The Riemann-Roch theorem for arithmetic curves

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Abstract

We define arithmetic Chow and Grothendieck groups, in order to prove a Grothendieck-Riemann-Roch theorem for these. The last section contains a discussion about the absolute Riemann-Roch theorem and analogues of the dimensions of the 0's and first cohomology groups in the arithmetic case.

Dies ist die Diplomarbeit, die ich an der ETHZ geschrieben habe.

D. R.

Contents

0.1 Acknowledgements

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0.2 Prerequisites

We assume only elementary classical algebra. References to the Algebraische Zahlentheorie of Neukirch ([15]) are given, for the results of algebraic number theory that we shall need. For the very few propositions from classical algebraic K-theory and the theory of ideles, we give references to the Algebraic Number Theory of Lang (cf. [12]) and the Algebraic K-theory of Milnor (cf. [14]).

0.3 Standard notations

$\tau|\sigma$ The embedding τ extends the embedding σ
$N_{L|K}$ The norm function for the K-algebra L
$N(\vec{x})$ The Norm (product of coordinates) of $\vec{x} \in \mathbb{R}^n$
$Tr(\vec{x})$ The Trace (sum of the coordinates) of $\vec{x} \in \mathbb{R}^n$
[.] Equivalence classes
p.i.d. Principal Ideal Domain
f.g. Finitely generated
ν_p The exponential valuation corresponding to the absolute value p ([15, p. 125])
I_K The idele group of the number field K ([15, p. 373])
s.e.s. Short Exact Sequence
$rg(M)$ Rank of the module M (if it exists)
$M_{\mathcal{O}_K}$ M considered as an $\mathcal{O}_K$-module
$\mathcal{AB}$ The category of abelian groups
B^* The polar body of a body B in some Euclidean space
$\#S$ The cardinality of the set S

d_K The discriminant of the number field K ([15, p. 15])
$W^\perp$ The orthogonal complement of the subspace W
$\sim$ Asymptotically equivalent
$D_{L|K}$ The different ideal of the extension $L|K$ of number fields ([15, Ch. III, Par. 2])
$\delta_{L|K}$ The discriminant ideal of the extension $L|K$ ([15, Ch. III, Par. 2])
$\mathcal{N}(I)$ The Norm of the ideal I
$diam(S)$ The Diameter of the set $S \subseteq \mathbb{R}^n$

1 Introduction

In this text, we shall develop the theory needed to formulate a Riemann-Roch theorem of the Grothendieck type for arithmetic curves. Such a theorem is concerned with the naturality of a Chern-character mapping from Grothendieck groups to Chow groups.

The term **arithmetic** means that our theorems hold for some schemes over $\mathbb{Z}$. This implies that the classical techniques of algebraic geometry have to be supplemented, in order to provide analogues to the higher cohomology groups, since the classical non-zero ones vanish over $\mathbb{Z}$ in the case of curves. The original impulse for the development of such techniques came from Weil (cf. the introduction of [3]), who had noticed strong analogies between the field of rational functions of curves over an algebraically closed field and number fields (cf. [22]). This encouraged transfers of theorems from the classical algebraic geometry of varieties to this case. Arakelov later (in 1974) provided (cf. [1]) the construction necessary to implement this transfer fully. He initiated thereby a whole program, which is concerned with the generalisation of his ideas to larger classes of schemes over $\mathbb{Z}$ and with the transfer of classical theorems. The main steps were carried through by Faltings (cf. [4]) and Gillet-Soulé (cf. [19]).

In the case of curves, the basic idea of Arakelov consists in the following; theprime ideals of the ring of integers of a number field correspond to discrete valuations on the number field, so that the number field can be considered as a field of functions on the spectrum of its ring of integers (as for the field of rational functions on a curve); Arakelov extends the domain of these functions to all the valuations (i.e. also the archimedean ones) of the number

field. It is on this "completed spectrum" that one has to carry through the geometrical constructions.
This also means that the Minkowski geometry of numbers will enter into the play; for instance, the fibers of an ideal of the ring of integers of a number field will become complex lines on the archimedean valuations; specifying hermitian metrics on these lines, the Riemann problem becomes "How many elements of the ideal are within given bounded subsets of the direct sum of the complex lines ?". The image of the ideal in the direct sum of all the complex lines being a lattice, this question is therefore strongly linked with Minkowski's theorem (see [15, p.22, 4.4], for instance).

2 Foundations

In the sequel, we shall currently denote a number field by K and its ring of integers by $\mathcal{O}_K$. *Spec* $\mathcal{O}_K$ is written X and is our general model of an *arithmetic curve*. If the underlying field is not K, we shall add a subscript to X (e.g. X_L if the field is L). The set of all the field embeddings of K into $\mathbb{C}$ will be denoted by X_∞. The elements of X_∞ are denoted by small greek letters (τ, σ etc.). We also make the following definition

Definition 2.1 *$\mathcal{NF}$ is the category of all number fields. The category structure is given by the data*
- *$Mor(K, L) := \{i\}$, if $K \subseteq L$, $i : K \to L$ being the inclusion*
- *$Mor(K, L) := \emptyset$, if $K \not\subseteq L$*

If σ is in X_∞ and M is an $\mathcal{O}_K$-module, M_σ will be $M \otimes_{\mathcal{O}_{K,\sigma}} \mathbb{C}$ considered as a complex vector space. If $L : M \to N$ is an $\mathcal{O}_K$-module homorphism, then we define $L_\sigma := L \otimes_{\mathcal{O}_{K,\sigma}} Id. : M_\sigma \to N_\sigma$. The subscript σ of the tensor product means that the $\mathcal{O}_K$ action on $\mathbb{C}$ is $\sigma(a).z$ ($a \in \mathcal{O}_K, z \in \mathbb{C}$). Note that we may view $(.)_\sigma$ as an exact functor from the category of f.g. $\mathcal{O}_K$-modules to the category of finite-dimensional complex vector spaces. With these notations we now define our basic working environment.

Definition 2.2 *Let M be a finitely generated projective $\mathcal{O}_K$-module.*
- *(Conjugation) The conjugation is the isomorphism $M_\sigma \to M_{\bar{\sigma}}$, given by*

$\iota \otimes z \longrightarrow \iota \otimes \overline{z}$, *for each* $\sigma \in X_\infty$.

- *(Metrized modules) We denote by* $\widehat{M}$, M *endowed with a conjugation-invariant hermitian scalar product* h_{M_σ} *on* M_σ, *for all* $\sigma \in X_\infty$. $\widehat{M}$ *is then called a metrized* $\mathcal{O}_K$*-module. One writes* $\widehat{M}$ *also as* $(M, (h_{M_\sigma})_\sigma)$.
- *(Morphisms) If* $\widehat{N}$ *is another metrized* $\mathcal{O}_K$*-module, a morphism of metrized modules* $\widehat{L} : \widehat{M} \rightarrow \widehat{N}$ *is an* $\mathcal{O}_K$*-module homomorphism* $L : M \rightarrow N$.

With this setting, the metrized $\mathcal{O}_K$*-modules clearly form a category, that will be denoted by* $\widehat{\mathcal{PF}}(X)$.

M may be viewed as a locally free sheaf on X. In view of the conjugation invariance requirement, we shall often have to consider X_∞ modulo conjugation, i.e. with conjugate embeddings identified. This new set will then be denoted by X'_∞. All the functions defined on X'_∞ will be written as functions on X_∞, but will implicitly be submitted to the supplementary requirement of conjugation invariance. The natural notion of exactness on $\widehat{\mathcal{PF}}(X)$ is the following:

Definition 2.3 *We say that an exact sequence of metrized* $\mathcal{O}_K$*-modules*

$$\widehat{\mathcal{M}} : 0 \rightarrow \widehat{M}' \stackrel{\widehat{d'}}{\rightarrow} \widehat{M} \stackrel{\widehat{d''}}{\rightarrow} \widehat{M}'' \rightarrow 0$$

(for some $\widehat{M}', \widehat{M}'' \in \widehat{\mathcal{PF}}(X)$*) splits orthogonally if the coresponding exact sequence of inner product spaces*

$$\mathcal{M} : 0 \rightarrow M'_\sigma \stackrel{d'_\sigma}{\rightarrow} M_\sigma \stackrel{d''_\sigma}{\rightarrow} M''_\sigma \rightarrow 0$$

splits orthogonally, at each $\sigma \in X_\infty$.

That is, we have $d'^{\,*}_\sigma(h_{M_\sigma}) = h_{M'_\sigma}$ and $d''_{\sigma *}(h_{M_\sigma}) = h_{M''_\sigma}$. In words: M'_σ embeds isometrically in M_σ, and M''_σ carries the quotient metric of M_σ and M'_σ.

Note that $\widehat{\mathcal{PF}}(X)$ is **not** an abelian category. In fact, there seems to be no way to define morphisms which make it into one, because of the "rigidity" of isometries. Therefore, we shall define another stronger isomorphism property.

Definition 2.4 *Let* $\widehat{M}, \widehat{N}$ *be metrized* $\mathcal{O}_K$*-modules. We say that* $\widehat{L} : \widehat{M} \rightarrow \widehat{N}$ *is an isometric isomorphism, if* $L : M \rightarrow N$ *is an isomorphism and* $L_\sigma : M_\sigma \rightarrow N_\sigma$ *is an isometry (for all* $\sigma \in X_\infty$*). We write* $\widehat{M} \widehat{\simeq} \widehat{N}$ *in that case.*

The symbol "$\widehat{\ }$" will in the sequel appear in all instances of an arithmetic analogon of a classical object.[1] Borrowing a term from sheaf theory, we shall call metrized modules of rank 1 **invertible** metrized modules.

Example. Let I be a fractional ideal of $\mathcal{O}_K$. Then we can specify a metric h_{I_σ} for each $\sigma \in X'_\infty$, by requiring the element $\iota_\sigma \otimes z_\sigma \in I_\sigma$ with $\sigma(\iota_\sigma).z_\sigma = 1$ to be an orthonormal basis for the metric. The mapping $\iota_\sigma \otimes z_\sigma \mapsto \sigma(\iota_\sigma).z_\sigma$ is clearly an $\mathcal{O}_K$-isomorphism: it is well-defined, since it is bilinear; furthermore, it is injective, since $\mathbb{C}$ is a domain, and surjective since z_σ can be chosen freely. This metric is called the **trivial metric** of I. The just described basis is called the **standard basis** of I at σ. Its only element is refered to as $I_\sigma[1]$. We shall often identify the metrics of metrized fractional ideals with their coefficients in that basis.

3 Operations in $\widehat{\mathcal{PF}}(X)$

We can extend the usual module operations to $\widehat{\mathcal{PF}}(X)$:
Let $\widehat{M}, \widehat{N}$ be metrized $\mathcal{O}_K$-modules and $\sigma \in X'_\infty$.
Taking into account the canonical isomorphisms

$$\bullet\ (Det(M))_\sigma \simeq Det_{\mathbb{C}}(M_\sigma) \tag{1}$$

$$\bullet\ (M \otimes N)_\sigma \simeq M_\sigma \otimes N_\sigma \tag{2}$$

$$\bullet\ (M \oplus N)_\sigma \simeq M_\sigma \oplus N_\sigma \tag{3}$$

$$\bullet\ (M_\sigma)^\vee \simeq (M^\vee)_\sigma$$

we define the operations

Definition 3.1

$$Det_{\mathcal{O}_K}(\widehat{M}) := (Det(M), (Det(h_{M_\sigma}))_\sigma)$$

$$\widehat{M} \otimes_{\mathcal{O}_K} \widehat{N} := (M \otimes_{\mathcal{O}_K} N, (h_{M_\sigma} \otimes h_{N_\sigma})_\sigma)$$

[1]We finally decided not to use the more precise Gillet-Soulé notation, which makes a distinction between "endowed with a metric" ("-") and arithmetic, by fear to overburden the reader with symbols (since we already try to be quite exhaustive).

$$\widehat{M} \oplus_{\mathcal{O}_K} \widehat{N} := (M \oplus_{\mathcal{O}_K} N, (h_{M_\sigma} \oplus h_{N_\sigma})_\sigma)$$

$$\widehat{M}^{\vee} := (M^\vee, (h_\sigma^\vee)_\sigma)$$

where for two metrics h, h' on given finite dimensional vector spaces we define

$$Det(h)(x_1 \wedge \ldots \wedge x_d, y_1 \wedge \ldots \wedge y_d) := Det((h(x_i, y_j))_{i,j})$$

$$(h \otimes h')(x \otimes y, u \otimes v) := h(x,u).h(y,v)$$

$$(h \oplus h')(x \oplus y, u \oplus v) := h(x,u) + h(y,v)$$

$$h^\vee(x,y) = \overline{h(x^\vee, y^\vee)}$$

for some vectors $x, y, x_1 \ldots x_d, y_1 \ldots y_d, u, v$.

(the dual of an element is the image of that element under the mapping provided by the Riesz representation theorem)
We have now an arithmetic version of the structure theorem for finitely generated projective modules over Dedekind rings ([9, II, p.626, Th. 10.14]). Recall that this theorem says that any such module is isomorphic to a finite direct sum of fractional ideals and that this representation is unique up to reordering of the factors and multiplication of each factor with a principal ideal.

Proposition 3.2 (An arithmetic Splitting Principle) *For any metrized $\mathcal{O}_K$-module $\widehat{M}$ there exists an invertible module $\widehat{I}$ and a metrized module $\widehat{N}$ such that the following sequence splits orthogonally:*

$$0 \to \widehat{N} \to \widehat{M} \to \widehat{I} \to 0$$

Proof: We know that we can find N, I so that the above sequence is algebraically exact. Endow $\widehat{N}$ with the pull-back metric from $\widehat{M}$ and $\widehat{I}$ with the push-forward metric. ♣

Here is another arithmetic version of a classical theorem (cf. [11, p. 591, Pr. 9.2]).

