that $P(0) = \pm 1$ and such that $P(\gamma_{\bar{s}}) = 0$.

Finally, choose a prime $l \neq p$. The Tate module $T_l(G_{\bar{s}})(\kappa(\bar{s}))$ is naturally a submodule of $T_l(A_{K_{\text{sh}}})(K_{\text{sht}})$, where $K_{\text{sht}}$ is the fraction field of the strict henselisation of the local ring of $S$ at $s$ (it is the "toric part" of the Tate module). Hence the endomorphism

$$P(T_l(\phi)) \in \text{End}_{\mathbb{Z}_l}(T_l(A_{K_{\text{sh}}})(K_{\text{sht}}))$$

vanishes. Since any infinite group of $l$-primary torsion points of $A(\bar{K})$ is dense in $A_{\bar{K}}$ (because $A$ is geometrically simple), this implies that $P(\phi) = 0$. Hence $\phi$ is an automorphism, which is a contradiction.

### 7 Proof of Theorem 1.1

Recall the statement:

(a) Suppose that $A$ is geometrically simple. If $A(K_{\text{perf}})$ is finitely generated and of rank $> 0$ then $\text{Tor}_p(A(K_{\text{sep}}))$ is a finite group.

(b) Suppose that $A$ is an ordinary (not necessarily simple) abelian variety. If $\text{Tor}_p(A(K_{\text{sep}}))$ is a finite group then $A(K_{\text{perf}})$ is finitely generated.

We shall need the following

**Lemma 7.1.** Let $B$ be an abelian variety over $K$ and let $\gamma : B \to B$ be a $K$-isogeny such that $\deg(\phi) > 1$. Suppose that $B$ is geometrically simple. Let $H \subseteq A(\bar{K})$ be a finitely generated subgroup. Then the set

$$\bigcap_{r \geq 0} \gamma^{or}(H)$$

25
is a finite group.

**Proof.** (of Lemma 7.1) Let $G := \bigcap_{r \geq 0} \gamma^r(H)$. Let $F := G / \text{Tor}(G)$ be the quotient of $G$ by its torsion subgroup. We may suppose without restriction of generality that $\text{rk}(G) > 0$ for otherwise the lemma is proven. Since $\gamma$ is a group homomorphism, we have $\gamma(\text{Tor}(G)) \subseteq \text{Tor}(G)$ and thus $\gamma$ gives rise to a group homomorphism $F \to F$ that we also denote by $\gamma$. By construction, we have $\gamma(F) = F$ and thus $\gamma : F \to F$ is a bijection, since $F$ is a finitely generated free $\mathbb{Z}$-module. Let

$$P(t) := t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \in \mathbb{Z}[t]$$

be the characteristic polynomial of $\gamma : F \to F$. We have $P(\gamma) = 0$ by the Cayley-Hamilton theorem and since $\gamma$ is an automorphism, we have have

$$P(0) = a_0 = \pm 1 = \det(\gamma).$$

Hence

$$(-a_0)^{-1} \cdot (\gamma^{\circ,n-1} + a_{n-1} \cdot \gamma^{\circ,n-2} + \ldots + a_1 \cdot \text{Id}_F)$$

is the inverse of $\gamma : F \to F$. Now let $\tilde{\gamma}$ be the $K$-group scheme homomorphism

$$\tilde{\gamma} := (-a_0)^{-1} \cdot (\gamma^{\circ,n-1} + a_{n-1} \cdot \gamma^{\circ,n-2} + \ldots + a_1 \cdot \text{Id}_B)$$

from $B$ to $B$. Suppose first that the morphism of $K$-group schemes $\tilde{\gamma} \circ \gamma - \text{Id}_B$ is not the zero morphism. Then it is surjective, because $B$ is simple. Furthermore the group $G$ is dense in $B_K$, since $B$ is geometrically simple. Thus the group $(\tilde{\gamma} \circ \gamma - \text{Id}_B)(G)$ is dense in $B_K$. On the other hand, by construction $(\tilde{\gamma} \circ \gamma - \text{Id}_B)(G) \subseteq \text{Tor}(G)$. Since $\text{Tor}(G)$ is a finite group, it is not dense in $B_K$ and thus we deduce that $\tilde{\gamma} \circ \gamma - \text{Id}_B$ must be the zero morphism. Hence $\gamma$ is invertible (with inverse $\tilde{\gamma}$), which contradicts the assumption that $\deg(\gamma) > 1$. We conclude that we cannot have $\text{rk}(G) > 0$ and thus $G = \text{Tor}(G)$ is a finite group. □

For the proof of Theorem 1.1 (a), suppose first that $\text{Tor}_p(A(K_{\text{sep}}))$ is not a finite group. Then by Theorem 2.9, there exists an abelian variety $B$ over $K$, which is $K$-isogenous to $A$ and which carries an étale $K$-endomorphism $B \to B$, whose degree is $> 1$ and is a power of $p$. The dual of $B$ hence carries an isogeny $\phi$, which is purely inseparable (because the dual of a finite étale group scheme over a field is an infinitesimal group scheme) and thus we have

$$B^\vee(K_{\text{perf}}) = \bigcap_{r \geq 0} \phi^r(B^\vee(K_{\text{perf}}))$$

26
By Lemma 7.1, $B^{\vee}(K^{\text{perf}})$ is thus either finite or not finitely generated and the same holds for $A$, since $A$ is isogenous to $B^{\vee}$. This proves (a).

We now turn to the proof of statement (b). Note that by Grothendieck’s semiabelian reduction theorem, we may (and do) assume that $A$ is semiabelian. Suppose that $A(K^{\text{perf}})$ is not finitely generated and that $A$ is ordinary. Then by Proposition 2.6, there is an abelian variety $B$ over $K$, which is $K$-isogenous to $A$ and which carries a $K$-isogeny $B \to B$, whose kernel is a multiplicative group scheme of order $> 1$. The dual $\phi$ of this isogeny is an étale isogeny of $B^{\vee}$, which has degree $p^r$ for some $r > 0$. Thus $\text{Tor}_p(B^{\vee}(K^{\text{sep}}))$ is an infinite group and the same holds for $A$, since $A$ is isogenous to $B^{\vee}$. This proves (b).

8 Proof of Theorem 1.2

Recall the statement:

Suppose that $A$ is a semiabelian scheme and that $A$ is a geometrically simple abelian variety over $K$. If $\text{Tor}_p(A(K^{\text{sep}}))$ is infinite, then

- (a) $A$ is an abelian scheme;
- (b) there is $r_A \geq 0$ such that $p^{r_A} \cdot \text{Tor}_p(A(K^{\text{sep}})) \subseteq \text{Tor}_p(A(K^{\text{unr}}));$

Furthermore, there is

- (c) an abelian scheme $B$ over $S$;
- (d) a generically étale $S$-isogeny $A \to B$, whose degree is a power of $p$;
- (e) an étale $S$-isogeny $B \to B$ whose degree is $> 1$ and is a power of $p$.

