

HALL ALGEBRAS

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1. INTRODUCTION

1.1. **History.** The classical Hall-Steinitz algebra of partitions goes back to Steinitz (1901) and Hall(1957). The authors were interested in lattices of abelian p -groups.

In 1990, Ringel defined the Hall algebra of an abelian category (with suitable finiteness conditions). Starting with the quiver algebra of a Dynkin diagram, this gives the positive nilpotent half of the corresponding quantum group.

In 1995, Green defined a coproduct on Hall algebras. This helped clarify the relation with quantum groups.

Around 1995, Xiao, Peng-Xiao and Kapranov constructed Hall algebras for certain versions of derived categories of hereditary algebras to construct the full quantum group.

In 2005, Toen defined the Hall algebra of a triangulated category via homotopical methods. This was simplified by Xiao and Xu in 2006.

Let us conclude this quick historical survey by some of the current directions of investigations.

In 2005, Joyce exhibited relations with Bridgeland's space of stability conditions.

Work of Caldero, Chapoton and Keller has shown close connections between cluster categories and Hall algebras.

Work in progress of the lecturer points in the direction of a different approach of Hall algebras based on higher representation theory (aka categorification).

There are geometrical approaches to Hall algebras. They eventually led to constructions of representations.

Lusztig constructed canonical bases in representations of Kac-Moody algebras using perverse sheaves on moduli spaces of representations of quivers (note that Kashiwara constructed independently such bases by combinatorial methods).

Nakajima constructed representations of (quantum) Kac-Moody algebras in the cohomology (equivariant K-theory) of moduli spaces of representations of "framed double quivers".

2. HALL-STEINITZ ALGEBRA OF PARTITIONS

2.1. Finite representations of discrete valuation rings.

2.1.1. *Discrete valuation rings.* Let us consider a discrete valuation ring \mathcal{O} with maximal ideal \mathfrak{p} and finite residue field $k = \mathcal{O}/\mathfrak{p}$ of characteristic p and with q elements. Recall that a discrete valuation ring is a principal ring with a unique non-zero prime ideal.

The theory gives the same result for \mathcal{O} replaced by its completion. Note that a complete discrete valuation ring with residue field k is

- $k[[t]]$ (here $\mathfrak{p} = tk[[t]]$) (type 1) or
- the ring of integers of a finite extension of the field of p -adic numbers \mathbf{Q}_p . For example $\mathcal{O} = \mathbf{Z}_p$ and $\mathfrak{p} = p\mathbf{Z}_p$ (type 2).

2.1.2. *Finite representations.* We will consider the category \mathcal{A} of finite \mathcal{O} -modules (we mean modules with finitely many elements). It has a unique simple object, namely k . Given $M \in \mathcal{A}$, we denote by $l(M)$ the length of a composition series of M . Note that $|M| = q^{l(M)}$.

In the type 1, a finite \mathcal{O} -module is a finite dimensional vector space V over k together with a nilpotent endomorphism.

Consider the type 2, *i.e.*, $\mathcal{O} = \mathbf{Z}_p$ the ring of p -adic numbers. Given a finite \mathcal{O} -module V , there is n such that $p^n\mathcal{O}$ acts by 0 on V , hence V is actually a finitely generated $\mathbf{Z}/(p^n)$ -module. So, a finite \mathcal{O} -module is a finitely generated abelian p -group. This was the case of interest to Steinitz and Hall.

2.1.3. *Classification.* Since \mathcal{O} is a principal ideal domain, every finitely generated module is a direct sum of cyclic modules. Here:

Theorem 2.1 (Jordan, Kronecker). *The indecomposable objects of \mathcal{A} are, up to isomorphism, the $\mathcal{O}/\mathfrak{p}^n$ with $n \geq 1$.*

Given $M \in \mathcal{O}$, there are unique integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m > 0$ such that

$$M \simeq \mathcal{O}/\mathfrak{p}^{\lambda_1} \oplus \dots \oplus \mathcal{O}/\mathfrak{p}^{\lambda_m}.$$

This Theorem gives a bijection between partitions and isomorphism classes of objects of \mathcal{A} : given $\lambda = (\lambda_1 \geq \dots \geq \lambda_m > 0)$ a partition, we put $M(\lambda) = \mathcal{O}/\mathfrak{p}^{\lambda_1} \oplus \dots \oplus \mathcal{O}/\mathfrak{p}^{\lambda_m}$.

2.2. Hall-Steinitz algebra.

2.2.1. *Definition.* Define $\mathcal{H} = \bigoplus_{\lambda} \mathbf{Z}u_{\lambda}$, a free \mathbf{Z} -module with basis elements indexed by partitions. Let us put an algebra structure on \mathcal{H} .

Let λ, μ, ν be three partitions. We denote by $F_{\lambda, \mu}^{\nu}$ the number of submodules N of $M(\nu)$ such that $N \simeq M(\mu)$ and $M(\nu)/N \simeq M(\lambda)$.

We define

$$u_{\lambda} * u_{\mu} = \sum_{\nu} F_{\lambda, \mu}^{\nu} u_{\nu}$$

2.2.2. *Iterated products.* The element u_{\emptyset} is the identity of the multiplication.

Let us now show that the multiplication is associative. We have

$$u_{\beta} * (u_{\lambda} * u_{\mu}) = \sum_{\nu'} \sum_{\nu} F_{\lambda, \mu}^{\nu} F_{\beta, \nu}^{\nu'} u_{\nu'}$$

while

$$(u_{\beta} * u_{\lambda}) * u_{\mu} = \sum_{\nu'} \sum_{\nu} F_{\beta, \lambda}^{\nu} F_{\nu, \mu}^{\nu'} u_{\nu'}$$

The sum $\sum_{\nu} F_{\lambda, \mu}^{\nu} F_{\beta, \nu}^{\nu'}$ computes the number of submodules $L' \subset L \subset M(\nu')$ such that $L' \simeq M(\mu)$, $L/L' \simeq M(\lambda)$ and $M(\nu')/L \simeq M(\beta)$. The sum $\sum_{\nu} F_{\beta, \lambda}^{\nu} F_{\nu, \mu}^{\nu'}$ computes the number of pairs (L', L_1) , where L' is a submodule of $M(\nu')$ and $L' \simeq M(\mu)$ and L_1 is a submodule of $M(\nu)/L'$ with $L_1 \simeq M(\lambda)$ and $(M(\nu)/L')/L_1 \simeq M(\beta)$. So, $u_{\beta} * (u_{\lambda} * u_{\mu}) = (u_{\beta} * u_{\lambda}) * u_{\mu}$.

$$\begin{array}{ccccccc} 0 & \subset & L' & \subset & L & \subset & M(\nu') \\ & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ & M(\mu) & & M(\lambda) & & M(\beta) & \end{array}$$

Iterated products compute number of filtrations where the isomorphism type of the successive quotients is prescribed:

Proposition 2.2. *Given $\lambda^1, \dots, \lambda^n$ partitions, we have*

$$u_{\lambda^1} * \dots * u_{\lambda^n} = \sum_{\nu} F_{\lambda^1, \dots, \lambda^n}^{\nu},$$

where $F_{\lambda^1, \dots, \lambda^n}^{\nu}$ is the number of filtrations

$$M(\nu) = M_0 \supset \dots \supset M_n = 0$$

with $M_{i-1}/M_i \simeq M(\lambda^i)$.

2.2.3. *Duality.* We have a duality $M \mapsto M^{\vee}$ on the category \mathcal{A} , given by the functor

$$\mathrm{Hom}_{\mathcal{O}}(-, \mathrm{colim}_n \mathcal{O}/\mathfrak{p}^n) = \mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}}(-, \mathcal{O}/\mathfrak{p}^n).$$

In type 1, this is the same as taking the k -linear dual. In type 2, we are taking the dual of a finite abelian p -group.

Since $(\mathcal{O}/\mathfrak{p}^n)^{\vee} \simeq \mathcal{O}/\mathfrak{p}^n$, we deduce that the multiplication is commutative.