Theorem 3.3 *Let*
$$0 \to \widehat{M}' \to \widehat{M} \to \widehat{M}'' \to 0$$
be an orthogonally splitting s.e.s. Then there is a canonical isometric isomorphism
$$Det(\widehat{M}) \tilde{\simeq} Det(\widehat{M}') \otimes Det(\widehat{M}'')$$

Proof: It is classical that for *any* splitting $s : M'' \to M$, the map $(m'_1 \wedge \ldots \wedge m'_{d'}) \otimes (m''_1 \wedge \ldots \wedge m''_{d''}) \mapsto m'_1 \wedge \ldots \wedge m'_{d'} \wedge s(m''_1) \wedge \ldots \wedge s(m''_{d''})$ is an isomorphism (d', d, d'' are the dimensions). The induced isomorphisms at $\sigma \in X_\infty$ won't change, if we replace s_σ by the orthogonal splitting $s^\perp$. Now let $v'_1 \ldots v'_{d'}, v''_1 \ldots v''_{d''}$ be orthonormal bases for M'_σ, M''_σ; clearly $(v'_1 \wedge \ldots \wedge v'_{d''}) \otimes (v''_1 \wedge \ldots v''_{d''})$ is an orthonormal basis of $M'_\sigma \otimes M''_\sigma$ and we compute

$$h_{Det(M_\sigma)}(v'_1 \wedge \ldots v'_{d'} \wedge s^\perp(v''_1) \wedge \ldots \wedge s^\perp(v''_{d''})) =$$

$$Det \begin{pmatrix} (h_{M'_\sigma}(v'_i, v'_j))_{i,j} & 0 \\ 0 & (h_{M''_\sigma}(v''_k, v''_l))_{k,l} \end{pmatrix} = 1$$

which ends the proof. ♣

After these preliminaries, we now come to the objects the arithmetic Grothendieck-Riemann-Roch will be concerned with.

4 Arithmetic Chow groups

4.1 Definitions

For the spectra of Dedekind rings, the algebraic 0-cycles correspond to the fractional ideals and the principal cycles are the principal ones. The first Chow group is then the idealclass group. The arithmetic analogs to these objects are the following:

Definition 4.1 *The group of (arithmetic) 0-cycles is defined as*

$$\widehat{Z}^1(X) := Z^1(X) \oplus \mathbb{R}^{X'_\infty}$$

Z^1 denoting the multiplicative group of fractional ideals of $\mathcal{O}_K$.
The group $\widehat{R}^1(X)$ of principal cycles form the image of the homomorphism

$$div : K^* \to \widehat{Z}^1(X)$$

defined by

$$div(a) := (a) \oplus (-2.\log(|\sigma(a)|))_\sigma$$

($(a) := \mathcal{O}_K.a$)

The homomorphy of div is clear from the equations $div(a.b) \stackrel{def.}{=} (a.b) \oplus (-2.\log(|\sigma(a.b)|))_\sigma \stackrel{def.}{=} ((a) \oplus (-2.\log(|\sigma(a)|))_\sigma) + ((b) \oplus (-2.\log(|\sigma(b)|))_\sigma) = div(a) + div(b)$.
With this setting

Definition 4.2 *The first Chow group is defined as*

$$\widehat{CH}^1(X) := \widehat{Z}^1(X)/\widehat{R}^1(X)$$

With these prescriptions the group

$$\widehat{CH}(X) := \mathbb{Z} \oplus \widehat{CH}^1(X)$$

($\mathbb{Z}$ corresponds to the free group generated by the unique 1-cycle) can be endowed with a ring structure $\odot$, described by

$$(r_1 \oplus cl_1) \odot (r_2 \oplus cl_2) := (r_1.r_2) \oplus (r_1.cl_2 + r_2.cl_1)$$

where $r_1, r_2 \in \mathbb{Z}$, and $cl_1, cl_2 \in \widehat{CH}^1$. This ring structure may be interpreted as an arithmetic intersection pairing. This accounts for the fact that the product vanishes on two elements of $\widehat{CH}^1(X)$, since the higher arithmetic Chow groups vanish.

4.2 $\widehat{Pic}$ and the first Chern class

In this paragraph, we show that the isomorphy between the Picard group and the first Chow group, valid on any compact complex manifold, also holds in our arithmetic context.

Definition 4.3 *The (arithmetic) Picard group $\widehat{Pic}(X)$ of $\mathcal{O}_K$ is the set of isometric isomorphism classes of invertible metrized modules. A group structure on $\widehat{Pic}(X)$ is given by the tensor product $\otimes$. The neutral element is $\mathcal{O}_K$ endowed with the trivial metric.*

The inverses are easily seen to be the metrized duals. The fact that $(\mathcal{O}_K, (1)_\sigma)$ is neutral may be checked directly, but is also an immediate consequence of the next theorem:

Theorem 4.4 (Structure Theorem for $\widehat{Pic}$) *There is an isomorphism*

$$c_1 : \widehat{Pic}(X) \to \widehat{CH}^1(X)$$

$$c_1([\widehat{M}]_{\widehat{Pic}}) := [I(M)^{-1} \oplus (-\log(|h_\sigma|))_{\sigma \in X'_\infty}]_{\widehat{CH}^1}$$

where $I(M)$ is a fractional ideal isomorphic to M. The metric h_σ of $\widehat{I(M)}$ is the one induced by the isomorphism.

$I(M)$ exists by the structure theorem. c_1 is called the *first Chern class.*

Proof: Part of the proof is the well-definedness of c_1. The structure theorem tells us that the only variation allowed on $I(M)$ is multiplication by $a \in K^*$. So if we inverse on both sides, we are reduced to prove the equation

$$[I(M) \oplus (\log(|h_\sigma|))_{\sigma \in X'_\infty}]_{\widehat{CH}^1} =$$

$$[a.I(M) \oplus (\log(|h_{(a.I(M))_\sigma}|))_{\sigma \in X'_\infty}]_{\widehat{CH}^1}$$

We compute $h_{(a.I(M))_\sigma} = |1/\sigma(a)^2|.h_{I(M)_\sigma}$. Indeed, if $\iota \otimes z$ ($\iota \in I(M), z \in \mathbb{C}$) is the standard basis of $I(M)_\sigma$, then $(a \otimes Identity)(\iota \otimes z) = (a.\iota) \otimes z$, so that the standard basis of $(a.I(M))_\sigma$ is $\frac{1}{\sigma(a)}((a.\iota) \otimes z)$, by the definition of the action. From this it should be clear that the difference between the two members of the last equation is $[div(a)]_{\widehat{CH}^1} = 0$.
The homomorphy of c_1 is evident. c_1 is clearly injective and surjective, so it is an isomorphism. ♣

4.3 Functoriality

We now turn to the functoriality of Chow groups.

Definition 4.5 *Let $i : K \subseteq L$ be an inclusion of number fields. The pull-back function is defined by*

$$i^*_{\widehat{CH}} : \widehat{CH}(X_K) \to \widehat{CH}(X_L)$$

$$i^*_{\widehat{CH}}(r \oplus [I \oplus (g_\sigma)_\sigma]_{\widehat{CH}^1}) := r.[L:K] \oplus [\mathcal{O}_L.I \oplus i^*(g)]_{\widehat{CH}^1}$$

where $i^*(g)_\tau := g_{\tau|K}$ for $\tau \in X_{L}{}'_\infty$.

We have of course to show that $i^*_{\widehat{CH}}$ is well-defined.

• **Proof of the well-definedness of $i^*_{\widehat{CH}}$.**

The above definition is clearly additive in all its arguments. Therefore, considering for a moment that $i^*_{\widehat{CH}}$ is defined from $\widehat{Z}^1(X_K)$ to $\widehat{Z}^1(X_L)$ instead, we only have to settle that $i^*_{\widehat{CH}}(\widehat{R}^1(X_K)) \subseteq \widehat{R}^1(X_L)$ (there is no ambiguity on $\mathbb{Z}$). Now the following straightforward computation

$$i^*_{\widehat{CH}}((a)_{\mathcal{O}_K} \oplus (-2.\log(|\sigma(a)|))_{\sigma \in X'_\infty}) = ((a)_{\mathcal{O}_L} \oplus (-2.\log(|\sigma(a)|))_{\tau|\sigma, \tau \in X_{L}{}'_\infty, \sigma \in X_{K}{}'_\infty}) =$$

$$((a)_{\mathcal{O}_L} \oplus (-2.\log(|\tau(a)|))_{\tau \in X_{L}{}'_\infty}) = div_{\mathcal{O}_L}(a)$$

($a \in K^*$) shows that this holds. Now the two expected results.

Theorem 4.6 *$\widehat{CH}$ becomes a covariant functor $\mathcal{NF} \longrightarrow \mathcal{AB}$, if an inclusion $i : K \longrightarrow L$ of number fields is mapped on the homomorphism $i^*_{\widehat{CH}} : \widehat{CH}(X_K) \to \widehat{CH}(X_L)$.*

Proof: The fact that $i^*_{\widehat{CH}}$ is a group homomorphism is clear from the above mentionned additivity. The functoriality is clear, if one remembers that $[L:K].[K:J] = [L:J]$ for three number fields $L \subseteq K \subseteq J$. ♣

We can also define a push-forward for Chow groups.

Definition 4.7 *The push-forward function is defined by*

$$i_{*\widehat{CH}} : \widehat{CH}(X_L) \to \widehat{CH}(X_K)$$

$$i_{*\widehat{CH}}(r \oplus [I \oplus (g_\tau)_{\tau \in X_{L}{}'_\infty}]_{\widehat{CH}^1}) := r \oplus [(N_{L|K}(I)) \oplus i_*(g)]_{\widehat{CH}^1}$$

where $i_*(g)(\sigma) = \sum_{\tau|\sigma} g(\tau)$ ($\sigma \in X'_\infty$).

Again, we have to check well-definedness.

• **Proof of the well-definedness of $i_{*\widehat{CH}}$:**

Consider for a moment that $i_{*\widehat{CH}}$ is defined from $\widehat{Z}^1(X_L)$ to $\widehat{Z}^1(X_K)$. $i_{*\widehat{CH}}$ is clearly additive, therefore, as before, we have to check that $i_{*\widehat{CH}}(\widehat{R}^1(X_L)) \subseteq \widehat{R}^1(X_K)$. The following computation

$$i_{*\widehat{CH}}((a) \oplus (-2.\log(|\tau(a)|))_{\tau \in X_{L\infty}'}) \stackrel{def.}{=}$$
$$(N_{L|K}(a)) \oplus (\sum_{\tau|\sigma} -2.\log(|\tau(a)|))_{\sigma \in X_\infty'} =$$
$$(N_{L|K}(a)) \oplus (-2.\log(|\prod_{\tau|\sigma} \tau(a)|))_\sigma =$$
$$(N_{L|K}(a)) \oplus (-2.\log(|\sigma(N_{L|K}(a))|))_\sigma$$

$(a \in L^*)$

settles the matter. For the last equation, remember that the norm of a appears as the constant coefficient of its characteristic polynomial over $\mathbb{Q}$ (cf. [9, p. 611, Pr. 10.8]).

Again we have

Theorem 4.8 *$\widehat{CH}$ becomes a contravariant functor $\mathcal{NF} \to \mathcal{AB}$, if an inclusion $i : K \to L$ of number fields is mapped on the homomorphism $i_{*\widehat{CH}}$: $\widehat{CH}(X_L) \to \widehat{CH}(X_K)$.*

Proof: Clear, if one remembers that $N_{J|L} \circ N_{L|K} = N_{J|K}$ for three number fields $L \subseteq K \subseteq J$ (cf. [15, p. 10, Kor. 2.7]). ♣

One can even show that the pull-back is a ring homomorphism, which shows that $\odot$ obeys the right axioms (cf. [7, Appendix I]).

4.4 The degree mapping

It is possible in classical algebraic geometry to calculate the degree of an invertible sheaf on a suitable scheme, **via** a morphism to a smooth projective variety (cf. [3, p. 9]). In the same fashion, we shall define the degree of an element of $\widehat{CH}^1(X)$ as the degree of its push-forward to $\widehat{CH}^1(X_{\mathbb{Q}})$. Therefore, we first need to compute $\widehat{CH}^1(X_{\mathbb{Q}})$.

Lemma 4.9 $\widehat{CH}^1(X_{\mathbb{Q}}) \overset{T}{\simeq} \mathbb{R}$
where $T : [(q) \oplus (g)]_{\widehat{CH}^1} \mapsto \frac{1}{2}(g + 2.\log(|q|))$
$(q \in \mathbb{Q})$.

(the $\frac{1}{2}$ factor is irrelevant in the computation; it has only been introduced to allow the elements of $\mathbb{R}$ to be interpreted as the norms - and not as the metrics)

Proof: First, we have to prove that T is well-defined. It is clearly well-defined and a homomorphism on $\widehat{Z}^1(X_{\mathbb{Q}})$, since all fractional ideals are principal and their generators are only ambiguous up to sign. Next, its kernal in $\widehat{Z}^1(X_{\mathbb{Q}})$ is by definition exactly $\widehat{R}^1(X_{\mathbb{Q}})$. Since it is clearly surjective, we are done. ♣

Now we can proceed with the

Definition 4.10 *The degree homomorphism is*

$$deg : \widehat{CH}^1(X) \to \mathbb{R}$$
$$deg(x) = T(i_{*\widehat{CH}}(x))$$

where $i : \mathbb{Q} \to K$.