Finally

- (f) if $A$ is ordinary then the Kodaira-Spencer rank of $A$ is not maximal;
- (g) if $\dim(A) \leq 2$ then $\text{Tr}_{K|k}(A_K) \neq 0$.
- (h) for all closed points $s \in S$, the $p$-rank of $A_s$ is $> 0$.

Proof of (a): this is a consequence of the criterion of Néron-Ogg-Shafarevich and Theorems 2.9 and 2.10.
Proof of (b): Let $H := \text{Gal}(K_{\text{sep}}|K_{\text{unr}})$. For $i \geq 0$, let $G_i := A(K_{\text{sep}})[p^i]$. The group $G_i$ is the group of $K$-rational points of an étale finite group scheme $G_i$ over $K$, which is naturally a closed subgroup scheme of $A$. Let $A_i := A/G_i$ and for $i \leq j$ let $\phi_{i,j} : A_i \to A_j$ be the natural morphism. Let $\mathcal{A}_i$ be the connected component of the zero section of the Néron model of $A_i$ over $S$. By (a) and the criterion of Néron-Ogg-Shafarevich, this is an abelian scheme. Furthermore, the morphisms $\phi_{i,j}$ extend to morphisms $\bar{\phi}_{i,j} : \mathcal{A}_i \to \mathcal{A}_j$ and if $i \leq j$, we have an exact sequence

$$0 \to \bar{\phi}_{i,j}^*(\Omega_{\mathcal{A}_j/S}) \to \Omega_{\mathcal{A}_i/S} \to \Omega_{\bar{\phi}_{i,j}} \to 0.$$  

(5)

Let $\pi_i : \mathcal{A}_i \to S$ be the structural morphism. We have a functorial isomorphism

$$\Omega_{\mathcal{A}_i} \cong \pi_i^*(\pi_i^*(\Omega_{\mathcal{A}_i/S}))$$

and thus there is a coherent sheaf $T_{i,j}$ on $S$, which is a torsion sheaf, such that $\pi_i^*(T_{i,j}) \cong \Omega_{\bar{\phi}_{i,j}}$ and the sequence (5) is the pull-back by $\pi_i^*$ of a sequence

$$0 \to \pi_j^*(\Omega_{\mathcal{A}_j/S}) \to \pi_i^*(\Omega_{\mathcal{A}_i/S}) \to T_{i,j} \to 0$$

and in particular

$$\deg_S(\pi_j^*(\Omega_{\mathcal{A}_j/S})) + \deg_S(T_{i,j}) = \deg_S(\pi_i^*(\Omega_{\mathcal{A}_i/S})).$$

Now recall that $\deg_S(\pi_i^*(\Omega_{\mathcal{A}_i/S})) \geq 0$ for all $i \geq 0$ (see [14, V, Prop. 2.2, p. 164]). Thus, for $i = 0, 1, \ldots$ the sequence $\deg_S(\pi_i^*(\Omega_{\mathcal{A}_i/S}))$ is a non-increasing sequence of natural numbers. Hence for large enough $i$, say $i_0$, it reaches its minimum. We conclude that $T_{i_0,j} = 0$ for $j > i_0$, so that the morphism $\bar{\phi}_{i_0,j}$ is étale. In particular, we have

$$\phi_{0,i_0}(G_j(K_{\text{sep}})) \subseteq A_{i_0}(K_{\text{unr}})$$

when $j > i_0$. In other words, for any $x \in G_j(K_{\text{sep}})$ and any $\gamma \in H$, we have

$$\gamma(x) - x \in G_{i_0}(K_{\text{sep}}).$$

In particular, we have

$$\gamma(p^{i_0} \cdot x) = p^{i_0} \cdot \gamma(x) = p^{i_0} \cdot x$$

In particular, since $j > i_0$ was arbitrary, we see that

$$\gamma(p^{i_0} \cdot x) = p^{i_0} \cdot x$$

for all $x \in \text{Tor}_p(A(K_{\text{sep}}))$ and all $\gamma \in H$. Setting $r_A = i_0$ concludes the proof of (b).
Proof of the existence statements (c), (d), (e): this is a consequence of (a) and Theorems 2.9 and 2.10.

Proof of (f): this is contained in a theorem of J.-F. Voloch; see [48, Proposition on p. 1093].

Proof of (g): this is a consequence of (b) and Proposition 2.8.

Proof of (h): This follows from (a) and (e).

9 Proof of Theorem 1.4

We now suppose that $l_0 = \overline{F}_p$ and we use the notations of Conjecture 1.3.

We start with some reductions.

Let $\lambda : \text{Tr}_{l_0}(C) \to C$ be the canonical morphism. Here we write $C/\text{Im}(\lambda)$ for the quotient of $C$ by $\text{Im}(\lambda)$. This is an abelian variety over $L$, which represents the quotient of $C$ by $\text{Im}(\lambda)$ in the category of fppf sheaves.

(0) **Theorem 1.4 holds if it holds for $C/\text{Im}(\lambda)$ in place of $C$.**

We shall need the following

**Lemma 9.1.** Let $N$ be a finite flat infinitesimal group scheme over a field $J$ of characteristic $p$. There is a finite field extension $J'|J$ such that for any $n \geq 0$ and any element $\alpha \in H^1(J, N(p^n))$, the image $\alpha_{J'}$ of $\alpha$ in $H^1(J', N(p^n))$ vanishes.

Here $H^1(J, N(p^n))$ is the first cohomology group of $N(p^n)$ viewed as a sheaf in the fppf topology. More concretely, it is the group of isomorphism classes of torsors of $N(p^n)$ over $J$.

In the following proof, we shall write $J^{p^{-m}} \subseteq J$ for the subfield of $J$ consisting of elements of the form $x^{p^{-m}}$, where $x \in J$.