2.2.4. *Algebra structure.*

Lemma 2.3. *Let λ be a partition and λ' be the transposed partition. Then, $\lambda'_i = \dim_k(\mathfrak{p}^{i-1}M/\mathfrak{p}^iM)$.*

Theorem 2.4. *The canonical map $\mathbf{Z}[u_1, u_{12}, \dots] \rightarrow \mathcal{H}$ is an isomorphism of algebras.*

Proof. Given a partition λ , let

$$x_{\lambda} = u_{1\lambda'_1} \cdots u_{s\lambda'_s} = \sum_{\nu} a_{\lambda, \nu} u_{\nu}$$

where a_{ν} is the number of filtrations

$$M(\nu) = M_0 \supset M_1 \supset \dots \supset M_s = 0$$

such that $M_{i-1}/M_i \simeq k^{\lambda'_i}$. Given such a filtration, we have $\mathfrak{p}M_{i-1} \subset M_i$, hence $\mathfrak{p}^iM \subset M_i$. It follows that $l(M/\mathfrak{p}^iM) \geq l(M/M_i)$, hence

$$\nu'_1 + \dots + \nu'_i \geq \lambda'_1 + \dots + \lambda'_i$$

by Lemma 2.3. So, if $a_{\lambda, \nu} \neq 0$, then $\nu' \geq \lambda'$. On the other hand, $a_{\lambda, \lambda} = 1$, hence the matrix expressing the x_{λ} 's in the basis of u_{ν} 's is unitriangular for the order on transposed partitions. \square

We have an isomorphism of algebras

$$\phi : \mathbf{Q} \otimes_{\mathbf{Z}} \mathcal{H} \xrightarrow{\sim} \mathcal{S}, \quad u_{1^r} \mapsto q^{-r(r-1)/2} e_r.$$

Theorem 2.7. *The morphism ϕ is an isomorphism of bialgebras.*

Proof. We have $\Delta(e_n) = \sum_{i=0}^n e_i \otimes e_{n-i}$. So, Lemma 2.6 shows that ϕ is compatible with coproducts. \square

Remark 2.8. The set $\{\phi(u_\lambda)\}$ is the basis of Hall-Littlewood functions.

Remark 2.9. There is an isomorphism

$$\mathbf{Q} \otimes_{\mathbf{Z}} \bigoplus_{n \geq 0} K_0(\mathbf{C}\mathfrak{S}_n) \xrightarrow{\sim} \mathcal{S}$$

sending an irreducible character χ_λ of a symmetric group to the corresponding symmetric function s_λ . Under this isomorphism, the product corresponds to induction $\text{Ind}_{\mathfrak{S}_i \times \mathfrak{S}_j}^{\mathfrak{S}_{i+j}}$, while the coproduct is a sum of restrictions $\bigoplus_{i=0}^n \text{Res}_{\mathfrak{S}_i \times \mathfrak{S}_{n-i}}^{\mathfrak{S}_n}$.

Exercice 2.1. Prove Lemmas 2.3 and 2.6.

3. HALL ALGEBRAS AND QUIVER REPRESENTATIONS

Before starting the general study of Hall algebras associated with abelian categories, we introduce our main source of examples, namely representations of quivers.

3.1. Quivers, beginning. Let Q be a *quiver* (= an oriented graph), *i.e.*,

- a finite set Q_0 (the vertices)
- a finite set Q_1 (the arrows)
- maps $p, q : Q_1 \rightarrow Q_0$ (tail=source and head=target of an arrow).

Let k be a field. A *representation* of Q over k is the data of $(V_s, \phi_a)_{s \in Q_0, a \in Q_1}$ where V_s is a k -vector space and $\phi_a \in \text{Hom}_k(V_{p(a)}, V_{q(a)})$.

A morphism from $(V_s, \phi_a)_{s,a}$ to $(V'_s, \phi'_a)_{s,a}$ is the data of a family $(f_s)_{s \in Q_0}$, where $f_s \in \text{Hom}_k(V_s, V'_s)$, such that for all $a \in Q_1$, the following diagram commutes:

$$\begin{array}{ccc} V_{p(a)} & \xrightarrow{\phi_a} & V_{q(a)} \\ f_{p(a)} \downarrow & & \downarrow f_{q(a)} \\ V'_{p(a)} & \xrightarrow{\phi'_a} & V'_{q(a)} \end{array}$$

The *quiver algebra* $k(Q)$ associated to Q is the k -algebra generated by the set $Q_0 \cup Q_1$ with relations

$$as = \delta_{q(a),s} a, \quad sa = \delta_{p(a),s} a, \quad ss' = \delta_{s,s'} s \quad \text{for } s, s' \in Q_0 \text{ and } a \in Q_1 \quad \text{and} \quad 1 = \sum_{t \in Q_0} t$$

Let $\gamma = (s_1, a_1, s_2, a_2, \dots, s_n)$ be a *path* in Q , *i.e.*, a sequence of vertices $s_i \in Q_0$ and arrows $a_i \in Q_1$ such that $p(a_i) = s_i$ and $q(a_i) = s_{i+1}$. We put $\tilde{\gamma} = s_1 a_1 s_2 \cdots a_{n-1} s_n \in k(Q)$.

The following Proposition is clear.

Proposition 3.1. *The set of $\tilde{\gamma}$, where γ runs over the set of paths of Q , is a basis of $k(Q)$.*

Note that $k(Q)$ is a graded algebra, with Q_0 in degree 0 and Q_1 in degree 1. In general, a path of length n is homogeneous of degree n .

There is an equivalence from the category of representations of Q to the category of right $k(Q)$ -modules: given (V_s, ϕ_a) a representation of Q , let $M = \bigoplus_s V_s$. We define a structure of right $k(Q)$ -module as follows: $s \in Q_0$ is the projection onto V_s . An element $a \in Q_1$ acts by zero on $\bigoplus_{s \neq p(a)} V_s$ and sends $V_{p(a)}$ to $V_{q(a)}$ via ϕ_a .

Given $s \in Q_0$, there is a simple representation $V = V(s)$ of Q given by $V_t = 0$ for $t \neq s$, $V_s = k$ and $\phi_a = 0$ for all $a \in Q_1$. When $k(Q)$ is finite dimensional, we obtain all simple representations of Q , up to isomorphism.

Example 3.2. For each of the following quivers, we give the list of finite dimensional indecomposable representations (up to isomorphism) and we indicate the isomorphism type of the quiver algebra.

- (i) $\bullet : (k)$. The quiver algebra is k .
- (ii) $1 \longrightarrow 2$: $V(1) = (V_1 = k, V_2 = 0, \phi = 0)$, $V(2) = (V_1 = 0, V_2 = k, \phi = 0)$ and $M = (V_1 = k, V_2 = k, \phi = 1)$. The quiver algebra is isomorphic to the algebra of 2×2 upper triangular matrices.

(iii) $\curvearrowright \bullet : (k^n, \phi(n, \lambda))_{n \geq 1, \lambda \in k}$ with $\phi(n, \lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$, assuming k algebraically

closed. The quiver algebra is isomorphic to $k[x]$.

Exercise 3.1. A representation of Q is said to be nilpotent if every loop $\tilde{\gamma} = (s_1 a_1 \cdots a_{n-1} s_1)$ acts nilpotently. Show that every simple nilpotent representation of a quiver is isomorphic to $V(s)$ for some vertex s .

Exercise 3.2. Check the statements of Example 3.2.

3.2. Definition of Hall algebras.

3.2.1. *Finitary categories.* Let \mathcal{A} be an abelian category: typical examples are representations of algebras or representations of quivers, coherent sheaves on an algebraic variety.

Let us assume that \mathcal{A} is *finitary*, i.e., for all $M, N \in \mathcal{A}$, then $\text{Hom}_{\mathcal{A}}(M, N)$ and $\text{Ext}_{\mathcal{A}}^1(M, N)$ are finite sets.

Recall that $\text{Ext}_{\mathcal{A}}^1(M, N)$ is the set of exact sequences $0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$ modulo equivalence. Two exact sequences $0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$ and $0 \rightarrow N' \rightarrow L' \rightarrow M' \rightarrow 0$ are equivalent if there is an isomorphism $f : L \xrightarrow{\sim} L'$ making the following diagram commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & M \longrightarrow 0 \\ & & \parallel & & \downarrow f \sim & & \parallel \\ 0 & \longrightarrow & N' & \longrightarrow & L' & \longrightarrow & M' \longrightarrow 0 \end{array}$$

Exercise 3.3. Show that the following categories are finitary:

- (a) representations of a quiver over a finite field
- (b) finite representations of a noetherian ring.

3.2.2. *Hall algebras.* Given $M, N \in \mathcal{A}$, let $F_{M,N}^L$ be the number of submodules N' of L such that $N' \simeq N$ and $L/N' \simeq M$.