From now on, we shall even extend the domain of definition of *deg* to any metrized module, by the formula $deg \circ c_1 \circ Det(.)$. Looking back on the definition of $i_{*\widehat{CH}}$, we see that an explicit formula for $deg((I, (h_\sigma)_\sigma))$ is given by

$$-\log(\mathcal{N}(I)) - \frac{1}{2} \sum_{\sigma \in X_\infty} \log(h_\sigma)$$

($\mathcal{N}(I) = |q|$ if $(q) = N_{L|\mathbb{Q}}(I)$; see [15, p. 37])
Note that we sum over **all** of X_∞. *deg* will appear again in the absolute Riemann-Roch theorem.

5 Arithmetic Grothendieck groups

The first section will describe a 0-dimensional specialisation of Bott-Chern forms.

5.1 0-dimensional Bott-Chern forms

We define

Definition 5.1 *Let*

$$\mathcal{V} : 0 \to V' \to V \to V'' \to 0$$

be an exact sequence of finite dimensional Euclidean spaces, with metrics h', h, h''. *The Bott-Chern form of* $\mathcal{V}$ *is defined as*

$$\widetilde{ch}(\mathcal{V}) := -\log(|det(h)/det(h' \oplus h'')|)$$

where the direct sum is taken under the natural orthogonal splitting.

By convention, the determinant of the metric of a 0-dimensional space will be 1.
The definition of $\widetilde{ch}$ is independant of the basis of V chosen to compute the determinants, since any basis change would induce a multiplication with the same factor in the numerator and in the denominator. It is clear from the definition that $\widetilde{ch}(\mathcal{V}) = 0$ if $\mathcal{V}$ splits orthogonally. On a compact Riemann surface, the Bott-Chern form of an exact sequence of vector bundles is the first Chern form of the middle term minus the first Chern forms of the first and final term. We have an analogous formula here. Let

$$\widehat{\mathcal{M}} : 0 \to \widehat{M}' \to \widehat{M} \to \widehat{M}'' \to 0$$

be an algebraically exact sequence of metrized modules and

$$\mathcal{M}_\sigma : 0 \to M'_\sigma \to M_\sigma \to M''_\sigma \to 0$$

the corresponding sequence at $\sigma \in X'_\infty$. We compute

$$c_1(Det(\widehat{M})) - c_1(Det(\widehat{M'})) + c_1(Det(\widehat{M}'')) =$$

$$c_1(Det(\widehat{M}) \otimes (Det(\widehat{M'} \oplus \widehat{M}''))^{\check{}}) \overset{def.}{=}$$

$$[\mathcal{O}_K \oplus (-\log(|det(h_{M_\sigma})/det(h_{M'_\sigma} \oplus h_{M''_\sigma})|))_\sigma]_{\widehat{CH}^1}$$

The Bott-Chern form has the following symmetry:

Proposition 5.2 *Let*

$$
\begin{array}{ccccccccc}
 & & \mathcal{V}' & & \mathcal{V} & & \mathcal{V}'' & & \\
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{V}_1 : 0 & \rightarrow & V_1' & \rightarrow & V_1 & \rightarrow & V_1'' & \rightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{V}_2 : 0 & \rightarrow & V_2' & \rightarrow & V_2 & \rightarrow & V_2'' & \rightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{V}_3 : 0 & \rightarrow & V_3' & \rightarrow & V_3 & \rightarrow & V_3'' & \rightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

be an s.e.s. of complexes of finite dimensional Euclidean spaces. Then

$$\widetilde{ch}(\mathcal{V}') - \widetilde{ch}(\mathcal{V}) + \widetilde{ch}(\mathcal{V}'') = \widetilde{ch}(\mathcal{V}_1) - \widetilde{ch}(\mathcal{V}_2) + \widetilde{ch}(\mathcal{V}_3)$$

Proof: In view of the fact that the logarithm swaps products and sums, this statement is in fact equivalent to this other one:

$$Det(h_{V_1})/(Det(h_{V'_1}).Det(h_{V''_1})).(Det(h_{V_2})/(Det(h_{V'_2}).$$

$$Det(h_{V''_2})))^{-1}.Det(h_{V_3})/(Det(h_{V'_3}).Det(h_{V''_3})) =$$

$$Det(h_{V'_1})/(Det(h_{V'_2}).Det(h_{V'_3})).(Det(h_{V_1})/(Det(h_{V_2}).$$

$$Det(h_{V_3})))^{-1}.Det(h_{V''_3})/(Det(h_{V''_3}).Det(h_{V''_3}))$$

(choosing arbitrary bases for the Euclidean spaces to compute the determinants)
But this is obvious, if one reorders the factors. ♣

As a corollary, the Bott-Chern forms are additive on short exact sequences of complexes of euclidean spaces.

5.2 Definitions

Next, we have to define Grothendieck groups, i.e. groups that are universal with respect to short exact sequences. Here two options are open. The first one consists in simply taking the quotient of all the orthogonally splitting s.e.s., as one does classically; the second one appears, when one tries to conserve as much as possible of the old category of f.g. projective modules; in this case, one takes the quotient of all the algebraic s.e.s and a Bott-Chern form. The two approaches turn out to yield isomorphic groups. We define

Definition 5.3 *The categorical Grothendieck group $\widehat{K}_C(X)$ is the group described by the following data:*
- **Group:** *Free $\mathbb{Z}$-module generated by all the objects of $\widehat{\mathcal{PF}}(X)$*
- **Relations:** $[\widehat{M}]_{\widehat{K}_C} = [\widehat{M}']_{\widehat{K}_C} + [\widehat{M}'']_{\widehat{K}_C}$ *for any orthogonally splitting s.e.s.*

$$0 \to \widehat{M}' \to \widehat{M} \to \widehat{M}'' \to 0$$

More explicitly, this notation means that $\widehat{K}_C(X)$ is the quotient of the mentioned free group by the subgroup generated by the relations. $\widehat{K}_C(X)$ also has a natural unitary ring structure under the tensor product $\otimes$ of representants, $[(\mathcal{O}_K, (1)_\sigma)]_{\widehat{K}_C}$ playing the role of the unit. It is well-defined because $\otimes$ is biadditive (universality of the group). We shall denote that product simply by .. Nevertheless we shall always refer to $\widehat{K}_C(X)$ as a group, unless we explicitly state the contrary. Since Det and rg are additive (see 3.3 for the former), they factor through $\widehat{K}_C(X)$ to become homomorphisms on this group, by universality. This universality removes any danger in denoting Det and rg as defined on $\widehat{K}_C(X)$ by the same symbols. We shall apply this convention from now on and **extend it without notice to any additive map.** Also, since the image of Det consists only of invertible modules, we may identify $Det(\widehat{M})$ and $[Det(\widehat{M})]_{\widehat{Pic}} \in \widehat{Pic}(X)$. Conversely, there is a mapping $\widehat{Pic}(X) \to \widehat{K}_C(X)$ given by $[\widehat{M}]_{\widehat{Pic}} \mapsto [\widehat{M}]_{\widehat{K}_C}$. This mapping is well-defined, since isometrically isomorphic metrized modules $\widehat{M}, \widehat{M}'$ have identical images in $\widehat{K}_C(X)$, as the orthogonally splitting s.e.s $0 \to \widehat{M} \stackrel{Is.Isom.}{\to} \widehat{M}' \to 0$ should make clear. Since Det is clearly an inverse to this last mapping, this means that $\widehat{Pic}(X)$ **may be viewed as embedded in** $\widehat{K}_C(X)$. Next

Definition 5.4 *The Gillet-Soulé Grothendieck group $\widehat{K}_{GS}(X)$ is the group described by the following data:*
• **Group:** $\{$*Free $\mathbb{Z}$-module generated by all the objects of* $\widehat{\mathcal{PF}}(X)\}\bigoplus\mathbb{R}^{X'_\infty}$
• **Relations:** $[\widehat{M}]_{\widehat{K}_{GS}}+(\widetilde{ch}(\widehat{\mathcal{M}}_\sigma))_\sigma=[\widehat{M}']_{\widehat{K}_{GS}}+[\widehat{M}'']_{\widehat{K}_{GS}}$
where

$$\widehat{\mathcal{M}}:0\to\widehat{M}'\to\widehat{M}\to\widehat{M}''\to 0$$

is an algebraically exact sequence for some $\widehat{M}',\widehat{M},\widehat{M}''\in\widehat{\mathcal{PF}}(X)$.

For simplicity of notation, we shall from now on often write $\widetilde{ch}(\widehat{\mathcal{M}})$ for $(\widetilde{ch}(\widehat{\mathcal{M}}_\sigma))_\sigma$.
Both of these groups can be defined on the level of all f.g. $\mathcal{O}_K$-modules without change (i.e. one obtains isomorphic groups under the natural inclusion mapping). This not entirely trivial topic is treated in the Appendix A. We shall sometimes use these seemingly larger groups in the sequel. They will be refered to as $\widehat{K}_C(X)'$ (for the categorical one) and $\widehat{K}_{GS}(X)'$ (for the "Gillet-Soulé" one).

5.3 Comparison of the groups

There are two natural questions arising now. What is the relation of our Grothendieck groups to the classical one ? What is the relation of our Grothendieck groups between them ? We shall bring these questions together in one diagram. Before coming to the diagram, we prove non-canonical structure theorems for the two groups. We need a lemma

Lemma 5.5 *Let $\widehat{I},\widehat{J}$ be metrized fractional ideals. Then $[I\oplus J,(h_{I_\sigma}\oplus h_{J_\sigma})_\sigma]_{\widehat{K}_C}=[I\oplus J,(h_{J_\sigma}\oplus h_{I_\sigma})_\sigma]_{\widehat{K}_C}$.*

Proof: As usual, we work at σ. Consider the s.e.s.

$$0\to(I_\sigma,\alpha_\sigma)\to(I_\sigma\oplus J_\sigma,H_\sigma)\to(J_\sigma,\beta_\sigma-\frac{\gamma_\sigma\overline{\gamma_\sigma}}{\alpha_\sigma})\to 0$$

(with the obvious maps, where $\alpha_\sigma,\beta_\sigma\in\mathbb{R}^*_+$)
where H_σ is the matrix $\begin{pmatrix}\alpha_\sigma & \gamma_\sigma\\ \overline{\gamma_\sigma} & \beta_\sigma\end{pmatrix}$ defining a metric of $I_\sigma\oplus J_\sigma$ in the standard basis. We contend that this s.e.s. splits orthogonally. Indeed, the

orthogonal complement of I_σ in $I_\sigma \oplus J_\sigma$ is described by the equation in a, b ${}^t(x,0)H_\sigma(a,b) = \alpha_\sigma.x.a + \gamma_\sigma.x.b = 0$ for all $x \in I_\sigma$, i.e. it consists of the elements $(-\gamma_\sigma/\alpha_\sigma).b, b)$ for all $b \in J_\sigma$. Then the simple computation

$$ {}^t(-\gamma_\sigma/\alpha_\sigma.b, b)H_\sigma(-\gamma_\sigma/\alpha_\sigma.b', b') = (\beta_\sigma - \frac{\gamma_\sigma.\overline{\gamma_\sigma}}{\alpha_\sigma}).b.b' $$

$(b, b' \in I_\sigma)$
demonstrates our contention.
Going through exactly the same reasoning on the first term, we see that the following s.e.s. also splits:

$$ 0 \to (J_\sigma, \beta_\sigma) \to (I_\sigma \oplus J_\sigma, H_\sigma) \to (I_\sigma, \alpha_\sigma - \frac{\gamma_\sigma\overline{\gamma_\sigma}}{\beta_\sigma}) \to 0 $$

By the definition of the $\widehat{K_C}(X)$, this means that $[(I \oplus J, (H_\sigma)_\sigma)]_{\widehat{K}_C} = [(I \oplus J, (\alpha_\sigma)_\sigma \oplus (\beta_\sigma - \frac{\gamma_\sigma\overline{\gamma_\sigma}}{\alpha_\sigma})_\sigma]_{\widehat{K}_C} = [(I \oplus J, (\alpha_\sigma - \frac{\gamma_\sigma\overline{\gamma_\sigma}}{\beta_\sigma})_\sigma \oplus (\beta_\sigma)_\sigma)]_{\widehat{K}_C}$. Now choose $\alpha_\sigma = h_{I_\sigma}$, $\beta_\sigma = \alpha_\sigma$ and $\gamma_\sigma.\overline{\gamma}_\sigma = \alpha_\sigma^2 - h_{J_\sigma}.\alpha_\sigma$ to read off our claim. ♣

Proposition 5.6 *Let Γ be the lattice $-2.\log(|\mathcal{O}_K{}^*|) \subseteq \mathbb{R}^{X'_\infty}$)[2]. Let $K_C(X)$ be the classical Grothendieck group of X. The following sequence is exact*

$$ 0 \to \mathbb{R}^{X'_\infty}/\Gamma \xrightarrow{kf} \widehat{K_C}(X) \xrightarrow{f} K_C(X) \to 0 $$

where f is the natural forgetful mapping (i.e. $f([\widehat{M}]_{\widehat{K}_C}) := [M]_{K_C}$) and kf is defined by

$$ kf([g]_{\mathbb{R}^{X'_\infty}/\Gamma}) := [(\mathcal{O}_K, (\exp(g_\sigma))_\sigma)]_{\widehat{K}_C} - [(\mathcal{O}_K, (1)_\sigma)]_{\widehat{K}_C} \qquad (4) $$

Put $I = J = \mathcal{O}_K$, $\alpha_\sigma = \exp(g_\sigma).\exp(g'_\sigma)$, $\beta_\sigma = \exp(g'_\sigma)$, $(\beta_\sigma - 1).\alpha_\sigma = |\gamma_\sigma|^2$, at the end of the last proof, to see that kf is a homomorphism.