**Proof.** (of Lemma 9.1) First suppose that $N$ has a filtration by finite closed subgroup schemes, whose quotients are isomorphic to either $\alpha_{p,J}$ or $\mu_{p,J}$. Let $m \geq 0$ be the number of non vanishing quotients. We shall prove by induction on $m$ that the image of $\alpha$ in $H^1(J^{p^{-m}}, N(p^n))$ vanishes for all $n \geq 0$ (under the supplementary assumption on $N$), for any field $J$ of characteristic $p$. If $m = 0$ the statement holds tautologically, so we shall suppose that it holds for $1, \ldots, m - 1$. Let

$$0 \to F_1 \to N_{J_1} \to F_2 \to 0$$

be a presentation of $N$ where $F_2$ is isomorphic to either $\alpha_{p,J}$ or $\mu_{p,J}$ and $F_1$ has a filtration as above, whose number of non vanishing quotients is $\leq m - 1$. This induces exact sequences

$$0 \to H^1(J^{p^{-1}}, (F_{1,J^{p^{-1}}})(p^n)) \to H^1(J^{p^{-1}}, (N_{J^{p^{-1}}})(p^n)) \to H^1(J^{p^{-1}}, (F_{2,J^{p^{-1}}})(p^n))$$

29
Now according to [25, par. 2.4, p. 28] there is a finite extension \( J \) of \( J \) such that \( N_{J_1} \) has a filtration by finite closed subgroup schemes, whose quotients are isomorphic to either \( \alpha_{p,J_1} \) or \( \mu_{p,J_1} \). This extension will by construction also work for all the group schemes \( N^{(p^n)} \) and the number of non vanishing quotients of all the group schemes \( N^{(p^n)}_{J_1} \) is constant, say it is \( m \). Hence the extension \( J' := J^{p^{-m}}_1 \) has the required property. \( \square \)

Now suppose that Theorem 1.4 holds for \( C/\operatorname{Im}(\lambda) \) in place of \( C \). We want to show that Theorem 1.4 holds (for \( C \)).

Write

\[
\lambda^{(p^n)} : \operatorname{Tr}_{L/\mathcal{O}}(C)^{(p^n)} \to C^{(p^n)}
\]

for the base change of \( \lambda \) by \( F^n_p \). We have an exact sequence

\[
0 \to \operatorname{Im}(\lambda)(L) \to C(L) \to (C/\operatorname{Im}(\lambda))(L)
\]

and we have \( (C/\operatorname{Im}(\lambda))^{(p^n)} \simeq C^{(p^n)}/\operatorname{Im}(\lambda^{(p^n)}) \). Let now

\[
x_0 \in C(L), x_1 \in C^{(p)}(L), x_2 \in C^{(p^2)}(L), \ldots
\]

be a sequence of points such \( V_{C^{(p)}(L)}(x_1) = x_0 \), \( V_{C^{(p^2)}(L)}(x_2) = x_1 \) etc. Then we know from the above supposition that the image of \( x_n \) in \( (C^{(p^n)}/\operatorname{Im}(\lambda^{(p^n)}))(L) \) is a prime to \( p \) torsion point for all \( n \geq 0 \). In particular, the order \( m \) of the image of \( x_n \) in \( (C^{(p^n)}/\operatorname{Im}(\lambda^{(p^n)}))(L) \) is independent of \( n \), because the degree of the Verschiebung is always a power of \( p \). Let \( m \) be the order of \( x_0 \) (and hence of all the \( x_n \)). Then \( m \cdot x_n \in \operatorname{Im}(\lambda^{(p^n)})(L) \) for all \( n \) and thus \( m \cdot x_0 \) is indefinitely Verschiebung divisible in \( \operatorname{Im}(\lambda)(L) \) (because the Verschiebung morphism commutes with morphisms of commutative group schemes). It now suffices to prove that \( m \cdot x_0 \) is of finite and prime to \( p \) order in \( \operatorname{Im}(\lambda)(L) \). Hence, we may and do assume that the morphism \( \lambda : \operatorname{Tr}_{L/\mathcal{O}}(C) \to C \) is a surjection.

Now \( \lambda \) is also finite and purely inseparable by [10, Th. 6.12] and it is thus a bijection. We are now given infinitely many \( L \)-morphisms

\[
\ldots (\lambda^{(p^n)})^*(x_n) \to \cdots \to (\lambda^{(p)})^*(x_1) \to \lambda^*(x_0)
\]
where \((\lambda^{(p^n)})^*(x_n)\) is the base change by \(\lambda^{(p^n)}\) of \(x_n\) viewed as a closed subscheme of \(C^{(p^n)}\). The \(L\)-scheme \((\lambda^{(p^n)})^*(x_n)\) is a torsor under the group scheme \((\ker \lambda)^{(p^n)} \simeq \ker \lambda^{(p^n)}\) and according to Lemma 9.1, there is a finite extension \(L'\), which splits all the \((\lambda^{(p^n)})^*(x_n)\). We thus obtain an indefinitely Verschiebung divisible point \(x'_0\) in \(\text{Tr}_{L'/l_0}(C)(L')\), whose image in \(C(L')\) is \(x_0\). Now \(\text{Tr}_{L'/l_0}(C)\) is by definition the base change to \(L'\) of an abelian variety over \(l_0\); so we are reduced to showing Theorem 1.4 for abelian varieties \(C\) that arise by base-change from \(l_0\). The next lemma thus concludes reduction step (0).

**Lemma 9.2.** Theorem 1.4 holds if \(C \simeq C_0 \times_{l_0} L\), where \(C_0\) is an abelian variety over \(l_0\).

**Proof.** Let \(n \geq 0\) be such that \(C^{(p^n)} \simeq C\) and let \(V := V^{(n)}_{C/L} : C^{(p^n)} \to C\) be the corresponding composition of Verschiebung morphisms. There is such an \(n\) because \(C_0\) (and hence \(C\)) has a model over a finite field. We identify \(V\) with a a morphism \(C \to C\) via a fixed isomorphism \(C^{(p^n)} \simeq C\). Let \(x_0 \in C(L)\) be an indefinitely Verschiebung divisible point. The point \(x_0\) is in particular indefinitely \(V\) divisible and \(V\) leaves the subgroup \(C_0(l_0) \subseteq C(L)\) invariant. Furthermore, \(V|_{C_0(l_0)} : C_0(l_0) \to C_0(l_0)\) is a surjection, since \(l_0\) is algebraically closed. Thus, by the snake lemma, \(V\) descends to an injective group morphism \(\psi : C(L)/C_0(l_0) \to C(L)/C_0(l_0)\). Now note that that there is a polynomial with integer coefficients \(P(t) \in \mathbb{Z}[t]\) such that \(P(V) = 0\) and (see [37, p. 182]). We choose \(P(t)\) so that \(\text{deg}(P(t))\) is minimal. There are no algebraic units among the roots of this polynomial (we leave the proof of this to the reader). Now let \(y_0\) be the image of \(x_0\) in \(C(L)/C_0(l_0)\). Let \(I := \bigcap_{j \geq 0} \psi^{(p)}(C(L)/C_0(l_0))\). Note that \(I\) is finitely generated, since \(C(L)/C_0(l_0)\) is finitely generated by the Lang-Néron theorem. The restriction \(\psi|_I : I \to I\) is by construction a surjection, and hence a bijection, since \(\psi\) is injective. If \(I\) is not a finite group, then the minimal polynomial of

\[
\psi_Q : (C(L)/C_0(l_0))_Q \to (C(L)/C_0(l_0))_Q
\]

has a non trivial factor, whose roots are algebraic units and this factor must divide \(P(t)\). This is a contradiction, so \(I\) is a finite torsion group. So there is an \(m \geq 1\) so that \(m \cdot x_0 \in C_0(l_0)\).