Let $P_{M,N}^L$ denote the number of exact sequences $0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$. Then,

$$(1) \quad F_{M,N}^L = \frac{P_{M,N}^L}{|\mathrm{Aut}(M)| \cdot |\mathrm{Aut}(N)|}.$$

Let $H'_\mathcal{A}$ be the free abelian group with basis the isomorphism classes of objects of \mathcal{A}

$$H'_\mathcal{A} = \bigoplus_{[L] \in \mathcal{A}/\sim} \mathbf{Z}[L].$$

We define a product in $H'_\mathcal{A}$ by

$$[M] * [N] = \sum_{[L] \in \mathcal{A}/\sim} F_{M,N}^L [L].$$

The class $[0]$ is a unit for the product. As in §2.2.2, the product is shown to be associative. The algebra $H'_\mathcal{A}$ is the *Hall algebra* of \mathcal{A} .

Given $N_1, \dots, N_n, L \in \mathcal{A}$, let F_{N_1, \dots, N_n}^L be the number of filtrations

$$L = L_0 \supset \dots \supset L_n = 0$$

with $L_{i-1}/L_i \simeq N_i$. Then, Proposition 2.2 generalizes:

$$\mathbf{Proposition 3.3.} \quad \text{We have } [N_1] * \dots * [N_n] = \sum_{[L] \in \mathcal{A}/\sim} F_{N_1, \dots, N_n}^L [L].$$

We say that A is a *finitary algebra* if its category \mathcal{A} of finite modules is finitary. When this happens, we put $H'_A = H'_\mathcal{A}$.

Proposition 3.4. *Let A and B be two finitary rings. Then, we have an isomorphism of rings*

$$H'_A \otimes_{\mathbf{Z}} H'_B \xrightarrow{\sim} H'_{A \times B}, \quad [M] \otimes [N] \mapsto [M \oplus N].$$

Note also that H'_A depends only on the category of finite representations of A . So,

Proposition 3.5. *If A and B are two Morita-equivalent finitary algebras, then their Hall algebras are isomorphic.*

Example 3.6. Let \mathcal{A} be the category of finite representations of a discrete valuation ring with finite residue field. Then, $H'_\mathcal{A}$ is the Hall-Steinitz algebra of §2.

Example 3.7. Let \mathcal{A} be a semi-simple category. Then, $H'_\mathcal{A}$ is commutative. Consider the case $A = \mathbf{F}_q$ is a finite field with q elements. Then, $[k^n] * [k^m] = \binom{m+n}{n}_q [k^{m+n}]$ where $\binom{r}{s}_q = \frac{[r-s+1]_q \dots [r-1]_q [r]_q}{[2]_q \dots [s]_q}$ and $[i]_q = \frac{q^i - 1}{q - 1}$.

Example 3.8. Let \mathcal{A} be the category of finite representations of the quiver $1 \longrightarrow 2$ over \mathbf{F}_q (cf Example 3.2, (ii)). Let $e_1 = [V(1)]$, $e_2 = [V(2)]$ and $e_{12} = [M]$. We find $e_1 * e_2 = e_{12} + [V(0) \oplus V(1)]$ and $e_2 * e_1 = [V(0) \oplus V(1)]$. The algebra $H'_\mathcal{A}$ is not commutative. We have $[e_1, e_2] = e_{12}$.

We have $e_1 * e_{12} = q[M \oplus V(1)]$ and $e_{12} * e_1 = [M \oplus V(1)]$. So, $e_1 * e_{12} = qe_{12} * e_1$. If we view q as an indeterminate and specialize it to 1, then the Lie subalgebra of H'_A generated by e_1, e_2 and e_{12} is isomorphic to the Lie algebra of strictly upper triangular 3×3 -matrices:

$$e_1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{12} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Exercise 3.4. Prove the equality (1).

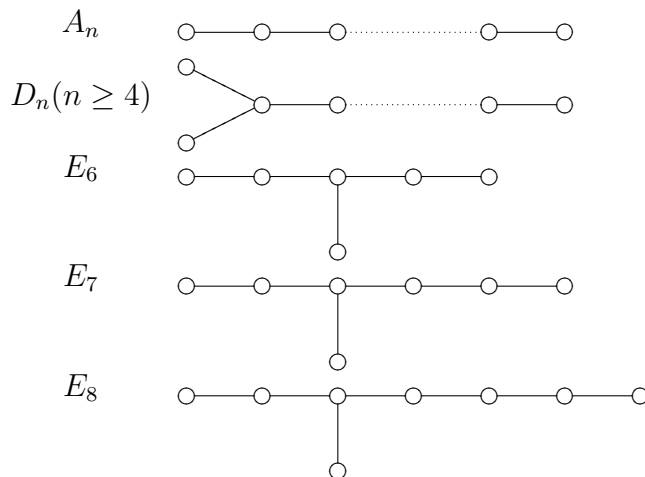
An important Theorem to be proved later is the existence of Hall polynomials, for Hall algebras associated to quivers. Our main interest will be the study of the Hall algebra associated to a Dynkin quiver (type ADE). We will show it is isomorphic to $U_q(\mathfrak{n}_+)$, the positive nilpotent part of the corresponding quantum group.

3.3. Quivers, continued. We continue our study of quivers. The main theme is the link between representations of ADE quivers and corresponding simple complex Lie algebras.

3.3.1. ADE classifications. Consider the classifications of the following

- Thom’s catastrophe theory
- Surface singularities
- Quivers with finite representation type
- Regular solids
- Simple connected Lie groups
- Finite simple groups.

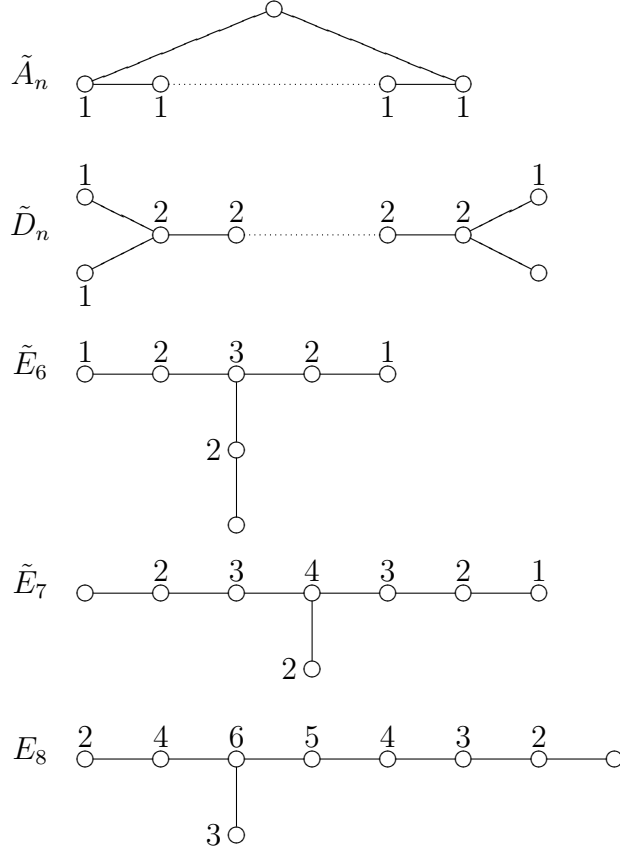
The classifications rely on the same A-D-E pattern. The objects are classified (up to some exceptional cases) by a diagram of type ADE, with possibly some additional data.



Note that the index attached to the name of the diagram is the number of vertices.

Let Γ be a graph (with no loops): this is the data of Q_0 (vertices), Q_1 (edges) and a map from Q_1 to the set of 2-element subsets of Q_0 (endpoints of the edge). We put $V_Q = \mathbf{R}^{Q_0}$ and denote by $\{e_s\}_{s \in Q_0}$ the canonical basis.

3.3.2. *Affine Dynkin diagrams.* The following graphs $\tilde{\Gamma}$ come with a subgraph Γ of type ADE (vertices with a coefficient). The *longest root* α_0 is the element of \mathbf{R}^{Q_0} whose coefficients are provided with the graph. We denote by $\tilde{\alpha}$ the additional vertex, the unique element of $\tilde{Q}_0 - Q_0$.



Remark 3.9. There is a unique way to complete (=add a vertex to) a Dynkin diagram of type ADE so that the new diagram is not a Dynkin diagram anymore and so that by removing any vertex of the completed graph, we obtain a Dynkin diagram.