Proof: To demonstrate exactness in the middle, we have to compute the kernel of f. Recall that $[M]_{K_C} = [N]_{K_C}$ iff $M \simeq N$, for f.g. projective $\mathcal{O}_K$-modules N, M (see [14, p.4]). Now suppose that

$$ f([n_1\widehat{M_1} + n_2\widehat{M_2}\dots n_k\widehat{M_k}]_{\widehat{K}_C}) = [n_1M_1 + n_2M_2\dots n_kM_k]_{K_C} = 0 $$

[2] $\mathcal{O}_K{}^*$ is embedded by σ at the component σ.

$(n_1 \ldots \in \mathbb{Z}, \widehat{M_1} \ldots \in \widehat{\mathcal{PF}}(X))$ in $K_C(X)$. Choose fractional ideals $I_1^{(1)}, \ldots I_1^{(rg(\widehat{M_1}))}$, $I_2^{(1)}, \ldots I_k^{(rg(\widehat{M_k}))}$ such that

$$\bigoplus_{i=1}^{rg(\widehat{M_j})} I_j^{(i)} = M_j$$

$(1 \leq j \leq k)$
Then we see that

$$[n_1 M_1 + n_2 M_2 \ldots n_k M_k]_{K_C} = [\bigoplus_{j=1}^{k} \bigoplus_{n_j : n_j \geq 0} \bigoplus_{i=1}^{rg(\widehat{M_j})} I_j^{(i)}]_{K_C} - [\bigoplus_{j=1}^{k} \bigoplus_{-n_j : n_j \leq 0} \bigoplus_{i=1}^{rg(\widehat{M_j})} I_j^{(i)}]_{K_C}$$

which shows that all the ideals $I_j^{(i)}$ with $n_j \geq 0$ must be pairwise isomorphic to the ideals $I_j^{(i)}$ with $n_j \leq 0$. Therefore the kernel of f is generated by expressions of the form $[(I, (h_{I_\sigma})_\sigma)]_{\widehat{K_C}} - [(I, (h'_{I_\sigma})_\sigma)]_{\widehat{K_C}} = ([(I, (h_{I_\sigma})_\sigma)]_{\widehat{K_C}} - [(I, (1)_\sigma]_{\widehat{K_C}}) - ([(I, (h'_{I_\sigma})_\sigma)]_{\widehat{K_C}} - [(I, (1)_\sigma]_{\widehat{K_C}})$. Comparing the definition of kf with the bracketed expressions, we see that exactness in the middle amounts to the statement

$$[(\mathcal{O}_K \oplus I, (h_{I_\sigma} \oplus 1)_\sigma)]_{\widehat{K_C}} = [(\mathcal{O}_K \oplus I, (1 \oplus h_{I_\sigma})_\sigma)]_{\widehat{K_C}}$$

The lemma above just tells us that this isomorphy holds, i.e. that we can swap h_{I_σ} and 1.
The surjectivity on the right is obvious. The injectivity on the left amounts to the statement $(\mathcal{O}_K, (exp((g_\sigma)_\sigma) \widehat{\simeq} (\mathcal{O}_K, (1)_\sigma)$ iff $(g_\sigma)_\sigma = (-2. \log(|\sigma(a)|))_\sigma$ $(a \in \mathcal{O}_K{}^*)$. Now it is clear from the structure theorem for projective modules, that the only automorphisms of $\mathcal{O}_K$ are multiplications by elements $a \in \mathcal{O}_K{}^*$; the metric $(exp(g_\sigma))_\sigma = (1/|\sigma(a)|^2)_\sigma$ is just the push-forward of the trivial metric by such an automorphism, which yields our claim. ♣

Note. Exactness in the middle could also have been easily proved using the isomorphism $\widehat{K_C}(X) \simeq \widehat{K_C}(X)'$. Just use the orthogonally splitting sequence

$$0 \to \widehat{I} \to \widehat{\mathcal{O}_K} \to \widehat{\mathcal{O}_K / I} \to 0$$

Note 2. This proposition already gives us full information on the structure of $\widehat{K_C}(X)$. As for the Chow groups, the injectivity of $\mathbb{R}^{X'_\infty}/\Gamma$ yields the non-canonical isomorphy $\mathbb{R}^{X'_\infty}/\Gamma \oplus K_C(X) \simeq \widehat{K_C}(X)$. In two sections, we shall

determine an isomorphism explicitly.
Before we proceed, note that c_1 is well-defined on $\widehat{K}_{GS}(X)$. Define it as usual on the metrized modules and as $(\mathcal{O}_K \oplus (g_\sigma)_\sigma)$ for $(g_\sigma)_\sigma \in \mathbb{R}^{X'_\infty}$; extend it then by linearity. Our computation (5.1) shows that the kernal contains the group of relations of $\widehat{K}_{GS}(X)$. Now the analogon of the last theorem for the Gillet-Soulé Grothendieck group is

Proposition 5.7 *The following sequence is exact*

$$0 \to \mathbb{R}^{X'_\infty}/\Gamma \xrightarrow{kf'} \widehat{K}_{GS}(X) \xrightarrow{f'} K_C(X) \to 0$$

where f' is the natural forgetful mapping (i.e. $f'([\widehat{M}]_{\widehat{K}_{GS}}) = [M]_{K_C}$ and $f'([g]_{\widehat{K}_{GS}}) = 0$ for $\widehat{M} \in \widehat{\mathcal{PF}}(X), g \in \mathbb{R}^{X'_\infty}$). kf' is defined by

$$kf'([g]_{\mathbb{R}^{X'_\infty}/\Gamma}) := [g]_{\widehat{K}_{GS}} \tag{5}$$

Proof: The surjectivity on the right is obvious. To prove exactness in the middle, we have again to determine the kernel of f'. Suppose that

$$[n_1 M_1 + n_2 M_2 + \ldots n_k M_k]_{K_C} = 0 \tag{6}$$

for some metrized modules $\widehat{M}_j$ $(1 \leq j \leq k)$. Then the definition of $\widehat{K}_{GS}(X)$ gives

$$[n_1 \widehat{M}_1 + n_2 \widehat{M}_2 + \ldots n_k \widehat{M}_k]_{\widehat{K}_{GS}} + [g]_{\widehat{K}_{GS}} = 0$$

(for some $g \in \mathbb{R}^{X'_\infty}$)
which proves our claim. To prove injectivity on the left amounts to the statement $[g]_{\widehat{K}_{GS}} = 0$ iff $g \in \Gamma$.
For the sufficiency of the condition $g \in \Gamma$, simply consider the two sequences

$$\widehat{\mathcal{S}}_1 : 0 \to (\mathcal{O}_K, (1)_\sigma) \xrightarrow{Id.} (\mathcal{O}_K, (1)_\sigma) \to 0$$

$$\widehat{\mathcal{S}}_2 : 0 \to (\mathcal{O}_K, (1)_\sigma) \xrightarrow{\frac{1}{\epsilon}} (\mathcal{O}_K, (1)_\sigma) \to 0$$

where $\epsilon \in \mathcal{O}_K{}^*$. There are clearly exact. One readily computes that $(\widetilde{ch}(\mathcal{S}_{2\sigma}))_\sigma - (\widetilde{ch}(\mathcal{S}_{1\sigma}))_\sigma = (\widetilde{ch}(\mathcal{S}_{2\sigma}))_\sigma = (-2.\log(|\sigma(\epsilon)|))_\sigma$, which implies that $[(-2.\log(|\sigma(\epsilon)|)_\sigma]_{\widehat{K}_{GS}} =$

0.
For the necessity, note that clearly $[g]_{\widehat{K}_{GS}} = [(\mathcal{O}_K, (\exp(g_\sigma))_\sigma)] - [(\mathcal{O}_K, (1)_\sigma)]$. Since c_1 is well-defined on $\widehat{K}_{GS}(X)$, this means that the last expression is 0 iff $(\mathcal{O}_K, (\exp(g_\sigma))_\sigma) \widehat{\simeq} (\mathcal{O}_K, (1)_\sigma)$. The computation at the end of the proof of 5.6 therefore settles the matter. ♣

Theorem 5.8 *There is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 \to & \mathbb{R}^{X'_\infty}/\Gamma & \stackrel{kf}{\to} & \widehat{K}_C(X) & \stackrel{f}{\to} & K_C(X) & \to 0 \\ & \downarrow Id. & & \downarrow & & \downarrow Id. & \\ 0 \to & \mathbb{R}^{X'_\infty}/\Gamma & \stackrel{kf'}{\to} & \widehat{K}_{GS}(X) & \stackrel{f'}{\to} & K_C(X) & \to 0 \end{array}$$

The map from $\widehat{K}_C(X)$ to $\widehat{K}_{GS}(X)$ is the natural repelling one (i.e. $[\widehat{M}]_{\widehat{K}_C} \to [\widehat{M}]_{\widehat{K}_{GS}}$).

Proof: It only remains to check the commutativity of the first square, i.e. to check the equation

$$[g]_{\widehat{K}_{GS}} = [(\mathcal{O}_K, (\exp(g_\sigma))_\sigma)]_{\widehat{K}_{GS}} - [(\mathcal{O}_K, (1)_\sigma)]_{\widehat{K}_{GS}} \qquad (7)$$

This is clear. The commutation of the other squares is immediate. ♣

This proposition has a satisfying corollary:

Corollary 5.9 *$\widehat{K}_C(X) \simeq \widehat{K}_{GS}(X)$ by the natural repelling map*

Proof: Apply the 5-lemma (see [11, p. 112]) to the commutative diagram of the last theorem. ♣

The preceding results show that we shall lose no information by formulating from now on all our results for $\widehat{K}_C(X)$ only, even if the other isomorphic Grothendieck groups will be of essential importance in some proofs. Another conceptually important consequence is that **one obtains isomorphic Grothendieck groups if one replaces orthogonal split exactness by volume exactness** (a volume on a metrized module is just a hermitian metric on its determinant).

We are now in position to tackle with the structure theorem for $\widehat{K}_C$.

5.4 The structure theorem for $\widehat{K}_C(X)$

We record as a proposition an immediate consequence of the splitting principle (3.2):

Proposition 5.10 *$\widehat{K}_C(X)$ is generated as a group by the elements of the form $[\widehat{M}]_{\widehat{K}_C}$, for $\widehat{M}$ invertible metrized modules.*

We define now a Chern character mapping

Definition 5.11 *The Chern character mapping $\widehat{ch}$ is defined by*

$$\widehat{ch} : \widehat{K}_C(X) \to \widehat{CH}(X)$$
$$\widehat{ch}(x) := rg(x) + c_1(Det(x))$$

The property of main interest is

Proposition 5.12 *$\widehat{ch}$ is a ring homomorphism.*

Proof: The proof is a computation:
• (commutation with the additive structure):

$$\widehat{ch}(x+y) \overset{def.}{=} rg(x+y) + c_1(Det(x+y)) =$$
$$rg(x) + rg(y) + c_1(Det(x).Det(y)) =$$
$$rg(x) + rg(y) + c_1(Det(x)) + c_1(Det(y))$$

$(x, y \in \widehat{K}_C(X))$
• (commutation with the multiplicative structure):
In view of 5.10 and the last set of equations, we can restrict the verification to the invertible case.

$$\widehat{ch}(x.y) \overset{def.}{=} rg(x.y) + c_1(Det(x.y)) =$$
$$1 + c_1(Det(x)) + c_1(Det(y)) = (1 + c_1(Det(x))) \odot (1 + c_1(Det(y)))$$

For the second equality, remember that the (arithmetic) determinant has no effect on an invertible metrized module. For the last one, remember that the $\odot$ product of elements of $\widehat{CH}^1$ vanishes. ♣

Note that this proposition shows that $\widehat{ch}$ obeys the axiomatic rules that a Chern character must obey (see [19, p.84, (ii),(iii),(iv)]). Finally we have the

Theorem 5.13 (The Structure Theorem for $\widehat{K}_C(X)$) *$\widehat{ch}$ is bijective, i.e. it is a ring isomorphism.*

Proof: We shall provide an inverse. By "abus de language" already define it to be

$$\widehat{ch}^{-1} : \widehat{CH} \to \widehat{K}_C(X)$$

$$\widehat{ch}^{-1}(r + cl) = r - 1 + [c_1{}^{-1}(cl)]_{\widehat{K}_C}$$

$(r \in \mathbb{Z} = \widehat{CH}^0, cl \in \widehat{CH}^1)$
The injectivity of $\widehat{ch}^{-1}$ is readily seen from the equations

$$\widehat{ch}(r-1+[c_1{}^{-1}(cl)]_{\widehat{K}_C}) \stackrel{5.12}{=} r-1+\widehat{ch}([c_1{}^{-1}(cl)]_{\widehat{K}_C}) \stackrel{def.of \widehat{ch}}{=} r-1+1+cl = r+cl$$

Now to the surjectivity. Anticipating further results, assume that $\widehat{ch}$ is additive. Then 5.10 tells us that we can restrict the verification of the equation $\widehat{ch}^{-1} \circ \widehat{ch} = Id.$ to invertible metrized modules, i.e to metrized fractional ideals. But in this case one can readily compute

$$\widehat{ch}^{-1}(\widehat{ch}((I,(h_{I_\sigma})_\sigma))) \stackrel{def.}{=} \widehat{ch}^{-1}(1 + (I \oplus (-\log(|h_{I_\sigma}|))_\sigma)) \stackrel{def.}{=} [\widehat{I}]_{\widehat{K}_C}$$

($\widehat{I}$ a metrized fractional ideal)
Therefore, we are reduced to verify the additivity of $\widehat{ch}^{-1}$.