Since \(l_0\) is algebraically closed, this implies that \(x_0 \in C_0(l_0)\), concluding the proof. \(\square\)

(1) **We may suppose that \(L\) is the function field of a smooth and proper curve \(B\) over \(l_0\).**

This follows from a standard spreading out argument together with Proposition B.1 in the Appendix. One could probably also carry out this reduction by appealing to Hilbert’s irreducibility theorem as in [30, chap. 9, cor. 6.3] but for lack of an adequate reference in the case of function fields, we prefer to use Proposition B.1. Note that reduction (1) works for any algebraically closed field \(l_0\) of characteristic \(p > 0\) (not only \(l_0 = \overline{F}_p\)).

31
(2) We may suppose that $C$ has a semiabelian model $\mathcal{C}$ over $B$.

This follows from Grothendieck’s semiabelian reduction theorem.

We now start with the proof. In view of reduction step (0), we may assume that $\text{Tr}_{L'/l_0}(C_{L'}) = 0$ for any finite extension $L'/L$ (if $\text{Tr}_{L'/l_0}(C_{L'}) \neq 0$ then replace $L$ by $L'$ and $C$ by $C_{L'}$ and go back to reduction step 0).

We first fix a model $B$ of $B$ over a finite field, say $k$. We also fix a model $C$ of $C$ over $B$ (increasing the size of $k$ if necessary).

By Lemma A.2 and the discussion preceding it we have canonical injection

$$C^{(p)}(L)/F_{C/L}(C(L)) \hookrightarrow \text{Hom}_B(\omega_{C^{(p)}}, \Omega_{B/l_0}(E)) \quad (6)$$

where $E = E(C)$ is the reduced divisor, which is the union of the closed point $b \in B$ such that $C_b$ is not proper over $\kappa(b)$. Note that we have $E(C) = E(C^{(p)}) = E(C^{(p^2)}) = \ldots$. The injection (6) is naturally compatible with isogenies (we skip the verification) and so there is an infinite commutative diagram

$$
\begin{align*}
C^{(p)}(L) &\longrightarrow \text{Hom}_B(\omega_{C^{(p)}}, \Omega_{B/l_0}(E)) \\
\uparrow V_{C^{(p^2)}/L} &\quad \uparrow V_{C^{(p^2)}/B} \\
C^{(p^2)}(L) &\longrightarrow \text{Hom}_B(\omega_{C^{(p^2)}}, \Omega_{B/l_0}(E)) \\
\vdots &\quad \vdots
\end{align*}
$$

(7)

Remember that we have

$$\omega_{C^{(p^n)}} \simeq F_S^{\text{on},*}(\omega_C).$$

Now choose $n_1 \geq 1$ so that

- $\omega_{C^{(p^{n_1})}}$ has a Frobenius semistable HN filtration;
- $(\omega_{C^{(p^{n_1})}}) = 0 \simeq (\omega_{C^{(p^{n_1})}}) = 0_{\text{bif}} \oplus (\omega_{C^{(p^{n_1})}}) = 0_{\text{mult}}$ splits into a biinfinitesimal and a multiplicative commutative coLie-algebra (see Lemmata 4.3 and 4.6).

Note that if some $n_1 \geq 1$ has the two above properties, than any higher $n_1$ will as well (by definition for the first property and tautologically for the second one).

Choose $n_2 > n_1$ so that

(I) the image of the map

$$V_{C^{(p^{n_2})}/B}^{(n_2-n_1)*} : \omega_{C^{(p^{n_1})}} \rightarrow \omega_{C^{(p^{n_2})}}.$$
lies in \((\omega_{\mathcal{C}(p^n_1)})_{\geq 0} \simeq F_B^{o(p^n_2-n_1)} \ast ((\omega_{\mathcal{C}(p^n_1)})_{\geq 0});
\)

(II) the image of the map of coLie algebras
\[
V_{\mathcal{C}(p^n_2)/B}^{(n_2-n_1),\ast} : (\omega_{\mathcal{C}(p^n_1)})_{=0} \to F_B^{o(p^n_2-n_1)} \ast ((\omega_{\mathcal{C}(p^n_1)})_{=0}) = (\omega_{\mathcal{C}(p^n_2)})_{=0}
\]
is \(F_B^{o(p^n_2-n_1)} \ast ((\omega_{\mathcal{C}(p^n_1)})_{=0}; \mu).
\]

Note that this is possible because the biinfinitesimal part of \((\omega_{\mathcal{C}(p^n_1)})_{=0}\) will be sent to 0 by sufficiently many composed Verschiebung morphisms (by definition).

Note that under (I) for any \(n_3 > n_2\) the image of the map
\[
V_{\mathcal{C}(p^n_3)/B}^{(n_3-n_2),\ast} : (\omega_{\mathcal{C}(p^n_2)})_{\geq 0} \to \omega_{\mathcal{C}(p^n_3)}
\]
and hence of the map
\[
V_{\mathcal{C}(p^n_3)/B}^{(n_3-n_2),\ast} : \omega_{\mathcal{C}(p^n_1)} \to \omega_{\mathcal{C}(p^n_3)}
\]
automatically lies in \((\omega_{\mathcal{C}(p^n_3)})_{\geq 0} \simeq F_B^{o(p^n_3-n_1)} \ast ((\omega_{\mathcal{C}(p^n_1)})_{\geq 0}).
\]

Choose \(n_3 > n_2\) so that

(III) the map
\[
\omega_{\mathcal{C}(p^n_3)} \to \Omega_{B/l_0}(E)
\]

given by \(x_{n_3}\) factors through its quotient \((F_B^{o(n_3-n_1)} \ast (\omega_{\mathcal{C}}))_{\leq 0} \simeq F_B^{o(n_3-n_1)} \ast ((\omega_{\mathcal{C}(p^n_1)})_{\leq 0});\)