Remark 3.10. The graph A_0 is defined to be the empty graph and \tilde{A}_0 is the graph with one vertex and one arrow.

3.3.3. *Quadratic form associated to a graph.* We define a quadratic form q on V by

$$q\left(\sum_s x_s e_s\right) = \sum_{s \in Q_0} x_s^2 - \sum_{\{t,t'\} \in Q_1} x_t x_{t'}.$$

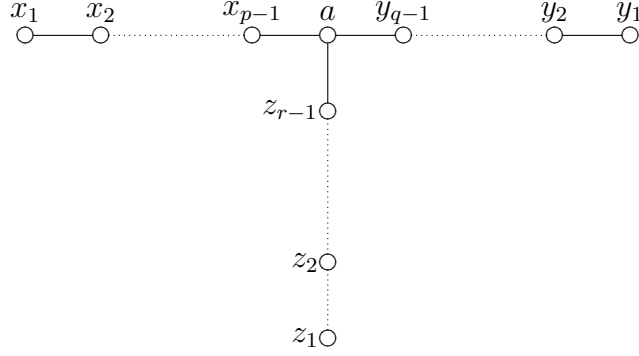
Let us compute the quadratic form in some simple cases.

Example 3.11. Graph $K(n)$ with 2 vertices and n edges



We have $q(x, y) = x^2 + y^2 - nxy$.

Graph with a vertex of valency 3 and all other vertices with valency 1 or 2.

$T(p, q, r)$


We have

$$q(x_i, y_i, z_i, a) = C_p(x_1, \dots, x_{p-1}, a) + C_q(y_1, \dots, y_{q-1}, a) + C_r(z_1, \dots, z_{r-1}, a) + \frac{a^2}{2} \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 \right)$$

where

$$C_n(t_1, \dots, t_n) = \sum_{i=1}^{n-1} \frac{i}{2(i+1)} \left(t_{i+1} - \frac{i+1}{i} t_i \right)^2.$$

So, q is positive definite if and only if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. It is non-negative if and only if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

Theorem 3.12. *Let Γ be a connected graph. Then, Γ is a Dynkin diagram of type ADE if and only if q is positive definite. The graph Γ is a completed Dynkin diagram of type $\tilde{A}\tilde{D}\tilde{E}$ if and only if q is positive semi-definite.*

Proof. The graphs of type ADE are characterized by the property of not having a subgraph of one of the following types

- $J(n)$, $n \geq 1$
- $K(n)$, $n \geq 2$
- \tilde{A}_n , $n \geq 2$
- \tilde{E}_n , $n = 6, 7, 8$
- \tilde{D}_n , $n \geq 4$.

For these graphs, q is not positive definite. So, if q is positive definite, then Γ is a Dynkin diagram of type ADE .

The study of the graph $T(p, q, r)$ (Example 3.11) shows that q_Γ is positive definite if and only if Γ has type ADE . This shows the first part of the Theorem.

Let Γ be of type ADE with completed graph $\tilde{\Gamma}$. Since $\tilde{q}(\alpha_0 + \tilde{\alpha}) = 0$, the quadratic form \tilde{q} associated with $\tilde{\Gamma}$ is positive semi-definite.

Let Γ' be a graph with a number of vertices minimal for the property that the associated quadratic form is positive semi-definite and assume Γ' is not of type ADE or $\tilde{A}\tilde{D}\tilde{E}$. Then, the subgraphs obtained by removing one vertex have type ADE or $\tilde{A}\tilde{D}\tilde{E}$. By Remark 3.9, one of those subgraphs is $\tilde{\Gamma}$ of type $\tilde{A}\tilde{D}\tilde{E}$.

The set of vertices of Γ' is $Q'_0 = \tilde{Q}_0 \cup \{z\}$ and the subgraph with vertices \tilde{Q}_0 is $\tilde{\Gamma}$. Then, $q'(2(\alpha_0 + \tilde{\alpha}) + e_z) \leq 1 - 2 \sum_{\{z,s\} \in Q'_1} x_s < 0$, where $\alpha_0 + \tilde{\alpha} = \sum_{s \in \tilde{Q}_0} x_s e_s$. So, the quadratic form q' associated with Γ' is not positive semi-definite.

It follows that if q is positive semi-definite, then Γ has type ADE or $\tilde{A}\tilde{D}\tilde{E}$. \square

3.3.4. *Quadratic form of a quiver.* Let k be a field, Q a quiver with underlying graph Γ and associated quadratic form q . We will give a geometrical and a homological interpretation of q .

Fix a non-zero map $d : Q_0 \rightarrow \mathbf{Z}_{\geq 0}$. The representations of Q over k of the form $(k^{d(s)}, \phi_a)$ are the k -points of the algebraic variety $\text{Rep}_d(Q) = \prod_a \mathbf{A}^{d(q(a)) \times d(p(a))}$. We denote by $V(\phi)$ the representation of Q corresponding to $\phi \in \text{Rep}_d(Q)$.

The group $G_d = \prod_{s \in Q_0} \text{GL}_{d(s)}$ acts on $\text{Rep}_d(Q)$ by $(\phi_a)_a \mapsto (g_{q(a)} \phi_a g_{p(a)}^{-1})$ for $(g_s) \in G_d$. The action of G_d factors through an action of $\bar{G}_d = G_d / \mathbf{G}_m$ where $\mathbf{G}_m(k) = \{(\alpha \cdot \text{id}), \alpha \in k^\times\}$ is the multiplicative group embedded diagonally in G_d . We have

$$(2) \quad \dim \bar{G}_d - \dim \text{Rep}_d(Q) = q(d) - 1.$$

Two representations $V(\phi)$ and $V(\phi')$ are isomorphic if and only if ϕ and ϕ' belong to the same G_d -orbit.

The stabilizer of ϕ in G_d is $C_{G_d}(\phi) = \text{Aut}(V(\phi))$. There is an isomorphism between the orbit $G_d \cdot \phi$ of ϕ and $G_d / C_{G_d}(\phi)$. So,

$$(3) \quad \dim \text{Rep}_d(Q) - \dim(G_d \cdot \phi) = \dim \text{Aut}(V(\phi)) - q(d).$$

Proposition 3.13. *Assume Q has no cycle. Let V and W be two finite dimensional representations of Q over k . Then, $\text{Ext}^i(V, W) = 0$ for $i > 1$, i.e., the category of finite dimensional representations of Q is hereditary.*

We have

$$(4) \quad \dim \text{Hom}(V, W) - \dim \text{Ext}^1(V, W) = \sum_{s \in Q_0} d(V)_s d(W)_s - \sum_{(p \rightarrow q) \in Q_1} d(V)_p d(W)_q.$$

In particular, $q(d(V)) = \dim \text{End}(V) - \dim \text{Ext}^1(V, V)$.

Proof. Let s be a vertex of Q . Let $P(s) := s \cdot k(Q)$. This is a projective cover of the simple right $k(Q)$ -module $V(s)$. Note that $P(s)_t$ has a basis given by paths from t to s . Let M be the radical of $P(s)$. We have $M_t = P(s)_t$ for $t \neq s$ and $M_s = 0$. It follows that $M = \bigoplus_a a \cdot k(Q)$, where a runs over the arrows of Q ending at s . Since $a \cdot k(Q) \simeq P(p)$, where p is the head of a , it follows that M is projective. We deduce that a submodule of a finitely generated projective module is projective. It follows that $\text{Ext}^i(V, W) = 0$ for all $i \geq 2$ and all finite dimensional representations V and W .

Assume V and W are simple. Then, $\dim \text{Ext}^1(V, W)$ is the number of arrows from V to W . Formula (4) follows in that case.

Now, since $\text{Ext}^{\geq 2}$ vanishes, the form defined on the left is the Euler form and it factors through a bilinear form defined on the Grothendieck group of finite dimensional representations of Q over k (cf §4.2.1). The result follows. \square

3.3.5. *Gabriel's Theorem.*

Theorem 3.14. *Let Q be a quiver and k a field. Then, Q has only finitely many isomorphism classes of indecomposable finite dimensional representations if and only if the underlying graph is a Dynkin diagram.*

proof of \Rightarrow . We show the first implication: finite representation type quivers give Dynkin diagrams.