- **Proof of the additivity of $\widehat{ch}^{-1}$:**

Taking into account the isomorphism 4.4, this amounts to prove the equality

$$(r - 1 + [\widehat{I}]_{\widehat{K}_C}) + (r' - 1 + [\widehat{I'}]_{\widehat{K}_C}) = r + r' - 1 + [\widehat{I}]_{\widehat{K}_C}.[\widehat{I'}]_{\widehat{K}_C} \qquad (8)$$

or

$$1 + [\widehat{I}]_{\widehat{K}_C}.[\widehat{I'}]_{\widehat{K}_C} = [\widehat{I}]_{\widehat{K}_C} + [\widehat{I'}]_{\widehat{K}_C} \qquad (9)$$

($\widehat{I}, \widehat{I'}$ metrized fractional ideals, $r, r' \in \mathbb{Z}$)
We can assume I, I' to be integral ideals that are coprime in O_K. Indeed, we can achieve integrality by tensoring I and I' with suitable trivially metrized principal fractional ideals. Then we can write the exact sequences

$$\widehat{\mathcal{I}}_1 : 0 \to (I.I', (h_{I_\sigma}.h_{I'_\sigma})_\sigma) \to (I, (h_{I_\sigma})_\sigma) \to (I/(I.I'), (0)_\sigma) \to 0$$

$$\widehat{\mathcal{I}}_2 : 0 \to (I', h_{I'_\sigma}) \to (\mathcal{O}_K, (1)_\sigma) \to (\mathcal{O}_K/I', (0)_\sigma) \to 0$$

(note that $((I/(I.I'))_\sigma = (\mathcal{O}_K/I')_\sigma = 0)$. Now we use the isomorphy $\widehat{K}_C(X) \simeq \widehat{K}_{GS}(X)'$ to compute in $\widehat{K}_{GS}(X)'$ instead; observing that $\mathcal{O}_K/I' \simeq I/(I.I')$ since $I + I' = \mathcal{O}_K$, these sequences allow us to write

$$[(\mathcal{O}_K/I', (0)_\sigma)]_{\widehat{K}_{GS}} = [(I/(I.I'), (0)_\sigma)]_{\widehat{K}_{GS}} = [\widehat{I}]_{\widehat{K}_{GS}} + \widetilde{ch}(\widehat{\mathcal{I}}_1) - [\widehat{I}]_{\widehat{K}_{GS}}.[\widehat{I'}]_{\widehat{K}_{GS}} =$$

$$1 + \widetilde{ch}(\widehat{\mathcal{I}}_2) - [\widehat{I'}]_{\widehat{K}_{GS}}$$

But we can calculate $\widetilde{ch}(\widehat{\mathcal{I}}_1) = (-\log(1/h_{I_\sigma}))_\sigma = \widetilde{ch}(\widehat{\mathcal{I}}_2)$, and therefore the last equation amounts to the equation 9 and we are done. ♣
[3]

5.5 Functoriality of $\widehat{K}_C$

We now define a pull-back and a push-forward for the Grothendieck groups.

Definition 5.15 *Let $M \in \widehat{\mathcal{PF}}(X_K)$. Let $i : K \to L$ be an inclusion of number fields. The pull-back function $i^*_{\widehat{\mathcal{PF}}}$ is defined by*

$$i^*_{\widehat{\mathcal{PF}}} : \widehat{\mathcal{PF}}(X_K) \to \widehat{\mathcal{PF}}(X_L)$$

$$i^*_{\widehat{\mathcal{PF}}}(\widehat{M}) := (M \otimes_{\mathcal{O}_K} \mathcal{O}_L, i^*((h_{M_\sigma})_\sigma))$$

(for the modules)

$$i^*_{\widehat{\mathcal{PF}}}(\widehat{f}) := f \otimes_{\widehat{\mathcal{O}_K}} Id.$$

[3]As a corollary, we can deduce the existence of a filtration of $\widehat{K}_C(X)$:

Corollary 5.14 *Let $F\widehat{K}_C(X)$ be $\widehat{ch}^{-1}(\widehat{CH}^1(X))$. Then $F\widehat{K}_C(X)$ is an ideal and $F\widehat{K}_C(X)^2 = 0$.*

Proof: $F\widehat{K}_C(X)$ is an ideal, since $\widehat{CH}^1(X)$ is an ideal in $\widehat{CH}(X)$ (it a subgroup by def. and $\odot$ multiplication by any element of $\widehat{CH}(X)$ doesn't lead out of it). Since $\widehat{ch}^{-1}$ is a ring homomorphism by 5.13, its image is also an ideal, and its square is 0, since $\widehat{CH}^1(X) \odot \widehat{CH}^1(X) = 0$ by the definition of $\odot$. ♣

(for the morphisms)
where $(i^*((h_{M_\sigma})_\sigma))_{\tau\in X_{L'_\infty}} := h_{M_{\tau|K}}$, *under the identification arising from the canonical complex vector space isomorphism*

$$(M \otimes_{\mathcal{O}_K} \mathcal{O}_L) \otimes_{\mathcal{O}_{L,\tau}} \mathbb{C} \simeq M \otimes_{\mathcal{O}_{K,\sigma}} \mathbb{C} \tag{10}$$

when $\tau|\sigma$ *and* $\tau \in X_{L'_\infty}$, $\sigma \in X_{K'_\infty}$.

The isomorphism is given by $(m \otimes_{\mathcal{O}_K} z_0) \otimes_{\mathcal{O}_{L,\tau}} z \to m \otimes_{\mathcal{O}_{K,\sigma}} (z_0.z)$ $(m \in M, z_0 \in \mathcal{O}_L, z \in \mathbb{C})$. This map is clearly $\mathbb{C}$-linear and surjective, and therefore injective, since the two spaces have the same dimension. The relevant property is

Proposition 5.16 $i^*_{\widehat{\mathcal{PF}}}$ *is an additive functor.*

Proof: The functoriality is merely the functoriality of the tensoring with a given module. As to the additivity, the algebraic part follows from the flatness of $\mathcal{O}_L$ over $\mathcal{O}_K$ (since $\mathcal{O}_L$ is torsion free); the metrical part follows readily from the easily verified commutation

$$\begin{array}{ccccccc} 0 \to & i^*_{\widehat{\mathcal{PF}}}(\widehat{M}')_\tau & \stackrel{i^*_{\widehat{\mathcal{PF}}}(\widehat{d'})_\tau}{\to} & i^*_{\widehat{\mathcal{PF}}}(\widehat{M})_\tau & \stackrel{i^*_{\widehat{\mathcal{PF}}}(\widehat{d''})_\tau}{\to} & i^*_{\widehat{\mathcal{PF}}}(\widehat{M}'')_\tau \to 0 \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & M'_\sigma & \stackrel{d'_\sigma}{\to} & M_\sigma & \stackrel{d''_\sigma}{\to} & M''_\sigma \to 0 \end{array}$$

where $\tau|\sigma$, $\tau \in X_{L_\infty}$, the vertical arrows are the isomorphisms of 10, and the bottom sequence splits orthogonally. ♣

Since $i^*_{\widehat{\mathcal{PF}}}$ is additive, we may view its domain of definition as $\widehat{K}_C(X)$ and even as a homomorphism $\widehat{K}_C(X_K) \to \widehat{K}_C(X_L)$. Therefore if we associate $i^*_{\widehat{\mathcal{PF}}}$ to the inclusion $i : K \to L$, then $\widehat{K}_C(X)$ becomes a covariant functor $\mathcal{NF} \to \mathcal{AB}$.
Similarly we have a push-forward.

Definition 5.17 *The push-forward* $i_{*\widehat{\mathcal{PF}}}$ *is defined as*

$$i_{*\widehat{\mathcal{PF}}} : \widehat{\mathcal{PF}}(X_L) \to \widehat{\mathcal{PF}}(X_K)$$

$$i_{*\widehat{\mathcal{PF}}}(\widehat{M}) := (M_{\mathcal{O}_K}, i_*((h_{M_\tau})_\tau)) \tag{11}$$

$$i_{*\widehat{\mathcal{PF}}}(\hat{f}) := \widehat{f_{\mathcal{O}_K}}$$

where $M_{\mathcal{O}_K}$ refers to M considered as an $\mathcal{O}_K$-module, $f_{\mathcal{O}_K}$ to f considered as homomorphism of $\mathcal{O}_K$-modules and $i_*((h_{M_\tau})_\tau)_{\sigma\in X_{K}{}'_\infty} := \sum_{\tau|\sigma} h_{M_\tau}$ under the identification arising from the canonical isomorphism

$$M \otimes_{\mathcal{O}_{K,\sigma}} \mathbb{C} \simeq \bigoplus_{\tau|\sigma} M \otimes_{\mathcal{O}_{L,\tau}} \mathbb{C}$$

($m \otimes_{\mathcal{O}_{K,\sigma}} z \to \bigoplus_{\tau|\sigma} m \otimes_{\mathcal{O}_{L,\tau}} z$).

Here the last mapping is clearly injective, and therefore surjective, since both spaces have the same dimension. We have again

Theorem 5.18 *$i_{*\widehat{\mathcal{PF}}}$ is an additive functor.*

Proof: The functoriality is obvious. As to the additivity, the algebraic part is clear, since exact sequences of $\mathcal{O}_L$-modules are also exact sequences of $\mathcal{O}_K$-modules. The metrical part again follows from an easily verifiable commutation as in the proof of 5.16. ♣

If we associate $i_{*\widehat{\mathcal{PF}}} : \widehat{K}_C(X_L) \to \widehat{K}_C(X_K)$ to the inclusion $i : K \to L$, then $\widehat{K}_C(X)$ becomes again a contravariant functor $\mathcal{NF} \to \mathcal{AB}$.

6 The Riemann-Roch theorem

6.1 The Grothendieck-Riemann-Roch theorem

The Grothendieck-Riemann-Roch theorem is concerned with the naturality of the push-forward mapping for $\widehat{K}_C$-groups with respect to the Chern-character, i.e. it examines the commutativity of the diagram

$$\begin{array}{ccc} \widehat{K}_C(X_L) & \xrightarrow{\widehat{ch}} & \widehat{CH}(X_L) \\ \downarrow i_{*\widehat{\mathcal{PF}}} & & \downarrow i_{*\widehat{CH}} \\ \widehat{K}_C(X_K) & \xrightarrow{\widehat{ch}} & \widehat{CH}(X_K) \end{array}$$

This diagram is **not** commutative. The Grothendieck-Riemann-Roch theorem asserts that a universal multiplicative correction factor must be inserted on the first row. We state

Theorem 6.1 (Grothendieck-Riemann-Roch) *The diagram*

$$\begin{array}{ccc} \widehat{K}_C(X_L) & \stackrel{Todd(Y|X)\odot\widehat{ch}}{\rightarrow} & \widehat{CH}(X_L)\otimes\mathbb{Q} \\ \downarrow i_{*\widehat{\mathcal{PF}}} & & \downarrow i_{*\widehat{CH}} \\ \widehat{K}_C(X_K) & \stackrel{\widehat{ch}}{\rightarrow} & \widehat{CH}(X_K) \end{array}$$

commutes.

$Todd(Y|X)$ is an element of $\widehat{CH}(X_L)\otimes\mathbb{Q}$, depending only on $\mathcal{O}_K and \mathcal{O}_L$, to be defined immediately.

Definition 6.2 *The (arithmetic) Todd class is* $Todd(Y|X) := 1+\frac{1}{2}df(Y|X)$, *where* $df(Y|X)$ *is the image in* $\widehat{CH}^1$ *of* $\widehat{D_{L|K}}$.

The tensoring by $\mathbb{Q}$ has only been introduced to allow the factor $\frac{1}{2}$, which will be swallowed in the computations; the group homomorphisms from $\widehat{CH}(X_L)$ extend to group homomorphisms from $\widehat{CH}(X_L)\otimes\mathbb{Q}$ by linearity. This theorem, when specialized over $\mathcal{O}_K = \mathbb{Z}$ is an essential tool to solve the Riemann-Roch problem. Before proving it, we need two propositions examining the effect of the push-forward on the direct summands of $\widehat{ch}$.

Proposition 6.3 *Let* $\widehat{M}$ *be a metrized* $\mathcal{O}_L$*-module. Then* $rg(i_{*\widehat{\mathcal{PF}}}(\widehat{M})) = [L:K].rg(\widehat{M})$.