(IV) the image of the map
\[
V_{\mathcal{C}(p^n_3)/B}^{(n_3-n_2),\ast} : F_B^{o(p^n_2-n_1)} \ast ((\omega_{\mathcal{C}(p^n_1)})_{=0}) \to F_B^{o(n_3-n_2)} \ast ((\omega_{\mathcal{C}(p^n_2)})_{=0})
\]
is \(F_B^{o(n_3-n_2)} \ast ((\omega_{\mathcal{C}(p^n_1)})_{=0}; \mu) \simeq F_B^{o(n_3-n_1)} \ast ((\omega_{\mathcal{C}(p^n_1)})_{=0}; \mu).
\]

Now we shall exploit the compatibility between the morphism
\[
\omega_{\mathcal{C}(p^n_1)} \xrightarrow{c(x_{n_1})} \Omega_{B/k}(E)
\]
induced by \(x_{n_1}\) and the morphism
\[
\omega_{\mathcal{C}(p^n_3)} \xrightarrow{c(x_{n_3})} \Omega_{B/k}(E)
\]
induced by \(x_{n_3}\). According to the diagram (7), this compatibility gives the equality
\[
c(x_{n_3}) \circ V_{\mathcal{C}(p^n_3-n_1)/B}^{\ast} = c(x_{n_1}).
\]

In other words the composition of morphisms
\[
\omega_{\mathcal{C}(p^n_1)} \xrightarrow{V_{\mathcal{C}(p^n_3-n_1)/B}^{\ast}} \omega_{\mathcal{C}(p^n_3)} \xrightarrow{c(x_{n_3})} \Omega_{B/k}(E)
\]

33
is \( c(x_{n_1}) \). Furthermore, in view of (I) and (III) the map \( c(x_{n_1}) \) factors as follows:

\[
\omega^\ast_{C(p^{n_1})} \rightarrow F_B^{\ast(n_3-n_1)}((\omega^\ast_{C(p^{n_1})})_\geq 0) \rightarrow F_B^{\ast(n_3-n_1)}((\omega^\ast_{C(p^{n_1})})_{=0}) \rightarrow \Omega_{B/k}(E)
\]

and by (I) the map

\[
\omega^\ast_{C(p^{n_1})} \rightarrow F_B^{\ast(n_3-n_1)}((\omega^\ast_{C(p^{n_1})})_{=0})
\]

factors as follows

\[
\omega^\ast_{C(p^{n_1})} \rightarrow F_B^{\ast(n_3-n_1)}((\omega^\ast_{C(p^{n_1})})_{=0}) \rightarrow \Omega_{B/k}(E)
\]

and thus by (IV) and (II) the image of this last map is precisely \( F_B^{\ast(n_3-n_1)}((\omega^\ast_{C(p^{n_1})})_{=0,\mu}) \).

We have thus constructed a multiplicative quotient of the \( p \)-coLie algebra \( \omega^\ast_{C(p^{n_1})} \). On the other hand the \( p \)-coLie algebra \( \omega^\ast_{C(p^{n_1})} \) is the \( p \)-coLie algebra of the finite flat group scheme \( \ker F_{C(p^{n_1})}/B \). By the equivalence of categories recalled in subsection 4.1, this quotient corresponds to a multiplicative subgroup scheme of \( \ker F_{C(p^{n_1})}/B \). By Lemma 4.7, this subgroup scheme embeds in the canonical largest multiplicative subgroup scheme (\( \ker F_{C(p^{n_1})}/B \)) of \( \ker F_{C(p^{n_1})}/B \) (in fact, it coincides with it, but we shall not need this). Finally note that

\[
(\ker F_{C(p^{n_1})}/B)_\mu \simeq ((\ker F_{C/B})_\mu(p^{n_1}))
\]

by the last part of Lemma 4.7.

Let \( G := (\ker F_{C/B})_\mu \). Note that \( G \) comes from a unique subgroup scheme \( G \) of \( C \), because it is \( \text{Gal}(\bar{k}/k) \)-invariant by unicity. Now consider the quotient \( C_1 := C/G \) (which is a semi-abelian scheme by 4.9) and let \( \psi_1 : C \rightarrow C_1 \) be the quotient morphism. The group scheme \( C_1 \) also a model over \( B \), namely \( C/G \). The point \( x_{n_1} \) and its image \( y_{n_1} \) in \( C_1(L) \) give a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & F_B^{\ast(n_3-n_1)}((\omega^\ast_{C(p^{n_1})})_{=0,\mu}) \\
\omega^\ast_{C(p^{n_1})} & \rightarrow & F_B^{\ast(n_3-n_1)}((\omega^\ast_{C(p^{n_1})})_{=0}) \\
\psi_1 & \rightarrow & \Omega_{B/k}(E) \\
\omega^\ast_{C(p^{n_1})} & \rightarrow & \Omega_{B/k}(E)
\end{array}
\]

where the left column is an exact sequence and \( c(y_{n_1}) \) is the morphism induced by \( y_{n_1} \).
Thus $c(y_{n_1})$ vanishes. In particular, $y_{n_1}$ lies in the image of $F_{C_1^{[p^{n_1-1}]/B}}(C_1^{(p^{n_1-1})}(L))$. Using the fact that

$$[p]_{C_1^{[p^{n_1-1}]/B}} = V_{C_1^{[p^{n_1}]/B}} \circ F_{C_1^{[p^{n_1-1}]/B}},$$

we conclude that $y_{n_1}$ has a $p$-th root in $C_1^{(p^{n_1})}(L)$. Hence $y_0$ also has a $p$-th root in $C_1(L)$. Now since $G$ is independent of $x_0$, we conclude that the image of any indefinitely Verschiebung divisible point of $C(L)$ in $C_1(L)$ has a $p$-th root. Since $G$ is compatible with twists, we also see that for any $n \geq 0$ the image of any indefinitely Verschiebung divisible point of $C^{(p^n)}(L)$ in $C_1^{(p^n)}(L)$ has a $p$-th root. From this, by an elementary combinatorial consideration, we see that the image of any indefinitely Verschiebung divisible point of $C(L)$ in $C_1(L)$ has a $p$-th root, which is indefinitely Verschiebung divisible.

Let $IVD(C)$ be the set of indefinitely Verschiebung divisible point in $C(L)$. By the discussion above, the image of $IVD(C)$ in $C_1(L)$ lies in $p \cdot IVD(C_{1,L})$. We now replace $C$ by $C_1$ and repeat all the above process to obtain a sequence $C_{1,L}, C_{2,L}, \ldots$

Now note that the degrees of the Hodge bundles of the $C_i$ are constant by Lemma 4.11. Furthermore, they all have a model over $B$. Thus there are only finitely many isomorphism classes in this sequence (the same reasoning was made in the proof of Theorem 2.5). In particular, we obtain a purely inseparable isogeny $\psi : C_{a,L} \to C_{a,L}$ for some $a \geq 0$, with the property that $\psi(IVD(C_{a,L})) \subseteq p \cdot IVD(C_{a,L})$.