Note that if a vertex s has a loop a , then the quiver has indecomposable representations of every dimension: we obtain such representations by letting all other arrows act by zero and by using the construction of Example 3.2 (iii), with $\lambda = 0$.

Assume now that Q has no loop. In that case, we can proceed with the weaker assumption that Q has only finitely many indecomposable representations of a given dimension, up to isomorphism. So, Q has only finitely many representations of a given dimension, up to isomorphism. Equivalently, given a non zero element $d \in \mathbf{Z}_{\geq 0}^{Q_0}$, then \bar{G}_d has only finitely many orbits on $\text{Rep}_d(Q)$. Consequently, it admits an open orbit. It follows that $\dim \text{Rep}_d(Q) \leq \dim \bar{G}_d$. So, $q(d) \geq 1$ by the equality (2).

We have shown that q restricted to $\mathbf{Z}_{\geq 0}^{Q_0} - \{0\}$ takes positive values. Since the off-diagonal coefficients of the matrix of q are non-positive, it follows that q takes positive values on $\mathbf{Z}^{Q_0} - \{0\}$, hence on $\mathbf{Q}^{Q_0} - \{0\}$ and finally on $\mathbf{R}^{Q_0} - \{0\}$. \square

Given a graph Γ with no loop, we define a *positive root* to be an element $\alpha \in \mathbf{Z}_{\geq 0}^{Q_0}$ such that $q(\alpha) = 1$.

Given M a finite dimensional representation of Q over k , the *dimension vector* of M is

$$d(M) = (\dim M_s)_{s \in Q_0} \in \mathbf{Z}_{\geq 0}^{Q_0}.$$

The second half of Theorem 3.14 is made more precise by the following parametrization of indecomposable representations.

Theorem 3.15. *Let Q be a Dynkin quiver and k a field. The application $M \mapsto d(M)$ induces a bijection from the set of isomorphism classes of finite dimensional indecomposable representations of Q and the set of positive roots.*

Proof (case k algebraically closed). We assume k is algebraically closed.

Let V be an indecomposable finite dimensional representation of Q . We show first that $\text{End}(V) = k$. Let ϕ be a non-zero endomorphism of V with $\phi^2 = 0$, chosen so that $L = \text{im } \phi$ has minimal possible dimension. Let V' be an indecomposable direct summand of $\ker \phi$. The minimality of L shows that the composition $f : L \hookrightarrow \ker \phi \twoheadrightarrow V'$ is injective for some indecomposable direct summand V' of $\ker \phi$. Note that the restriction of the element of $\text{Ext}^1(L, \ker \phi)$ given by the canonical extension to $\text{Ext}^1(L, V')$ is non-zero, since V is indecomposable. Since $\text{Ext}^2(V'/L, V') = 0$, it follows that the canonical map $\text{Ext}^1(V', V') \rightarrow \text{Ext}^1(L, V')$ is surjective. So, $\text{Ext}^1(V', V') \neq 0$. By induction, we know that $\text{End}(V') = k$. By Proposition 3.13, we obtain $q(d(V')) \leq 0$. This is impossible, since q is positive definite (Theorem 3.12).

Let $d = d(V)$ and fix ϕ such that $V \simeq V(\phi)$. The equality (3) shows that $\dim \text{Rep}_d(Q) - \dim(G_d \cdot \phi) = 1 - q(d)$. Since q is positive definite (Theorem 3.12), it follows that $q(d) = 1$ and that $G_d \cdot \phi$ is a dense open orbit. We have already shown that d is a root. Now, there can't be more than one dense open orbit in $\text{Rep}_d(Q)$, hence all indecomposable representations with dimension vector d are isomorphic.

Fix now d a positive root. There are only finitely many indecomposable representations, so G_d has only finitely many orbits on $\text{Rep}_d(Q)$. So, one of the orbits must be open. Let ϕ be a point in the open orbit. The equality (3) shows that $\dim \text{Aut}(V(\phi)) = 1$. So, $V(\phi)$ is indecomposable. \square

We will denote by $M(\gamma)$ an indecomposable representation associated with a positive root γ (it is unique up to isomorphism).

Remark 3.16. Note that given an indecomposable representation M of a Dynkin quiver over a field k , then $\text{End}(M) = k$ and $\text{Ext}^1(M, M) = 0$.

4. HALL ALGEBRAS AND QUANTUM GROUPS

4.1. Quantum groups. In this section, we give a quick introduction without proofs to complex semi-simple Lie algebras and quantum groups.

4.1.1. Complex semi-simple Lie algebras. We provide here some key facts on semi-simple Lie algebras.

Let k be a field. Given a k -algebra A , we define a Lie algebra A_{Lie} with underlying k -module A and bracket $[a, b] = ab - ba$. This defines a functor from the category of algebras to the category of Lie algebras. That functor admits a left adjoint: it sends a Lie algebra \mathfrak{g} to its enveloping algebra $U(\mathfrak{g})$. Note that there is a canonical Lie algebra injection $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$.

Theorem 4.1 (Poincaré-Birkhoff-Witt). *Let \mathcal{B} be a basis of \mathfrak{g} . Put a total order on \mathcal{B} . Then, the family of monomials $b_1 \cdots b_n$ with $b_1 \leq \cdots \leq b_n$ in \mathcal{B} and $n \geq 0$ is a basis of $U(\mathfrak{g})$.*

In more intrinsic terms, consider the filtration of $U(\mathfrak{g})$ given by $U(\mathfrak{g})^{\leq i} = \sum_{j \leq i} \mathfrak{g}^j$. Then, $\text{gr } U(\mathfrak{g})$ is commutative and the canonical map of vector spaces $\mathfrak{g} \rightarrow \text{gr } U(\mathfrak{g})$ extends to an isomorphism $S(\mathfrak{g}) \xrightarrow{\sim} \text{gr } U(\mathfrak{g})$.

Note that a \mathfrak{g} -module can be extended uniquely to a $U(\mathfrak{g})$ -module and the corresponding functor is an isomorphism from the category of \mathfrak{g} -modules to the category of $U(\mathfrak{g})$ -modules.

We take now for k a field of characteristic 0 and we consider only finite dimensional Lie algebras.

Recall that a Lie algebra \mathfrak{g} is semi-simple if it has no non-zero abelian ideal. It is simple if it is non-abelian and its only ideals are 0 and \mathfrak{g} .

The Lie algebra \mathfrak{g} is semi-simple if and only if the Killing form is non-degenerate. Recall that the Killing form is given by the trace in the adjoint representation

$$\beta : \mathfrak{g} \times \mathfrak{g} \rightarrow k, \quad \beta(x, y) = \text{Tr}(\text{ad } x \cdot \text{ad } y).$$

Every semi-simple Lie algebras is a product of simple Lie algebras. If \mathfrak{g} is semi-simple, then all finite dimensional representations of \mathfrak{g} are semi-simple.

An element x of \mathfrak{g} is semi-simple if $\text{ad } x$ is semi-simple. A Cartan subalgebra of \mathfrak{g} is a maximal commutative subalgebra whose elements are semi-simple. All Cartan algebras are conjugate in $\text{Aut}(\mathfrak{g})$. Fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . Given α a non-zero element of \mathfrak{h}^* , we put

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \text{ad}(y)x = \alpha(y)x \text{ for all } y \in \mathfrak{h}\}.$$

We denote by Δ the set of elements of non-zero elements α of \mathfrak{h}^* such that $\mathfrak{g}_\alpha \neq 0$. These are the roots. We have a decomposition of \mathfrak{g} in weight spaces

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

The Killing form on \mathfrak{h} is non-degenerate. Let $\mathfrak{h}_{\mathbf{R}}^*$ be the real subspace of \mathfrak{h} generated by Δ . Then, $\mathfrak{h}_{\mathbf{R}}^* \otimes_{\mathbf{R}} \mathbf{C} \xrightarrow{\sim} \mathfrak{h}^*$ and the restriction of the Killing form to $\mathfrak{h}_{\mathbf{R}}^*$ is positive definite.

We choose a hyperplane H of $\mathfrak{h}_{\mathbf{R}}^*$ containing no root and denote by $\mathfrak{h}_{\mathbf{R}}^{*+}$ and $\mathfrak{h}_{\mathbf{R}}^{*-}$ the corresponding half-spaces. We put $\Delta^\pm = \Delta \cap \mathfrak{h}_{\mathbf{R}}^{*\pm}$ (positive and negative roots). There is a unique minimal subset Π of Δ^+ such that every element of Δ^+ is a linear combination with positive

coefficients of elements of Π . We order the elements of Π as $\alpha_1, \dots, \alpha_n$, where α_i is closer to H than α_{i+1} . The set $\Pi = \{\alpha_1, \dots, \alpha_n\}$ of simple roots is a basis of \mathfrak{h}^* .