Proof: This equation is in fact classical. We have to show that $rg(M_{\mathcal{O}_K}) = [L:K]rg(M_{\mathcal{O}_L})$. This can be found in [9, p. 416]. ♣

Before we proceed to the next two propositions, we list some computations of the image of certain morphisms under the σ-functor:
(a) Trace
Let I be an integral ideal of $\mathcal{O}_L$. We may view the $Tr_{L|K}$ as an $\mathcal{O}_K$-homomorphism $I \rightarrow \mathcal{O}_K$ (the image is in $\mathcal{O}_K$ because the characteristic

polynomials of elements of $\mathcal{O}_L$ have coefficients in $\mathcal{O}_K$). What is Tr_σ ? We have natural isomorphisms $I_\sigma \overset{\phi}{\simeq} \oplus_{\tau|\sigma} I_{\mathcal{O}_{L_\tau}} \overset{\psi}{\simeq} \oplus_{\tau|\sigma}\mathbb{C}$ given by $\phi(\iota\otimes z) = \oplus_{\tau|\sigma}(\iota\otimes_{\mathcal{O}_{L_\tau}} z)$ and $\psi(\oplus_{\tau|\sigma}(\iota_\tau\otimes_{\mathcal{O}_{L_\tau}} z_\tau)) = \oplus_{\tau|\sigma}\tau(\iota_\tau).z_\tau$. These isomorphisms will be often used in the sequel and the subscript $(_\tau)$ notation will be used to denote the components as above. We now compute Tr_σ:

$$\sigma(Tr(\iota))\otimes_{\mathcal{O}_{K,\sigma}} z \overset{def.of Tr}{=} \sigma(\sum_{\tau|K}\tau(\iota))\otimes_{\mathcal{O}_{K,\sigma}} z = (\sum_{\tau|\sigma}\tau(\iota))\otimes_{\mathcal{O}_{K,\sigma}} z$$

which means that under the identification provided by ψ, we may calculate Tr_σ as the usual trace.
(b) Multiplication
Same setting as before with another integral ideal J of $\mathcal{O}_L$. We consider the multiplication $I\otimes_{\mathcal{O}_K} J \to \mathcal{O}_{L\mathcal{O}_K}$ as an $\mathcal{O}_K$ homomorphism. We compute for $\iota\in I, \rho\in J$:

$$(\iota\otimes_{\mathcal{O}_{K,\sigma}} u).{}_\sigma(\rho\otimes_{\mathcal{O}_{K_\sigma}} v) \overset{2}{=} (\iota.\rho\otimes_{\mathcal{O}_{K,\sigma}} u.v)$$

and applying $\psi\circ\phi$ to the last expression we get

$$\bigoplus_{\tau|\sigma}\tau(\iota).\tau(\rho).u.v$$

which means that under the identification provided by ψ, we may calculate $._\sigma$ by componentwise multiplication.
(d) Norm
With the same setting as for the trace, $N_{L|K_\sigma}$ corresponds to the usual norm, under the ψ identification.
With these computational data in hand, we can easily prove

Proposition 6.4 $c_1(Det(i_{*\widehat{\mathcal{PF}}}(\widehat{M}))) = i_{*\widehat{CH}}(c_1(Det(\widehat{M})))+rg(\widehat{M}).c_1(Det(i_{*\widehat{\mathcal{PF}}}(1)))$.

Proof: Again 5.10 allows us to assume $rg(M)=1$, since both sides of the equation are manifestly additive. Therefore, let $\widehat{M}=\widehat{I}$ ($\widehat{I}$ a fractional $\mathcal{O}_L$-ideal). If we use the isomorphism 4.4 to write our equation in $\widehat{Pic}(X)$, we obtain

$$((N_{L|K}(I)),(\prod_{\tau|\sigma}h_\tau)_\sigma)\otimes_{\mathcal{O}_K} Det(\mathcal{O}_{L\mathcal{O}_K},(\bigoplus_{\tau|\sigma}1)_\sigma)\cong Det(I_{\mathcal{O}_K},(\bigoplus_{\tau|\sigma}h_\tau)_\sigma)$$

$(h_\tau)_{\tau\in X_{L'_\infty}}$ is the metric of $\widehat{I}$ (we dropped subscripts for notational convenience) and $\prod$ refers to the product of the coefficients in the standard basis. An obvious candidate for an isometric isomorphism is $\widehat{L}$ where

$$L : a\otimes(b_1\wedge\ldots\wedge b_{[L:K]})\to a.(b_1\wedge\ldots b_{[L:K]})$$

(to be read in $Det_K L$)
To verify that L is an algebraic isomorphism, we first assume that $\mathcal{O}_K$ and $\mathcal{O}_L$ are both p.i.d. Then we could choose a generator of c' of I and an $\mathcal{O}_K$-basis $b'_1\ldots b'_{[L:K]}$ of $\mathcal{O}_L$. Then $N_{L|K}(c')$ is a generator of $N_{L|K}(I)$ (by multiplicativity of the norm) and $N_{L|K}(c')\otimes(b'_1\wedge\ldots b'_{[L:K]})$ a basis of the source set of L. We can compute

$$L(N_{L|K}(c')\otimes(b'_1\wedge\ldots b'_{[L:K]}))=(c'.b'_1\wedge\ldots c'.b'_{[L:K]})$$

using the definition of the norm, so that L sends a basis on a basis. The usual localisation procedure then yields the general result (i.e. we localise for any two prime ideals $p\subseteq\mathcal{O}_K$, $P\subseteq\mathcal{O}_L$, with $p\subseteq P$; the localisations are discrete valuation rings, since $\mathcal{O}_K,\mathcal{O}_L$ are Dedekind domains), since injectivity and surjectivity are local properties (see [2]). To verify that L induces an isometry, we exhibit orthonormal bases for the first and second member. We calculate at a fixed σ. Straightforward application of the rules (cf. 3.1) yields

$$(N_{L|K}(I))_\sigma[1]/(\prod_{\tau|\sigma}h_\tau^{\frac{1}{2}})\otimes(\bigwedge_{\tau|\sigma}\mathcal{O}_{L_\tau}[1])$$

and for the second one

$$\bigwedge_{\tau|\sigma}(I_\tau[1]/h_\tau^{\frac{1}{2}})$$

if we use 6.1 to compute in $\bigoplus_{\tau|\sigma}\mathbb{C}$, we see that $\psi(I_\tau[1]).\psi(\mathcal{O}_{L_\tau}[1])=\psi(I_\tau[1]).1$ so that $I_\tau[1]._\sigma\mathcal{O}_{L_\tau}[1]=I_\tau[1]$. Therefore the computation of $N_{L|K_\sigma}$ shows that L_σ sends the first basis on the second one. ♣

This theorem provides a kind of "additive" Riemann-Roch formula; such theorems are in general refered to as Riemann-Roch Theorems without denominators. As it points out, we shall need an explicit formula for $c_1(Det(i_{*}\widehat{_{\mathcal{P}\mathcal{F}}}(1)))$. In fact, we have

Proposition 6.5 $c_1(Det(i_{*\widehat{\mathcal{PF}}}(1))) = \frac{1}{2} i_{*\widehat{CH}}(2.Todd(Y|X) - 2) := \frac{1}{2} i_{*\widehat{CH}}(df(Y|X))$.

Proof: Writing this equation out and using the definition of the Todd class, we see that it is equivalent to the equation

$$Det(i_{*\widehat{\mathcal{PF}}}(\mathcal{O}_L))^{\otimes 2} \simeq \widehat{\delta_{L|K}}$$

Here we use the classical relation $\delta_{L|K} = (N_{L|K}(D_{L|K}))$; the fact that $\delta_{L|K}$ is trivially metrized follows directly from the definition of $i_{*\widehat{CH}}$.
Define an $\mathcal{O}_K$ homomorphism

$$T : Det(\mathcal{O}_L) \otimes Det(\mathcal{O}_L) \to \mathcal{O}_K$$

$$T((\alpha_1 \wedge \ldots \wedge \alpha_{[L:K]}) \otimes (\beta_1 \wedge \ldots \wedge \beta_{[L:K]})) \to Det(Tr_{L|K}(\alpha_i.\beta_j)_{1 \leq i,j \leq [L:K]})$$

It is well-known that the image is the discriminant ideal (by definition) and T is then injective, since $Det_{\mathcal{O}_K}(\mathcal{O}_L) \otimes Det_{\mathcal{O}_K}(\mathcal{O}_L)$ is also projective of rank 1.
For the geometrical part, we again exhibit orthonormal bases at σ. We work entirely in the complex identification $\prod_{\tau|\sigma} \mathbb{C}$. Since the metric is trivial, if $\mathcal{O}_{L\tau}[1] = x_\tau$ then

$$(\bigwedge_{\tau|\sigma} x_\tau)^{\otimes 2}$$

is an orthonormal basis for $(Det(\mathcal{O}_L) \otimes Det(\mathcal{O}_L))_\sigma$. We compute

$$|T_\sigma((\bigwedge_{\tau|\sigma} x_\tau)^{\otimes 2})| = |Det((x_{\tau_i}.x_{\tau_j})_{\tau_i,\tau_j|\sigma})| = \prod_{\tau|\sigma} x_\tau^2 = 1$$

since $x_{\tau_i}.x_{\tau_j} = 0$, if $i \neq j$ and $Tr_\sigma(x_\tau.x_\tau) = 1$ (see the computation 6.1). Since the absolute value in $\mathbb{C}$ of the resulting basis is 1, it is orthonormal for the usual metric of $\mathbb{C}$, i.e. of $\delta_{L|K_\sigma}$. ♣

Finally we come to the proof

Proof: (of the Grothendieck-Riemann-Roch Theorem)
The proof is a simple computation. Since both sides of the equation are evidently additive, we can restrict ourselves to the case $rg(x) = 1$, by 5.10. We compute

$$i_{*\widehat{CH}}(Todd(Y|X).\widehat{ch}(x)) = i_{*\widehat{CH}}((1 + \frac{1}{2}df(Y|X))(c_1(Det(x)) + 1)) =$$

$$i_{*\widehat{CH}}(c_1(Det(x)) + 1 + \frac{1}{2}df(Y|X)) \stackrel{6.5}{=}$$

$$i_{*\widehat{CH}}(1)+c_1(Det(i_{*\widehat{\mathcal{PF}}}(x)))-c_1(Det(i_{*\widehat{\mathcal{PF}}}(1)))+c_1(Det(i_{*\widehat{\mathcal{PF}}}(1))) = [L:K]+c_1(Det(i_{*\widehat{\mathcal{PF}}}(x)))$$

$$rg(i_{*\widehat{\mathcal{PF}}}(x)) + c_1(Det(i_{*\widehat{\mathcal{PF}}}(x))) \stackrel{def.}{=} \widehat{ch}(i_{*\widehat{\mathcal{PF}}}(x))$$

♣

6.2 The absolute Riemann-Roch Theorem via θ-functions

6.2.1 Preliminaries

Now recall the classical Riemann-Roch theorem for compact Riemann surfaces. It enounces that

$$\chi(\mathcal{L}(D)) = dim\ H^0(\mathcal{L}(D)) - dim\ H^1(\mathcal{L}(D)) =$$

$$dim\ H^0(\mathcal{L}(D)) - dim\ H^0(\Omega \otimes (\mathcal{L}(D))^\vee) = deg(D) + 1 - g$$

where D is a divisor on a compact Riemann surface of genus g, Ω is the canonical bundle (dualizing sheaf over $\mathbb{C}$). $\mathcal{L}$ sends D on its corresponding holomorphic line bundle and $\chi(.)$ takes the Euler-Poincaré characteristic. The second equation holds by Serre Duality. We can rewrite this formula in the form

$$\chi(\mathcal{L}(D)) = deg(D) + \chi(\mathcal{O}) = deg(D) + dim\ H^0(\mathcal{O}) - dim\ H^1(\mathcal{O}) \quad (12)$$

where $\mathcal{O}$ is the structure sheaf. We shall try to derive a formula analogous to 12. To perform this, we shall of course need an analog of χ.

6.2.2 Exact and approximate Riemann-Roch theorem

Before starting with the definitions, recall that $\mathbb{C}^{X_\infty}$ is endowed with a natural involution, given by $(z_\sigma)_\sigma \mapsto (\overline{z_{\overline{\sigma}}})$. The subspace of fixed points under this involution will be refered to as $K_\mathbb{R}$. It is naturally isomorphic to $\mathbb{R}^{X_\infty}$ and the natural embeding of the integer ideals of $\mathcal{O}_K$ sends them inside $K_\mathbb{R}$ (cf. [15, p. 31]). In the sequel, we shall consider the integer ideals as embedded

in $K_{\mathbb{R}}$, without mentioning it. h_I will refer to the vector of the coefficients of the metric of the metrized ideal $\widehat{I}$, viewed as an element of $K_{\mathbb{R}}$. n_I will refer to $\sqrt{h_I}$ (the square roots being taken componentwise).
Now we can define

Definition 6.6 *The (arithmetic) Euler-Poincaré characteristic, or Euler-Minkowski characteristic is*

$$\chi_K : \widehat{\mathcal{PF}}(X) \to \mathbb{R}$$

$$\chi_K(\widehat{M}) := deg(i_{*\widehat{\mathcal{PF}}}(\widehat{M}))$$

where $i : \mathbb{Q} \to K$ is the inclusion.

This mapping is clearly additive so we may view it as a map from $\widehat{K_C}(X)$. Applying deg on both sides of 6.4 we get

$$\chi_K(\widehat{I}) = deg(\widehat{M}) + \chi_K(i_{*\widehat{\mathcal{PF}}}(1)) \tag{13}$$

where $\widehat{I}$ is a metrized fractional $\mathcal{O}_K$-ideal. This last formula as the required appearance. χ_K can be computed as follows

Proposition 6.7 *Let I be integral. Then*
$\chi_K(\widehat{I}) = -\log(vol(n_I.I)) + \log(\sqrt{|d_K|})$

(vol takes the volume of a fundamental domain of a lattice in the standard metric)

Proof: Using a standard result of algebraic number theory (cf. [15, p. 33, Th. 5.2]), we compute $-\log(vol(n_I.I))$=$-\log(\sqrt{|d_K|}) + -\log(\mathcal{N}(I)) + -\log(N(n_I))$=$-\log(\sqrt{|d_K|})+-\log(\mathcal{N}(I))+-\frac{1}{2}\sum_\sigma \log(h_{I_\sigma}) = -\log(\sqrt{|d_K|})+ deg(\widehat{I})$ which yields our claim. ♣

We still need an interpretation of the dimensions of the cohomology groups. The analog of $dim\ H^0(.)$ we crave to introduce for a metrized integral ideal is of course the number of intersection points of the ideal with a convex compact centrally symmetric body within $K_{\mathbb{R}}$. We could define more generally

Definition 6.8

$$\widehat{H^0}(\widehat{I}, B) := \log(\#(n_I.I \cap B))$$

where $\widehat{I}$ is a metrized integral $\mathcal{O}_K$-ideal, B is a (convex, compact, centrally symmetric) body in $K_{\mathbb{R}}$. If B is the unit ball for the metric induced by $\widehat{I}$ on $K_{\mathbb{R}}$, then we write $\widehat{H^0}(\widehat{I})$ for $\widehat{H^0}(\widehat{I}, B)$.