Recall now that we want to show that $IVD(C) \subseteq \text{Tor}_p(C(L))$.

We shall first show that $IVD(C) \subseteq \text{Tor}(C(L))$.

To show this, we might assume without restriction of generality that $C$ is simple. This follows from the fact that by $C$ is in general isogenous to a product of simple abelian varieties over $L$ by Poincaré’s complete reducibility theorem. This supplementary assumption now also implies that in the situation above the $C_{i,L}$ are also simple for any $i \geq 0$. It is now sufficient to show that $IVD(C_{a,L})$ is a torsion group. Recall that $IVD(C_{a,L})$ is finitely generated by the Lang-Néron theorem, since $C$ has trace 0 by assumption and hence all the $C_{i,L}$ have trace 0 as well. Suppose now to obtain a contradiction that $IVD(C_{a,L})$ is infinite. Let

$$P(t) = t^e + c_{e-1}t^{e-1} + \cdots + c_0$$

be the characteristic polynomial of $\psi$ acting on $IVD(C_{a,L})_Q$. Since

$$\psi(IVD(C_{a,L})) \subseteq p \cdot IVD(C_{a,L}),$$

we see that all the $c_i$ are divisible by $p$. Furthermore, since $IVD(C_{a,L})$ is dense in $C_{a,L}$ (because $C_{a,L}$ is simple and $IVD(C_{a,L})$ is a group), we see that $P(\psi) = 0$ on $C_{a,L}$. Let
\( \psi^* : \omega_{C_a} \rightarrow \omega_{C_a} \) be the pull-back map. The fact that \( P(\psi) = 0 \) on \( C_{a,L} \) implies that the map \( P(\psi^*) : \omega_{C_a} \rightarrow \omega_{C_a} \) also vanishes and from the fact the \( c_i \) are divisible by \( p \), we deduce that we have \( (\psi^*)^e = 0 \). In other words \( \psi^* \) is nilpotent of order \( e \). By the correspondence between \( p \)-coLie algebras and finite flat group schemes of height one described at the beginning of subsubsection 4.1, this implies that

\[
\ker \psi^o \supseteq \ker F_{C_a/B}
\]

Since \( \ker \psi^o \) is multiplicative, this implies that \( \ker F_{C_a/B} \) is a multiplicative group scheme. In particular, by Lemma 4.11, we have that

\[
\deg(\omega_{C_1}) = \deg(\omega_{C_1(p)}) = p \cdot \deg(\omega_{C_1})
\]

and so \( \deg(\omega_{C_1}) = 0 \) (the same reasoning is made in the proof of Raynaud’s theorem [36, XI, Th. 5.1, p. 237]). By [14, chap. 5, Prop. 2.2], this implies that after a finite field extension \( L'|L \), the variety \( C_{1,L'} \) is the base-change of a variety over \( l_0 \). This contradicts the assumptions and concludes the proof of the fact that \( \text{IVD}(C) \subseteq \text{Tor}(C(L)) \).

We now relax the condition that \( C \) is simple and we show that \( \text{IVD}(C) \subseteq \text{Tor}^p(C(L)) \).

Again, it is sufficient to show that \( \text{IVD}(C_{a,L}) \) is a finite group of order prime to \( p \), since all the \( C_i \) are related by injective isogenies. Since we now know that \( \text{IVD}(C_{a,L}) \) is a finite group, we see that the restriction \( \psi|_{\text{IVD}(C_{a,L})} : \text{IVD}(C_{a,L}) \rightarrow \text{IVD}(C_{a,L}) \) of \( \psi \) to \( \text{IVD}(C_{a,L}) \) is a bijection, since it is an injection. Hence some power of \( \psi|_{\text{IVD}(C_{a,L})} \) is the identity and so we see that every element of \( \text{IVD}(C_{a,L}) \) is divisible by \( p \). By the structure theorem for finite abelian groups, this implies that no non zero element of \( \text{IVD}(C_{a,L}) \) has an order divisible by \( p \).

\section{Ampleness and nefness of the Hodge bundle of generically ordinary semiabelian schemes}

In this appendix, we shall prove a mild extension of the main result of [42]. The terminology of this appendix is independent of the terminology of the main body of the article.

Let \( k \) be a perfect field and let \( S \) be a geometrically connected, smooth and proper curve over \( k \). Let \( K := \kappa(S) \) be its function field. Suppose from now on that \( k \) has characteristic \( p > 0 \).

Let \( \pi : \mathcal{A} \rightarrow S \) be a smooth commutative group scheme and let \( A := \mathcal{A}_K \) be the generic fibre of \( \mathcal{A} \). Let \( \epsilon_{A/S} : S \rightarrow \mathcal{A} \) be the zero-section and let \( \omega := \epsilon_{A/S}^*(\Omega^{1}_{\mathcal{A}/S}) \) be the Hodge bundle of \( \mathcal{A} \) over \( S \).
Theorem A.1. Suppose that $A/S$ is semiabelian and that $A$ is an abelian variety. Suppose that $\bar{\mu}_{\min}(\omega) > 0$. Then there exists $\ell_0 \in \mathbb{N}$ such that the natural injection $A(K^{p^{-\ell_0}}) \hookrightarrow A(K_{\text{perf}})$ is surjective (and hence a bijection).

N.B. In [42, Th. 1.1], Theorem A.1 was proven under the assumption that $A$ is principally polarised and that $k$ is algebraically closed. It can be shown that the condition $\bar{\mu}_{\min}(\omega) > 0$ is equivalent to the requirement that $\omega$ is an ample bundle (see [42, Introduction] for detailed references).

Proof. Notice first that in our proof of Theorem A.1, we may replace $K$ by a finite extension field $K'$ without restriction of generality. We may thus suppose that $A$ is endowed with an $m$-level structure for some $m \geq 3$ with $(m, p) = 1$.

If $Z \to W$ is a $W$-scheme and $W$ is a scheme of characteristic $p$, then for any $n \geq 0$ we shall write $Z[n] \to W$ for the $W$-scheme given by the composition of arrows $Z \to W \xrightarrow{F_n} W$.

Now fix $n \geq 1$ and suppose that $A(K^{p^{-n}}) \setminus A(K^{p^{-n+1}}) \neq \emptyset$.