We put $\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$. This provides a triangular decomposition of \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-.$$

The spaces \mathfrak{g}_{α_i} and $\mathfrak{g}_{-\alpha_i}$ are one-dimensional. Let $\mathfrak{h}_i = [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}]$, a one-dimensional subspace of \mathfrak{h} . There is a unique $h_i \in \mathfrak{h}_i$ such that $\alpha_i(h_i) = 2$. We choose $e_i \in \mathfrak{g}_{\alpha_i}$ a non-zero element. There is a unique $f_i \in \mathfrak{g}_{-\alpha_i}$ such that $[e_i, f_i] = h_i$.

Let $a_{ij} = \alpha_i(h_j)$. The Cartan matrix of \mathfrak{g} is $(a_{ij})_{1 \leq i \leq n}$.

Note that $a_{ii} = 2$. The possibilities for the pairs (a_{ij}, a_{ji}) are (up to swapping i and j):

- $(0, 0)$ $\begin{array}{cc} \circ & \circ \\ & i \quad j \end{array}$
- $(-1, -1)$ $\begin{array}{cc} \circ & \text{---} \circ \\ & i \quad j \end{array}$
- $(-2, -1)$ $\begin{array}{cc} \circ & \text{====} \circ \\ & i \quad j \end{array}$
- $(-3, -1)$ $\begin{array}{cc} \circ & \text{====} \circ \\ & i \quad j \end{array}$

Example 4.2. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{C})$. The Killing form is given by $\beta(x, y) = \text{Tr}(xy)$. We take for \mathfrak{h} the subalgebra of trace zero diagonal matrices, which we identify with the hyperplane of \mathbf{C}^n of vectors (z_1, \dots, z_n) with $z_1 + \dots + z_n = 0$. Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the dual basis of $(\mathbf{C}^n)^*$. Then, \mathfrak{h}^* is identified with the quotient $(\bigoplus_i \mathbf{C}\varepsilon_i) / \mathbf{C}(\varepsilon_1 + \dots + \varepsilon_n)$. The root system is $\Delta = \{\varepsilon_i - \varepsilon_j\}_{1 \leq i \neq j \leq n}$ (more precisely, we should take the images in \mathfrak{h}^*). The set $\Pi = \{\varepsilon_i - \varepsilon_{i+1}\}_{1 \leq i \leq n-1}$ is a set of simple roots. Then, \mathfrak{n}^+ (resp. \mathfrak{n}^-) is the subalgebra of strictly upper (resp. lower) triangular matrices.

Let us recall the list of Dynkin diagrams. In addition to the simply laced ones (types ADE, of §3.3.1), we have the types B_n, C_n, F_4 and G_2 .

Theorem 4.3. *Isomorphism classes of finite dimensional complex simple Lie algebras are in bijection with Dynkin diagrams (types A...G).*

Given Γ a Dynkin diagram with Cartan matrix $(a_{ij})_{1 \leq i, j \leq n}$, the corresponding Lie algebra has a presentation by generators $e_1, \dots, e_n, f_1, \dots, f_n, h_1, \dots, h_n$ and relations

$$\begin{aligned} [h_i, h_j] &= 0, [h_i, e_j] = a_{ij}e_j, [h_i, f_j] = -a_{ij}f_j, [e_i, f_j] = \delta_{ij}h_i \\ \text{ad}^{1-a_{ij}}(e_i)e_j &= 0, \text{ad}^{1-a_{ij}}(f_i)f_j = 0. \end{aligned}$$

When $a_{ij} = 0$, then $[e_i, e_j] = 0$. When $a_{ij} = -1$, we have $[e_i, [e_i, e_j]] = 0$. In the enveloping algebra, that last relation becomes $e_i^2 e_j - 2e_i e_j e_i + e_j e_i^2 = 0$.

4.1.2. *Quantum groups.* Let v be an indeterminate. Given a positive integer n , we put

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}}, [n]! = [2] \cdots [n] \text{ and } \begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!}.$$

Consider the $\mathbf{Q}(v)$ -algebra $\tilde{U}_v(\mathfrak{g})$ generated by $E_1, \dots, E_n, F_1, \dots, F_n, K_1^{\pm 1}, \dots, K_n^{\pm 1}$ with relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, K_i K_j = K_j K_i$$

$$K_i E_j K_i^{-1} = v^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = v^{-a_{ij}} F_j, \quad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{v - v^{-1}}$$

$$(5) \quad \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix} E_i^r E_j E_i^{1-a_{ij}-r} = 0$$

$$(6) \quad \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix} F_i^r F_j F_i^{1-a_{ij}-r} = 0$$

Note that for $a_{ij} = 0$, we get $E_i E_j = E_j E_i$, while if $a_{ij} = -1$, we obtain

$$E_j E_i^2 - (v + v^{-1}) E_i E_j E_i + E_i^2 E_j = 0.$$

Here we should view $K_i = v^{h_i}$. Taking the limit $v \rightarrow 1$ gives the defining relations of the ordinary enveloping algebra $U(\mathfrak{g})$ (cf below for a precise statement).

Let $\tilde{U}_v^+(\mathfrak{g})$ (resp. $\tilde{U}_v^-(\mathfrak{g})$) be the $\mathbf{Q}(v)$ -subalgebra of $\tilde{U}_v(\mathfrak{g})$ generated by E_1, \dots, E_n (resp. F_1, \dots, F_n). Let $\tilde{U}_v^0(\mathfrak{g})$ be the subalgebra generated by $K_1, \dots, K_n, K_1^{-1}, \dots, K_n^{-1}$.

Theorem 4.4 (Triangular decomposition). *The canonical map is an isomorphism*

$$\tilde{U}_v^-(\mathfrak{g}) \otimes_{\mathbf{Q}(v)} \tilde{U}_v^0(\mathfrak{g}) \otimes_{\mathbf{Q}(v)} \tilde{U}_v^+(\mathfrak{g}) \xrightarrow{\sim} \tilde{U}_v(\mathfrak{g}).$$

The algebra $\tilde{U}_v^+(\mathfrak{g})$ has a presentation with generators E_1, \dots, E_n and relations (5).

The algebra $\tilde{U}_v^-(\mathfrak{g})$ has a presentation with generators F_1, \dots, F_n and relations (6).

The algebra $\tilde{U}_v^0(\mathfrak{g})$ is a Laurent polynomial algebra $\mathbf{Q}(v)[K_1^{\pm 1}, \dots, K_n^{\pm 1}]$.

We introduce now an ‘‘integral form’’ of the quantum group. We denote by $U_v(\mathfrak{g})$ the $\mathbf{Z}[v, v^{-1}]$ -subalgebra of $\tilde{U}_v(\mathfrak{g})$ generated by $K_i^{\pm 1}$, $E_i^{(r)} := \frac{E_i^r}{[r]!}$ and $F_i^{(r)} := \frac{F_i^r}{[r]!}$ for $i = 1, \dots, n$ and $r \geq 1$. This is a free $\mathbf{Z}[v^{\pm 1}]$ -algebra and $\tilde{U}_v(\mathfrak{g}) = U_v(\mathfrak{g}) \otimes_{\mathbf{Z}[v^{\pm 1}]} \mathbf{Q}(v)$: we have obtained a lattice in $\tilde{U}_v(\mathfrak{g})$. Given $v_0 \in \mathbf{C}^\times$, we put $U_{v_0}(\mathfrak{g}) = U_v(\mathfrak{g}) \otimes_{\mathbf{Z}[v^{\pm 1}]} \mathbf{C}[v^{\pm 1}]/(v - v_0)$.

The assignment $E_i \mapsto e_i$, $F_i \mapsto f_i$ induces a surjective morphism of algebras $U_1(\mathfrak{g}) \otimes_{\mathbf{Z}} \mathbf{C} \rightarrow U(\mathfrak{g})$. Its kernel is the ideal generated by the central elements $K_i - 1$. So, $U_v(\mathfrak{g})$ is a deformation of a central extension of $U(\mathfrak{g})$.