The analog of *dim* $H^1(.)$ would then simply be $\widehat{H^0}((\widehat{D_{L|K}}.\widehat{I})^{-1}, B^*)$, $\widehat{D}^{-1}_{L|K}$ playing the role of the dualizing sheaf. Nevertheless, it is intuitively quite obvious that if we tried to formulate an analog of 12 with the present definition of $\widehat{H^0}(\widehat{I}, B)$, we wouldn't be able to obtain an **equality**, since the intersection numbers of lattices with bodies can only be estimated from below, as is well-known. The impossibility to reach equalities in Minkowski's geometry of numbers is a sort of instance of Heisenberg's uncertainty principle[4]. The article [6] gives the best inequalities reached so far. For instance, they get

$$\sqrt(|d_K|) - C(r,s) \leq \widehat{H^0}(\widehat{I}) - \widehat{H^1}(\widehat{I}) - deg(\widehat{I}) - \chi(\widehat{\mathcal{O}_K}) \leq \sqrt(|d_K|) + C(r,s) \quad (14)$$

where r is the number of real embeddings of K, s the number of complex ones, $C(r,s) = (r+s).\log(3) - s.\log(2.\pi) + \log((r+s)!) + s.\log(2)$. 14 can be considered as an "approximate Riemann-Roch theorem".

To remedy this situation we replace the counting of intersection points by a global measure of the concentration of a lattice around the origin, by means of θ-functions. We define

Definition 6.9 *The θ-function of a metrized integral ideal $\widehat{I}$ of $\mathcal{O}_K$ is*

$$\theta(\widehat{I}, \vec{x}) = N(\vec{x}). \sum_{\iota \in I} \exp(-\pi. < \iota.h_I.\vec{x}^2, \iota >)$$

where $\vec{x} \in K_{\mathbb{R}}$ (to be suitably chosen). We write $\theta(\widehat{I})$ for $\theta(\widehat{I}, 1)$.

If we endow $\mathbb{C}^n$ with the metric of $\widehat{I}$ then $\theta(\widehat{I})$ is just the usual θ-function of the lattice I, at the complex vector $-i$. $\widehat{\mathcal{H}^0}(\widehat{I}, \vec{x}) := \log(\theta(\widehat{I}, \vec{x}))$ will now replace $\widehat{H^0}(\widehat{I}, B)$. Again, we define $\widehat{\mathcal{H}^1}(\widehat{I}, \vec{x}) := \log(N(\vec{x})\theta((\widehat{D}_{L|K}.\widehat{I})^{-1}, \frac{1}{\vec{x}^2}))$. An important invariance property of θ is

[4]This idea seems to go back to M. Atiyah

Lemma 6.10 *θ depends on the isometry class of $\widehat{I}$ only.*

Proof: As usual, let $\widehat{J} := \widehat{a.I}$ with the induced metric. Then we can compute

$$\theta(\widehat{J},\vec{x}) \stackrel{def.}{=} N(\vec{x}) \sum_{\lambda\in a.I} \exp(-\pi. < \lambda.\vec{x}^2.h_I.1/a^2, \lambda >) =$$

$$N(\vec{x})\sum_{\iota\in I} \exp(-\pi. < a.\iota.\vec{x}^2.h_I.1/a^2, a.\iota >) = \theta(\widehat{I},\vec{x})$$

♣

The natural domain of definition of $\theta(.,\vec{x})$ is therefore $\widehat{Pic}(X)$. We could of course have defined the θ of an entire metrized module $\widehat{M}$ by defining it to be $\theta(Det(\widehat{M}),\vec{x})$. The domain of definition of that θ is then $\widehat{K_C}(X)$. With this setting, we can finally prove

Theorem 6.11 *Let $\widehat{I}$ be a metrized integral ideal of $\mathcal{O}_K$. Then*

$$\widehat{\mathcal{H}^0}(\widehat{I},\vec{x}) - \widehat{\mathcal{H}^1}(\widehat{I},\vec{x}) = \chi(\widehat{I})$$

Proof: Just apply the classical θ-transformation formula (cf. [15, Chap. VII, p. 472, Prop. 3.6]) with $\vec{z} = -i.h_I.\vec{x}^2$, using 6.7. ♣

We can now compute the **genus** of L by means of the formula

$$\widehat{\mathcal{H}^0}(\widehat{\mathcal{O}_K}) - \widehat{\mathcal{H}^1}(\widehat{\mathcal{O}_K}) = \tag{15}$$

$$\chi(\widehat{\mathcal{O}_K}) = -\log(vol(\mathcal{O}_K)) + \log(|d_K|)$$

So that we obtain $g_K = \widehat{\mathcal{H}^1}(\widehat{\mathcal{O}_K}) = \widehat{\mathcal{H}^0}(\widehat{\mathcal{O}_K}) - \log(|d_K|) + \log(vol(\mathcal{O}_K))$. For example the genus of $\mathbb{Q}$ is $\log(\sum_{n\in\mathbb{Z}}\exp(-\pi.n^2))$. This definition is **absolute and does not depend on the choice of a standard body**. The choice of the value $\vec{1}$ for $\vec{x}$ is justified by the fact that it is the only value for which the correction $N(\vec{x})$ is not necessary in the formula for $\widehat{\mathcal{H}^1}$. We can finally formulate our Riemann-Roch theorem as follows:

Theorem 6.12 (Riemann-Roch) $\widehat{\mathcal{H}^0}(\widehat{I}) - \widehat{\mathcal{H}^1}(\widehat{I}) = deg(\widehat{I}) + \widehat{\mathcal{H}^0}(\widehat{\mathcal{O}_K}) - g_K$

We shall now show that the θ-functions a related to a *quantum* view of Minkowski's First Theorem. We may describe the situation as follows; let an integral ideal I be given, and a (compact, convex, centrally symmetric) body $B \subseteq K_{\mathbb{R}}$. Imagine that very small particles are located at each lattice point. We are interested in the number of particles inside B. If we think classically, this amounts precisely to the basic question of the geometry of numbers. Whereas if we think quantum-mechanically, the question may be formulated in the following terms. Let $P_B(\iota)$ be the probability of the particle ι of being found inside B; what is the sum of all the $P_B(\iota)$? We assume that the particle waves all have Gaussian probability distributions of the type

$$N(\vec{b})\exp(-\pi. \parallel (\vec{x}-\vec{n}).\vec{b} \parallel^2)$$

The passage to the classical view point is then performed by the passage to the limit $\vec{b} \to \infty$. In formulas, the quantum Minkowski problem amounts then to compute the following integral:

$$\int_B \sum_{\iota \in I} N(\vec{b})\exp(-\pi \parallel (\vec{x}-\iota).\vec{b} \parallel^2) d\vec{x}$$

As the θ-function gave certain weights to the elements of I, the last integral gives still other ones. The classical approach gives of course weight 0 to the elements of I outside B.

We can gain a vantage view-point on all these different approaches, if we formalize them in terms of measures. We list

- The θ-measure on I is defined by

$$\mu_{\theta_{\vec{x}}}(\iota) := N(\vec{x})\exp(-\pi. < \iota.h_I.\vec{x}^2, \iota >)$$

- The quantum measure on I is defined by

$$\mu_{Q_{\vec{b},B}}(\iota) := \int_B N(\vec{b})\exp(-\pi \parallel (\vec{x}-\iota).\vec{b} \parallel^2) d\vec{x}$$

- The classical measure on I is defined by

$$\mu_{C_B}(\iota) := \psi_{B\cap I}(\iota)$$

where ψ_S is the characteristic function of the set S.

In all three cases the Minkowski problem amounts to the computation of the measure of the entire I. We have the following asymptotic relations between the quantum and θ-measures:

Proposition 6.13

$$\lim_{diam(B)\to 0} \frac{\mu_{Q_{\vec{\iota},B}}(\iota)}{vol(B)} = \mu_{\theta_{\vec{\iota}}}(\iota)$$

$$\lim_{\iota\to\infty} \frac{\mu_{Q_{\vec{\iota},B}}(\iota)}{vol(B).\mu_{\theta_{\vec{\iota}}}(\iota)} = 1$$

Proof: The two assertions follow trivially from the mean-value theorem and the fact that the first derivative of the Gaussian function (around the origin, say) converges to 0 at ∞. ♣

These relations show that we may always summarize in a θ-measure the asymptotic properties of any quantum measure by taking the first limit. The resulting measure has then the advantage of being independent of the body B.

Of course the following relation between the quantum and classical measure holds

Proposition 6.14

$$\lim_{\vec{\iota}\to\infty} \mu_{Q_{\vec{\iota},B}}(\iota) = \mu_{C_B}(\iota)$$

Proof: This is merely the assertion that the integral of a Dirac δ-function on any set away from its peak is 0. ♣

If we are given a measure μ' on $K_{\mathbb{R}}$ we may canonically induce a measure μ on I by means of the formula

$$\mu(\iota) = \lim_{r\to 0} \frac{1}{vol(B(\iota,r))} \int_{B(\iota,r)} d\mu'$$

($B(\iota, r)$ is the ball of radius r (in the canonical metric) around ι)

The reader will easily find out that all the measures on I listed above can be obtained that way. In the classical case μ' is then simply the canonical measure inside B and 0 outside. We can therefore naturally speak of quantum, classical or θ-measures on $K_{\mathbb{R}}$.

Moreover, in this setting, we may interpret 6.13 in the following way: **the θ-measure is to the quantum measure what the Haar measure of**

$K_{\mathbb{R}}$ **is to the classical measure.** Indeed, the classical measure is asymptotically 0 when $diam(B) \to 0$, and is also asymptotically 0 when $\iota \to \infty$. The 0-measure being a multiple of the usual Haar measure, this means that we have similar asymptotic relationships between the Haar measure and the classical measure as between the θ-measure and the quantum measure.

7 Conclusion

The natural continuation of this work would be a generalisation of the θ-function approach to the Riemann-Roch problem described above. The general impression we gather from a quantum approach to the Minkowski problem (alias the arithmetic Riemann-Roch problem) is that it might be fruitful to replace the Haar measures at the infinite places by *θ-measures.* The θ-relation above shows that if one does so in the case of curves, it is possible to consider *separately* the elements of the extended determinant of the cohomology (i.e. with $\widehat{\mathcal{H}}^1$ included in our case) and write down a Riemann-Roch theorem in a classical guise. Since the work of Faltings (cf. [4]) it has become clear that one could only hope to compute the Euler-Minkowski characteristic of the determinant of the cohomology, and that one can not even endow the cohomology groups separately with Haar measures. We conjecture that it is possible to endow them with θ-measures, so that the alternating sum of the measures of the cohomology groups would give back the Euler-Minkowski characteristic.

Notes. The material presented here was essentially developed in the eighties. Grayson first proved the structure theorem for $\widehat{K}_C(X)$ ([5]); Tamme seems to have developed the Grothendieck formalism (in his lectures, nevertheless see [21]); Hübschke investigated the connections between the idelclass group and the arithmetic Chow group ([8]). The seminars of Szpiro ([20]) and Faltings-Wüstholz ([23]) made Arakelov geometry more widely known. The isomorphy $\widehat{K}_{GS}(X) \simeq \widehat{K}_C(X)$ is due to us, as well as the proof that $\widehat{K}_{GS}(X)$ can also be defined without projectivity assumptions. Our proof of the structure theorem for $\widehat{K}_C(X)$ is also new in that it is very much simplified by our use of $\widehat{K}_{GS}(X)'$ in the computation. The θ-function approach to the

Riemann-Roch problem is due to us[5].

Appendix A

.1 Grothendieck groups without projectivity assumptions

In this appendix, we carry out the proof of the assertion $\widehat{K}_{GS}(X)' \simeq \widehat{K}_{GS}(X)$. As we mentioned before, it is also true that $\widehat{K}_C(X) \simeq \widehat{K}_C(X)'$, but since we didn't use that statement, we refer to [15, Th. 5.4, p. 251] for the proof. Recall that the definition $\widehat{K}_{GS}(X)'$ is completly analogous to the definition of $\widehat{K}_{GS}(X)$, with only a requirement of finite generation on the modules. We begin with a lemma which will provide us with a inverse to the natural map. From now on, **metrized modules may be non-projective** .

Lemma .1 *Let $\widehat{M}$ be a metrized module. Then there is an orthogonally splitting resolution*

$$0 \to \widehat{E} \to \widehat{F} \to \widehat{M} \to 0$$

where $\widehat{E}, \widehat{F}$ are projective metrized modules.

Proof: The general structure theorem for f.g. modules over a Dedekind ring ([9, Th. 10.15, p.628]), tells us that we have an isomorphism

$$M \simeq I_1 \oplus \ldots \oplus I_r \oplus \mathcal{O}_K/I'_1 \oplus \ldots \oplus \mathcal{O}_K/I'_{r'}$$

for some fractional ideals. We let $F := (\bigoplus_{1\leq i\leq r} I_i) \bigoplus {\mathcal{O}_K}^{\oplus r'}$ and $E := \bigoplus_{1\leq i\leq r'} I'_i$. The maps are the obvious ones. We choose any metrics on the E_σ. We choose any splitting s_σ of the sequence

$$0 \to E_\sigma \to F_\sigma \overset{s_\sigma}{\leftrightarrows} M_\sigma \to 0$$

and we choose the metric on F_σ to be $h_{E_\sigma} \oplus s_{\sigma *}(h_{M_\sigma})$. ♣

The following lemma provides us with a link between two possible projective resolutions:

[5]We have to thank C. Soulé, for having brought to our knowledge the idea that θ-functions are relevant of a "quantum" approach, as well as having indicated to us the article of Siegel [18].