Fix $P \in A^{(p^n)}(K) \setminus A^{(p^{n-1})}(K) = A(K^{p^{-n}}) \setminus A(K^{p^{-n+1}})$. The point $P$ corresponds to a commutative diagram of $k$-schemes

\[
\begin{array}{ccc}
P & \longrightarrow & A \\
\Spec K[n] & \xrightarrow{\overline{F_n}} & \Spec K
\end{array}
\]

such that the residue field extension $K|\kappa(P(\Spec K[n]))$ is of degree 1 (in other words $P$ is birational onto its image). In particular, the map of $K$-vector spaces $P^*(\Omega^1_{A/k}) \to \Omega^1_{K[n]/k}$ arising from the diagram is non zero.

Now recall that there is a canonical exact sequence

\[0 \to \pi_K^*(\Omega^1_{K/k}) \to \Omega^1_{A/k} \to \Omega^1_{A/K} \to 0.\]

Furthermore the map $F_{K[n]}^*(\Omega^1_{K/k}) \xrightarrow{\phi_n} \Omega^1_{K[n]/k}$ vanishes. Also, we have a canonical identification $\Omega^1_{A/K} = \pi_K^*(\omega_K)$ (see [7, chap. 4., Prop. 2]). Thus the natural surjection $P^*(\Omega^1_{A/k}) \to \Omega^1_{K[n]/k}$ gives rise to a non-zero map

\[\phi_n = \phi_{n,P} : F_{K[n]}^*(\omega_K) \to \Omega^1_{K[n]/k}.\]

The next crucial lemma examines the poles of the morphism $\phi_n$.
We let $E$ be the reduced closed subset, which is the union of the points $s \in S$, such that the fibre $A_s$ is not complete.

**Lemma A.2.** The morphism $\phi_n$ extends to a morphism of vector bundles

$$F_S^{n,*}(\omega) \rightarrow \Omega^1_{S[n]/k}(E).$$

**Proof.** (of A.2). First notice that there is a natural identification $\Omega^1_{S[n]/k}(\log E) = \Omega^1_{S[n]/k}(E)$, because there is a sequence of coherent sheaves

$$0 \rightarrow \Omega_{S[n]/k} \rightarrow \Omega^1_{S[n]/k}(\log E) \rightarrow \mathcal{O}_E \rightarrow 0$$

where the morphism onto $\mathcal{O}_E$ is the residue morphism. Here the sheaf $\Omega^1_{S[n]/k}(\log E)$ is the sheaf of differentials on $S[n]\setminus E$ with logarithmic singularities along $E$. See [26, Intro.] for this result and more details on these notions.

We may also suppose without restriction of generality that $A$ is principally polarised. Indeed, consider the following reasoning. By Zarhin’s trick, the abelian variety $B := (A \times_K A^\vee)^4$ is principally polarised. Also, $B$ can be endowed with an $m$-level structure compatible with the given $m$-level structure on $A$, since $A^\vee$ is isogenous to $A$. Let $B := (A \times_K A^\vee)^4$, where (abusing language) we have written $A^\vee$ for the connected component of the zero-section of the Néron model of $A^\vee$. The group scheme $A^\vee$ is also semiabelian, since $A^\vee$ is isogenous to $A$ over $K$. The morphism $P \times 0 \times 0 \times \cdots \times 0$ (seven times) gives a point in $B^{(p^n)}(K)$ and there is a commutative diagram

$$
\begin{array}{ccc}
F_S^{n,*}(\omega, B, K) & \xrightarrow{\phi_n, p \times 0 \times \cdots} & \Omega^1_{K[n]/k} \\
\downarrow & & \uparrow \\
F_S^{n,*}(\omega, A, K) & \xrightarrow{\phi_n, p} & \Omega^1_{K[n]/k}
\end{array}
$$

where the vertical arrow on the left is the pull-back map induced by the closed immersion $\lambda \mapsto \lambda \times 0 \times 0 \times \cdots \times 0$ (seven times). Now since $B$ is principally polarised, we know that if Lemma A.2 holds for principally polarised abelian varieties, the upper row of the diagram (8) extends to a morphism $F_S^{n,*}(\omega_B) \rightarrow \Omega^1_{S[n]/k}(E)$ (note that the set of points, where $B$ is not complete coincides with the set of points, where $A$ is not complete). Since $F_S^{n,*}(\omega_A)$ is a direct summand of $F_S^{n,*}(\omega_B)$, we see that Lemma A.2 holds for $A$ if it holds for $B$, thus completing the reduction of Lemma A.2 to the principally polarised case. \qed

The rest of the proof of Theorem A.1 is identical word for word with the proof of Theorem 1.1 in [42] (from the beginning of the proof of Lemma 2.1). \qed
B Specialisation of the Mordell-Weil group

In this section, we shall prove a geometric analog of Néron’s result on the specialisation of the generic Mordell-Weil group to a fibre in a family of abelian varieties over number fields (see [30, chap. 9, Cor. 6.3]). The following results are reminiscent of some results proven by Hrushovski in a mixed characteristic context (see [21]) and they are probably already known to many people but we include complete proofs for lack of a reference.

The terminology of this section is independent of the terminology of the introduction.

Let $l_0$ be an algebraically closed field. Let $U$ be a smooth and connected quasi-projective variety over $l_0$. Let $B$ be an abelian scheme over $U$. Suppose given an immersion $\iota : U \to \mathbb{P}^N$ for some $N \geq 0$. Let $K$ be the function field of $U$ and let $B := B_K$.

**Proposition B.1.** Suppose that $B(U)$ is finitely generated. For almost all linear subspaces $L \subseteq \mathbb{P}^N$ of codimension $\dim(U) - 1$, the intersection $C := L \cap U$ is smooth and connected, the specialisation map

$$B(U) \to B_C(C)$$

is injective and $\text{Tr}_{\kappa(C)｜l_0}(B_{\kappa(C)}) = 0$.

Recall that the linear subspaces $L \subseteq \mathbb{P}^N$ of codimension $\dim(U) - 1$ are classified by the Grassmannian $\text{Gr}(\dim(U) - 1, N)$, which is smooth and projective over $l_0$. The words ”almost all” stand for ”for all the $l_0$-rational points of some Zariski open subset of $\text{Gr}(\dim(U) - 1, N)$”.

Recall that by a theorem of Weil, the restriction map $B(U) \to B(K)$ is a bijection. Thus, by the Lang-Néron theorem, the condition that $B(U) = B(K)$ is finitely generated is equivalent to the condition $\text{Tr}_{K｜l_0}(B) = 0$.