4.2. The Ringel-Hall algebra of a Dynkin quiver.

4.2.1. Euler forms and twisted multiplication. Let \mathcal{A} be an abelian category. The Grothendieck group $K_0(\mathcal{A})$ is the quotient of the free abelian group with basis (e_M) the isomorphism classes of objects of \mathcal{A} by the relation

$$e_M = e_L + e_N \quad \text{whenever there is an exact sequence } 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0.$$

Let k be a field and assume \mathcal{A} is k -linear. Assume that for all objects M, N of \mathcal{A} , we have $\dim_k \bigoplus_{i \geq 0} \text{Ext}_{\mathcal{A}}^i(M, N) < \infty$.

We put

$$\langle M, N \rangle = \sum_{i \geq 0} (-1)^i \dim_k \text{Ext}^i(M, N).$$

This depends only on the classes of M and N and induces a bilinear form, the *Euler form*, $K_0(\mathcal{A}) \times K_0(\mathcal{A}) \rightarrow \mathbf{Z}$.

Assume now $k = \mathbf{F}_q$ and put $t = \sqrt{q}$.

Definition 4.5. *The Ringel-Hall algebra $H_{\mathcal{A}}$ is the \mathbf{C} -vector space $\mathbf{C} \otimes_{\mathbf{Z}} H'_{\mathcal{A}}$ with the product*

$$[M] \cdot [N] = t^{\langle M, N \rangle} [M] * [N].$$

Exercise 4.1. Check that the definition of the Euler form in §4.2.1 makes sense, i.e., that $\sum_{i \geq 0} (-1)^i \dim_k \text{Ext}^i(M, N)$ depends only on the classes of M and N in $K_0(\mathcal{A})$.

Exercise 4.2. Check the associativity of the twisted multiplication.

Exercise 4.3. Give a definition of the twisted product in the case of an abelian category such that for all objects M and N , then $\coprod_{i \geq 0} \text{Ext}^i(M, N)$ is a finite set.

4.2.2. *Rank 2.* Let us start with \mathcal{A} the category of finite dimensional vector spaces over k (equivalently, finite dimensional representations of the quiver \bullet). We have $\langle k, k \rangle = 1$, so $[k] \cdot [k] = t[k] * [k]$. We have

$$[k]^n = t^{n(n-1)} [n]! [k^n].$$

Let us now consider the quiver $1 \longrightarrow 2$ of type A_2 (cf Example 3.8). We have

$$e_2 e_1 = [V(1) \oplus V(2)], \quad e_1 e_2 = t^{-1}([V(1) \oplus V(2)] + e_{12}).$$

We have

$$\begin{aligned} e_2 e_1^2 &= (t + t^3)[V(1)^{\oplus 2} \oplus V(2)] \\ e_1^2 e_2 &= (t^{-1} + t)([V(1) \oplus M] + [V(1)^{\oplus 2} \oplus V(2)]) \\ e_1 e_2 e_1 &= (t^2 + 1)[V(1)^{\oplus 2} \oplus V(2)] + [M \oplus V(1)] \end{aligned}$$

It follows that

$$e_1^2 e_2 - (t + t^{-1}) e_1 e_2 e_1 + e_2 e_1^2 = 0$$

So, e_1 and e_2 satisfy the quantum \mathfrak{sl}_3 -relations.

Exercise 4.4. Check the calculation of $[k]^n$ in §4.2.2.

4.2.3. *Ringel's Theorem.* Let Q be a quiver of type ADE and let \mathfrak{g} be the corresponding simple complex Lie algebra. The Cartan matrix coincides with the matrix of the symmetric bilinear form associated with $2q$. The definition of positive roots in §3.3.5 coincides with that given in §4.1.1.

Theorem 4.6. *There is a unique isomorphism of rings*

$$U_t^+(\mathfrak{g}) \xrightarrow{\sim} H_{\text{Rep}_k(Q)}, \quad E_i \mapsto [V(i)].$$

Proof (construction). Let us first show there is a morphism with $E_i \mapsto [V(i)]$.

We need to check that the relation (5) holds in the Hall-Ringel algebra. Consider two vertices i, j . The full subcategory of representations of Q supported at $\{i, j\}$ is equivalent to the category of representations of the subquiver with vertices $\{i, j\}$. So, the quantum Serre relations can be checked in rank 2: this has been done in §4.2.2. \square

Proposition 4.7. *Let Q be a Dynkin quiver of type ADE and k a field. There are no sequences of morphisms $M_0 \xrightarrow{f_1} M_1 \rightarrow \dots \xrightarrow{f_r} M_r = M_0$ with $r \geq 1$, f_i non-zero and non-invertible and M_i an indecomposable representation of Q over k for all i .*

Proof. Let $f : L \rightarrow M$ be a non-zero and non-invertible morphism between indecomposable representations. Let N be an indecomposable direct summand of $\text{im } f$, and consider the compositions of canonical maps $g : L \rightarrow \text{im } f \rightarrow N$ and $h : N \rightarrow \text{im } f \rightarrow M$. These are not zero and non-invertible maps. It follows that given a sequence as in the Proposition, we can construct a sequence

$$\begin{array}{ccccccc}
 & & N_1 & & N_3 & & \cdots & & N_{2m-1} & & \\
 & \nearrow^{g_0} & & \searrow^{g_1} & \nearrow^{g_2} & \searrow^{g_3} & \nearrow^{g_4} & & \nearrow^{g_{2m-2}} & \searrow^{g_{2m-1}} & \\
 N_0 & & & & N_2 & & N_4 & & \cdots & & N_{2m} = N_0
 \end{array}$$

where N_0, \dots, N_{2m-1} are indecomposable, $g_0, g_2, \dots, g_{2m-2}$ are injective but non-invertible and $g_1, g_3, \dots, g_{2m-1}$ are surjective but non-invertible. We will prove that this is impossible by induction of m , then by induction on $\dim N_0$ and finally by induction on $\dim \text{Hom}(N_1, N_3)$.

Since $\text{Ext}^2(N_3/N_2, \ker g_1) = 0$ (Proposition 3.13), the canonical map $\text{Ext}^1(N_3/N_2, N_1) \rightarrow \text{Ext}^1(N_3/N_2, N_2)$ is surjective. Consequently, there is a representation L and a commutative diagram whose horizontal sequences are exact:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & N_1 & \longrightarrow & L & \longrightarrow & N_3/N_2 & \longrightarrow & 0 \\
 & & \downarrow^{g_1} & & \downarrow^f & & \parallel & & \\
 0 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_3/N_2 & \longrightarrow & 0
 \end{array}$$

Note that L cannot be indecomposable, for otherwise the sequence

$$N_0 \hookrightarrow L \twoheadrightarrow N_4 \rightarrow \cdots$$

would contradict the minimality of m .

Let L' be an indecomposable direct summand of L such that the composition $g : N_0 \rightarrow N_1 \rightarrow L \rightarrow L'$ is non-zero. Consider the composition $h : N_0 \xrightarrow{g} L' \rightarrow L$.

Let $h' : N_0 \xrightarrow{h} L' \xrightarrow{f} N_3$. Assume $h' \neq 0$ and let N'_1 be an indecomposable direct summand of $\text{im } h'$. We have a sequence of maps $N'_1 \hookrightarrow N_3 \twoheadrightarrow N_4 \rightarrow \cdots \rightarrow N_{2m-1} \twoheadrightarrow N'_1$ contradicting the minimality of m . It follows that $h' = 0$.

We have an exact sequence

$$(7) \quad 0 \rightarrow N_1 \rightarrow L \oplus N_2 \rightarrow N_3 \rightarrow 0$$

Since $h' = 0$, it follows that h factors through a map $h'' : N_0 \rightarrow N_1$. If h'' is not injective, let N'_0 be an indecomposable direct summand of $\text{im } h''$. Then the sequence $N'_0 \hookrightarrow N_1 \twoheadrightarrow N_2 \rightarrow \cdots \rightarrow N_{2m-1} \twoheadrightarrow N'_0$ contradicts the minimality of $\dim N_0$. So, h'' is injective, hence g is injective as well.