Lemma .2 *Let M be an f.g. $\mathcal{O}_K$-module and E'', F'', E', F' the elements of two projective resolutions. Then there exist a projective resolution E, F and surjective homomorphisms α', β', α'', β'' so that the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \to & E' & \to & F' & \xrightarrow{f'} & M & \to 0 \\ & & \uparrow \alpha' & & \uparrow \beta' & & \| & \\ 0 & \to & E & \to & F & \xrightarrow{f} & M & \to 0 \\ & & \downarrow \alpha'' & & \downarrow \beta'' & & \| & \\ 0 & \to & E'' & \to & F'' & \xrightarrow{f''} & M & \to 0 \end{array}$$

Proof: This lemma in fact holds in a much more general sheaf-theoretic context and is used to prove a proposition similar to the following one, in a classical context. In our case, the solution is easy: choose $E = E' \oplus E''$ and $F = \{(x', x'') \in F' \oplus F'' | f'(x') = f''(x'')\}$, $f : (x', x'') \mapsto f'(x') = f''(x'')$ and $\alpha', \beta', \alpha'', \beta''$ to be the projections on the factors. The diagram then clearly commutes and the α, β are surjective, by construction. Since E, F are clearly torsion free, therefore projective, we are done. ♣

Theorem .3 $\widehat{K}_{GS}(X) \simeq \widehat{K}_{GS}(X)'$ *by the natural map*

Proof: We proceed to construct an inverse to this mapping. We define

$$\pi : \widehat{K}_{GS}(X)' \to \widehat{K}_{GS}(X)$$

$$\pi([\widehat{M}]_{\widehat{K}'_{GS}}) := [\widehat{F'}]_{\widehat{K}_{GS}} - [\widehat{E'}]_{\widehat{K}_{GS}} \tag{16}$$

$$\pi([g]_{\widehat{K}'_{GS}}) := [g]_{\widehat{K}_{GS}}$$

where $\widehat{E'}, \widehat{F'}$ is a resolution of $\widehat{M}$ as in the first lemma. We now confront a double well-definedness problem: is π well-defined as a mapping **into** $\widehat{K}_{GS}(X)$? is it well-defined as a mapping **from** $\widehat{K}_{GS}(X)'$? The two questions are obviously only problematic for 16. As to the first question, we shall use the last lemma. Let $\widehat{E}'', \widehat{F}''$ be another resolution and E, F be a (non-metrized) resolution dominating both of them. We endow E, F with metrics as in .1 and then induce some metrics on *ker* β', *ker* α'. We have then two sequences:

$$\widehat{\mathcal{A}} : 0 \to \widehat{\ker \alpha'} \to \widehat{E} \to \widehat{E'} \to 0$$

$$\widehat{\mathcal{B}} : 0 \to \widehat{\ker \beta'} \to \widehat{F} \to \widehat{F'} \to 0$$

By commutativity, it is clear that $\widehat{ker\ \alpha}' \cong \widehat{ker\ \beta}'$. We can then compute

$$[\widehat{F}]_{\widehat{K}_{GS}} - [\widehat{E}]_{\widehat{K}_{GS}} =$$

$$[\widehat{F'}]_{\widehat{K}_{GS}} + [\widehat{Ker\ \beta'}]_{\widehat{K}_{GS}} - \widetilde{ch}(\widehat{\mathcal{B}}) - ([\widehat{E'}]_{\widehat{K}_{GS}} + [\widehat{Ker\ \alpha'}]_{\widehat{K}_{GS}} - \widetilde{ch}(\widehat{\mathcal{A}}) =$$

$$[\widehat{F'}]_{\widehat{K}_{GS}} - [\widehat{E'}]_{\widehat{K}_{GS}}$$

since the Bott-Chern forms are equal by 5.2. Replacing $\widehat{E}', \widehat{F}'$ by $\widehat{E}'', \widehat{F}''$ and going through the same computation then settles the first question.
As to the second question, it amounts to show that π is additive on s.e.s modulo a Bott-Chern form by the universality of the group. We proceed to prove this in two steps. First we prove that it is merely additive on orthogonally splitting s.e.s. Let

$$\widehat{\mathcal{M}} : 0 \to \widehat{M'} \to \widehat{M} \xrightarrow{\widehat{g}} \widehat{M''} \to 0 \tag{17}$$

be such a sequence. Choose a metrized projective resolution

$$0 \to \widehat{E} \to \widehat{F} \xrightarrow{\widehat{f}} \widehat{M} \to 0$$

of $\widehat{M}$. Then

$$0 \to \widehat{E}'' \to \widehat{F} \xrightarrow{\widehat{f''}} \widehat{M''} \to 0$$

$(f'' = g \circ f,\ E'' = Ker\ f'')$
is also a metrized projective resolution. $\widehat{E}''$ carries the metric induced by $\widehat{F}$. By construction, we obtain a commutative diagram of non-metrized modules with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & E & \to & F & \xrightarrow{f} & M & \to & 0 \\ & & \downarrow & & \downarrow Id. & & \downarrow g & & \\ 0 & \to & E'' & \to & F & \xrightarrow{f''} & M'' & \to & 0 \end{array}$$

The Snake Lemma (cf. [9, p. 337]) then immediately yields the algebraic exactness of the following sequence:

$$0 \to \widehat{E} \to \widehat{E}'' \xrightarrow{\widehat{f}} \widehat{M'} \to 0$$

(the snake sequence is $0 \to M' \to E''/E \mapsto 0$)
Actually, it splits orthogonally, as can easily be checked. With these resolutions of $\widehat{M}', \widehat{M}, \widehat{M}''$ in hand, the additivity is clear.
Now to the general case: suppose we are given a sequence like 17, that is only algebraically exact. Then leaving the metric on $\widehat{M}$ fixed, we can uniquely prescribe new metrics on $\widehat{M}'$ and $\widehat{M}''$, obtaining new metrized modules $\widehat{M}_1'$ and $\widehat{M}_1''$ so that the sequence splits orthogonally. We can then draw the following commutative diagram with algebraically exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & \widehat{M}' & \to & \widehat{M} & \to & \widehat{M}'' & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & \widehat{M}_1' & \to & \widehat{M} & \to & \widehat{M}_1'' & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

If we apply 5.2 to this diagram, we readily see that the computation of the Bott-Chern form of the first row corresponds to the computation of the Bott-Chern form of first and last column. So it should be clear that if we can achieve additivity modulo Bott-Chern on the first and last column only, we would obtain the full additivity modulo Bott-Chern. We prove this in a sublemma:

Lemma .4 *Let $\widehat{\mathcal{M}} : 0 \to \widehat{M'} \to \widehat{M_1'} \to 0$ be an algebraically exact sequence of metrized modules. Then $\pi(\widehat{M}') = \pi(\widehat{M}_1') + \widetilde{ch}(\widehat{\mathcal{M}})$.*

Proof: Just apply 5.2 to the following diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \to & \widehat{E}' & \stackrel{id.}{\to} & \widehat{E}_1' & \to & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \to & \widehat{F}' & \stackrel{id.}{\to} & \widehat{F}_1' & \to & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \to & \widehat{M'} & \stackrel{id.}{\to} & \widehat{M_1'} & \to & 0 \end{array}$$

where the columns are metrized projective resolutions. ♣

♣

Appendix B

.1 More about the arithmetic Chow groups

How can we describe the first arithmetic Chow group in terms of the ideal-class group ? We are provided with a natural forgetful mapping from the latter to the former; its kernel can be described as follows:

Theorem .5 *The following sequence is exact:*

$$1 \to \mu(\mathcal{O}_K) \to \mathcal{O}_K{}^* \xrightarrow{\rho} \mathbb{R}^{X'_\infty} \xrightarrow{a} \widehat{CH}^1(X) \xrightarrow{J} Cl(X) \to 1 \qquad (18)$$

where $\mu(\mathcal{O}_K)$ is the (cyclic) group of roots of unity lying in $\mathcal{O}_K$, $\mathcal{O}_K{}^*$ the group of units of $\mathcal{O}_K$. $Cl(X)$ is the usual ideal class group of $\mathcal{O}_K$.
The mappings are defined as follows:

$$\rho(\alpha) := (-2.\log(|\sigma(\alpha)|))_\sigma$$

($\alpha \in \mathcal{O}_K{}^*, \sigma \in X'_\infty$)

$$a(g) := [(\mathcal{O}_K \oplus (g_\sigma)_\sigma)]_{\widehat{CH}^1}$$

($g \in \mathbb{R}^{X'_\infty}$) and

$$J([I \oplus (g_\sigma)_\sigma]_{\widehat{CH}^1}) := [I]_{Cl(X)}$$

Proof: The exactness at the end of the two first arrows is clear (see the proof of Dirichlet's Unit Theorem, cf. [15]). The exactness at the end of the third arrow may be reformulated as $(g \in \Gamma) \iff \exists a \in K^*(div(a) \stackrel{def.}{=} ((a) \oplus (-2.\log(|\sigma(a)|))_\sigma) = (\mathcal{O}_K \oplus (g_\sigma)_\sigma)$. This is clear, since $(a) = \mathcal{O}_K \iff a \in \mathcal{O}_K{}^*$ and then $\rho(a) \in \Gamma$. The exactness at the end of the 4th arrow amounts to the statement $\exists g \in \mathbb{R}^{X'_\infty}([\mathcal{O}_K \oplus (g_\sigma)_\sigma]_{\widehat{CH}^1} = [I \oplus (g'_\sigma)_\sigma]_{\widehat{CH}^1}) \iff \exists a \in K^*(I = (a))$ (for some fractional ideal I of $\mathcal{O}_K$, and $(g'_\sigma)_\sigma \in \mathbb{R}^{X'_\infty}$). For the $\leftarrow$ direction, choose $g = g' - (-2.\log(|\sigma(a)|))_\sigma$, if some a is given. The $\to$ direction is obvious. The exactness at the end of the 5th arrow is clear. ♣

It is clear that we may topologize $\widehat{Z}^1(X)$, with the discrete topology on the first factor and the usual one on the second one. A topological characterization of $div(K^*)$ runs as follows:

Proposition .6 *$div(K^*)$ lies discretly in $\widehat{Z}^1(X)$*

Proof: Since $Z^1(X)$ is discrete, it is obviously sufficient to prove that $div(K^*) \cap (I \oplus \mathbb{R}^{X'_\infty})$ lies discretly in $(I \oplus \mathbb{R}^{X'_\infty})$, for any fractional ideal I. We can even replace I by $\mathcal{O}_K$, since multiplication by a trivially metrized $\widehat{I}$ is an homeomorphism of $\widehat{Z}^1$ onto itself. The statement for $I = \mathcal{O}_K$ is clear: the only $k \in K^*$ sent within $(I \oplus \mathbb{R}^{X'_\infty})$ are the units of $\mathcal{O}_K$. They form a lattice in that set and by definition a lattice is a discrete set. ♣

Note The former proposition implies that we have $\mathbb{R}^{X'_\infty}/\rho(\mathcal{O}_K{}^*) \oplus Cl(X) \simeq \widehat{CH}^1(X)$, since $\mathbb{R}^{X'_\infty}/\rho(\mathcal{O}_K{}^*)$ is a divisible abelian group, which is therefore injective.

Note 2 The latter proposition implies that $\widehat{CH}^1(X)$ has a natural structure of Lie group (it has a natural differentiable structure as a quotient manifold, since by .6 $div(K^*)$ is closed). Its Lie algebra is isomorphic to $\mathbb{R}^{X'_\infty}$ as a trivial Lie algebra, since the group is commutative. With that setting, the mapping a is the exponential mapping, since it is a mapping onto the quotient. As a corollary, we trivially obtain that a is differentiable.

We can compare the Chow groups with the much older idele groups (for basic definitions see [12, Par. 3, Ch. VII]). In fact we have

Theorem .7 $\widehat{CH}^1(X) \simeq I_K/(I_K^0.K^*)$
the isomorphism being canonical and topological.

I_K^0 is to be defined.

Proof: Let M_K be the set of all normalized (so that their restriction to $\mathbb{Q}$ is standard) absolute values of K, M_K^1 the set of all non-archimedean normalized absolute values of K. We define a mapping

$$U : I_K \to \widehat{Z}^1(X)$$

$$U((a_p)_{p \in M_K}) = (\nu_p(a))_{p \in M_K^1} \oplus (2.\nu_{p_\sigma}(a))_{\sigma \in X'_\infty}$$

(the factor 2 has only been introduced, because we want to interpret ν_{p_σ} as the norm of the standard basis at σ - and not as its square)
where we implicitly use the fact that (1) the group of fractional ideals is naturally isomorphic to the free group generated by all the prime ideals of $\mathcal{O}_K$ (2) each normalized non-archimedean absolute value corresponds to exactly one prime ideal (3) each normalized archimedean absolute value (denoted by p_σ) corresponds to exactly one pair of conjugate embeddings $\sigma, \overline{\sigma}$.
U is clearly continuous (since the image carries the product topology, we must only check for each absolute value separately and then it is obvious). It follows readily from the definition that this mapping also sends K^* (as embedded in the idele group) on $div(K^*)$. We can therefore consider U as a mapping $I_K/K^* \to \widehat{CH}^1(X)$. What is its kernel ? We have

$$ker\ U =: I_K^0 = \{(a_p)_p \in I_K : \nu_p(a_p) = 0\ \forall p\}$$

This yields the result. ♣

From this proposition, one can infer the compactness of the kernal of *deg*. Indeed, it is easily checked (just compare the formulae) that the degree mapping onI_K (minus the logarithm of the norm) is compatible through U with *deg*, so that the kernal of *deg* is just the image through U of the kernal of the degree mapping on I_K. This last kernal is compact, as is well-known (cf. [12]).

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