For the proof of Proposition B.1, we shall need a few lemmata:

**Lemma B.2.** Let $N$ be a finite étale group scheme over $U$. Let $t \in H^1_{\text{et}}(U, N)$ and suppose that $t \neq 0$. Then for almost all linear subspaces $L \subseteq \mathbb{P}^N$ of codimension $\dim(U) - 1$, the intersection $C := L \cap U$ is smooth and connected and the restriction $t_C \in H^1_{\text{et}}(C, N_C)$ of $t$ to $C$ does not vanish.

**Proof.** Let $T \to U$ be a torsor under $N$. Note that the torsor $T$ is non trivial iff for all the irreducible components $T'$ of $T$, the (automatically flat and finite) morphism $T' \to U$ has degree $> 1$. The same remark applies to the restriction of $T$ to a smooth and connected closed subscheme of $U$. 

39
Let \((T_i)\) be the set of irreducible components of \(T\).

By Bertini’s theorem in Jouanolou’s presentation (see [27, p. 89, Cor. 6.11]), for almost all linear subspaces \(L \subseteq \mathbb{P}^N\) of codimension \(\dim(U) - 1\),
- the intersection \(C := L \cap U\) is smooth and connected;
and
- all the \(T_{i,C}\) are irreducible.

Let \(C\) be in this class. Suppose that \(T \to U\) is not trivial. By construction, the irreducible components of \(T_C\) are the \(T_{i,C}\). Since \(T_{i,C} \to C\) is flat and finite of the same degree as \(T_i \to U\), we see that the irreducible components of \(T_C\) all have degree > 1 over \(C\). Hence the torsor \(T_C\) is not trivial. \(\square\)

**Lemma B.3.** Let \(N\) be a finite \(\acute{e}tale\) group scheme over \(U\). Suppose that \(N(U) = 0\). Then for almost all linear subspaces \(L \subseteq \mathbb{P}^N\) of codimension \(\dim(U) - 1\), the intersection \(C := L \cap U\) is smooth and connected and \(N_C(C) = 0\).

**Proof.** Let \((N_i)\) be the set of irreducible components of \(N\), excluding the component of the identity. The condition that \(N(U) = 0\) is equivalent to the condition that for all \(i\), the morphism \(N_i \to U\) has degree > 1.

As before, by Bertini’s theorem, for almost all linear subspaces \(L \subseteq \mathbb{P}^N\) of codimension \(\dim(U) - 1\),
- the intersection \(C := L \cap U\) is smooth and connected;
and
- all the \(N_{i,C}\) are irreducible.

Let \(C\) be in this class. By construction, the irreducible components of \(N_C\) outside of the component of the identity are the \(N_{i,C}\). Since \(N_{i,C} \to C\) is flat and finite of the same degree as \(N_i \to U\), we see that the irreducible components of \(N_C\) outside of the component of the identity all have degree > 1 over \(C\). Hence \(N_C(C) = 0\). \(\square\)

**Lemma B.4.** Let \(G \subseteq B(U)\) be a finite group. For almost all linear subspaces \(L \subseteq \mathbb{P}^N\) of codimension \(\dim(U) - 1\), the intersection \(C := L \cap U\) is smooth and connected and the reduction map

\[ G \to B_C(C) \]

is injective.

**Proof.** Left to the reader. \(\square\)
Finally, we need an elementary but very insightful lemma, due to in essence to Néron. The following version is due to Hrushovski (see [21, lemma 1]):

**Lemma B.5** (Néron-Hrushovski). Let \( r : G \to H \) be a map of abelian groups. Let \( l \) be a prime number. Suppose that \( \text{Tor}_l(H) = 0 \) and such that the induced map \( G/lG \to H/lH \) is injective. Then \( \ker r \subseteq \bigcap_{j \geq 0} l^jG \).

**Proof.** Let \( g \in \ker r \). Suppose for contradiction that \( g \notin \bigcap_{j \geq 0} l^jG \). Let \( m \geq 0 \) be the smallest natural number such that \( g \not\in l^mG \). Then there is \( g' \in G \) such that \( l^{m-1}g' = g \) and thus \( r(g') \in \text{Tor}_l(H) \) so that from the assumptions we have \( r(g') = 0 \). Since the map \( G/lG \to H/lH \) is injective, there is \( g'' \in G \) such that \( lg'' = g' \). Hence \( g = l^m g'' \), a contradiction.

**Proof.** (of Proposition B.1). Let \( l \) be a prime number such that \( \text{Tor}_l(\mathcal{B}(U)) = 0 \) and such that \( l \) is not the characteristic of \( l_0 \). Note that for any closed subscheme \( C \) of \( U \), we have an injection \( \delta_C : \mathcal{B}(C)/l\mathcal{B}(C) \to H^1_{\text{et}}(C, \ker [l]_{\mathcal{B},C}) \) and this injection is functorial for restrictions to smaller closed subschemes \( C_1 \to C \). According to Lemmata B.3, B.2 and B.4, for almost all linear subspaces \( L \subseteq \mathbb{P}^N \) of codimension \( \text{dim}(U) - 1 \),

- the intersection \( C := L \cap U \) is smooth and connected;
- the restriction map \( H^1(U, \ker [l]_{\mathcal{B}}) \to H^1(C, \ker [l]_{\mathcal{B},C}) \) is injective on the image of \( \delta_U \);
- \( (\ker [l]_{\mathcal{B},C})(C) = 0 \);
- the restriction map \( \text{Tor}(\mathcal{B}(U)) \to \mathcal{B}(C) \) is injective.

Let \( C \) be in this class. By construction, the map \( \mathcal{B}(U)/l\mathcal{B}(U) \to \mathcal{B}(C)/l\mathcal{B}(C) \) is injective and \( \text{Tor}_l(\mathcal{B}(C)) = 0 \). Let \( F \) be a free subgroup of \( \mathcal{B}(U) \), which is a direct summand of \( \text{Tor}(\mathcal{B}(U)) \). We have \( F \cap (\bigcap_{j \geq 0} l^j\mathcal{B}(U)) = 0 \) since \( \mathcal{B}(U) \) is finitely generated and \( F \) is free. Applying Lemma B.5 to \( G = \mathcal{B}(U) \) and \( H = \mathcal{B}(C) \), we see that the restriction map \( F \to \mathcal{B}(C) \) is injective. Since the restriction map \( \text{Tor}(\mathcal{B}(U)) \to \mathcal{B}(C) \) is also injective, we thus see that the restriction map \( \mathcal{B}(U) \to \mathcal{B}(C) \) is injective. Finally, we have \( \text{Tr}_{n(C)/l_0}(\mathcal{B}_{n(C)}) = 0 \), for otherwise, we would have \( \text{Tor}_l(\mathcal{B}(C)) \neq 0 \).

**References**


43