We have an exact sequence (cf (7))

$$\text{End}(N_3) \rightarrow \text{Hom}(N_2, N_3) \oplus \text{Hom}(L', N_3) \oplus \text{Hom}(L/L', N_3) \rightarrow \text{Hom}(N_1, N_3) \rightarrow \text{Ext}^1(N_3, N_3).$$

We have $\text{End}(N_3) = k$ and $\text{Ext}^1(N_3, N_3) = 0$ (Remark 3.16). Furthermore, $\text{Hom}(N_2, N_3) \neq 0$ and $\text{Hom}(L/L', N_3) \neq 0$. It follows that $\dim \text{Hom}(N_1, N_3) > \dim \text{Hom}(L', N_3)$.

The indecomposability of N_1 shows that $f(L') \neq 0$ (cf (7)). Let L'' be an indecomposable direct summand of $f(L')$. We have a sequence of maps $N_0 \hookrightarrow L' \twoheadrightarrow L'' \hookrightarrow N_3 \rightarrow \cdots$ contradicting the minimality of $\dim \text{Hom}(N_1, N_3)$. \square

This proposition shows there is a total ordering $\gamma_1, \dots, \gamma_N$ of the positive roots such that $\text{Hom}(M(\gamma_i), M(\gamma_j)) \neq 0$ implies $i \leq j$.

Given $\phi : \Delta_+ \rightarrow \mathbf{Z}_{\geq 0}$, we put $M(\phi) = \bigoplus_{\gamma} M(\gamma)^{\phi(\gamma)}$. The set $\{[M(\phi)]\}_{\phi}$ is a basis of $H_{\mathcal{A}}$. We put $d(\phi) = d(M(\phi))$, a linear combination of simple roots.

Lemma 4.8. *If $i \leq j$, then $\text{Ext}^1(M(\gamma_i), M(\gamma_j)) = 0$.*

Proof. Note that if $i = j$, then the conclusion is provided by Remark 3.16. Consider a non-split extension $0 \rightarrow M(\gamma_j) \rightarrow N \rightarrow M(\gamma_i) \rightarrow 0$. Let $M(\gamma_r)$ be an indecomposable direct summand of N . Then, we have non zero maps $M(\gamma_j) \rightarrow M(\gamma_r)$ and $M(\gamma_r) \rightarrow M(\gamma_i)$. So, $j \leq r \leq i$. \square

Lemma 4.9. *Consider $a_1, \dots, a_N \in \mathbf{Z}_{\geq 0}$. Then, $[\bigoplus_i M(\gamma_i)^{a_i}] = [M(\gamma_1)^{a_1}] * \dots * [M(\gamma_N)^{a_N}]$.*

Proof. Let us compute the right hand side. We look for modules M with a filtration $M = M_0 \supset M_1 \supset \dots \supset M_N = 0$ with $N_{i-1}/N_i \simeq M(\gamma_i)^{a_i}$. By Lemma 4.8, we obtain $M \simeq \bigoplus_i M(\gamma_i)^{a_i}$.

Conversely, let $M = \bigoplus_i M(\gamma_i)^{a_i}$. Since $\text{Hom}(M(\gamma_i), M(\gamma_j)) = 0$ for $i > j$, there is a unique filtration as above. \square

Note that the total order on positive roots induces a total order on simple roots: $\alpha_1, \dots, \alpha_n$.

Lemma 4.10. *Let $d_1, \dots, d_n \in \mathbf{Z}_{\geq 0}$. Then,*

$$[M(\alpha_n)^{d_n}] * \dots * [M(\alpha_1)^{d_1}] = \bigoplus [M(\phi)]$$

where ϕ runs over maps $\Delta_+ \rightarrow \mathbf{Z}_{\geq 0}$ such that $d(\phi) = \sum_i d_i \alpha_i$.

Proof. It is clear that only those ϕ with $d(\phi) = \sum_i d_i \alpha_i$ occur in the decomposition.

Consider now $M = M(\phi)$. The simple module $M(\alpha_i)$ occurs in a composition series of M with multiplicity d_i . By Lemma 4.8, we have $\text{Ext}^1(M(\alpha_i), M(\alpha_j)) = 0$ if $i \leq j$. It follows that M admits a unique filtration $M = M_0 \supset M_1 \supset \dots \supset M_n = 0$ with $M_{i-1}/M_i \simeq M(\alpha_{n-i+1})^{d_{n-i+1}}$. \square

End of proof of Theorem 4.6. Let us show the surjectivity of the morphism Ψ of the Theorem.

We show by induction on $d(\phi)$ that $[M(\phi)]$ is in the image.

Using Lemma 4.9, we know by induction that if the support of ϕ has at least two elements, then the result holds.

Assume there is a unique r such that $\phi(\gamma_r) \neq 0$. Write $\phi(\gamma_r)d(\gamma_r) = \sum_{i=1}^n d_i \alpha_i$. It is a general property of root systems that if $\mathbf{Z}\gamma_i = \mathbf{Z}\gamma_j$ then $i = j$. So, Lemma 4.10 shows that $[M(\alpha_1)^{d_1}] * \dots * [M(\alpha_n)^{d_n}] - [M(\gamma_r)^{\phi(\gamma_r)}]$ is a sum of $[M(\phi)]$'s, where the support of ϕ contains at least two elements. As shown above, these $[M(\phi)]$'s are in $\text{im } \Psi$, so $M(\phi)$ is as well.

We deduce now that Ψ is an isomorphism. There is a grading on $U_t^+(\mathfrak{g})$ by the group $R = \bigoplus_i \mathbf{Z}\alpha_i$: it is given by assigning the degree α_i to E_i . We also have a grading on $H_{\mathcal{A}}$ by assigning the degree $d(\phi)$ to $M(\phi)$. Note that Ψ respects the grading.

Fix $d \in R$. The homogeneous component of $U_t^+(\mathfrak{g})$ of degree d has the same dimension as that for $U^+(\mathfrak{g})$. Thanks to the Poincaré-Birkhoff-Witt Theorem, this dimension is the number of ϕ 's with $d(\phi) = d$. It is clear that this coincides with the dimension of the homogenous component of degree d of $H_{\mathcal{A}}$. It follows that Ψ is an isomorphism. \square

Note that the proof above shows actually the existence of a basis of $U_v^+(\mathfrak{g})$ where v is generic ("PBW basis"):

Theorem 4.11. *There exists a basis $\{e_\phi\}_\phi$ of $U_v^+(\mathfrak{g})$ that specializes to the basis*

$$(t^{\dim \text{End}(M_\phi) - \dim M_\phi} [M_\phi])_\phi$$

for every q and $v \mapsto t = \sqrt{q}$.

There exists polynomials $P_{\phi_1, \phi_2}^\phi \in \mathbf{Q}[v]$ such that for every q , we have $P_{M(\phi_1), M(\phi_2)}^{M(\phi)} = P_{\phi_1, \phi_2}^\phi(t)$.

Exercise 4.5. Write down a proof of Theorem 4.11

The basis $(e_\phi)_\phi$ of $U_v^+(\mathfrak{g})$ depends on the choice of an orientation of the Dynkin diagram.

Consider the ring involution $a \mapsto \bar{a}$ of $U_v^+(\mathfrak{g})$ given by

$$v \mapsto v^{-1}, E_i \mapsto E_i.$$

There is a basis independent of the orientation. It is called canonical basis (Lusztig) or global crystal basis (Kashiwara).

Define a total order on $\mathbf{Z}_{\geq 0}^{\Delta^+}$ by setting $\phi < \psi$ if there is r with $\phi(i) = \psi(i)$ for $i < r$ and $\phi(r) > \psi(r)$.

The following Theorem has two proofs: a combinatorial one and a geometrical one, based on realizing the canonical basis via perverse sheaves on moduli spaces of representations.

Theorem 4.12. *There is a unique basis $(C_\phi)_\phi$ of $U_v^+(\mathfrak{g})$ such that*

$$\bar{C}_\phi = C_\phi \text{ and } C_\phi \in e_\phi + \sum_{\psi < \phi} v^{-1} \mathbf{Z}[v^{-1}] e_\psi.$$

The basis (C_ϕ) is independent of the orientation of the Dynkin diagram.

5. CONSTRUCTION OF REPRESENTATIONS

5.1. Constructible functions and Euler characteristic.

5.1.1. *Euler-Poincaré characteristic.* We fix an algebraically closed field k . Let X be a variety (=separated scheme of finite type) over k . A locally closed subset of X is the intersection of an open subset and a closed subset. A constructible subset of X is a finite disjoint union of locally closed subsets. An important property is that the image of a constructible subset by a morphism of varieties is constructible (Chevalley's Theorem).

Example 5.1. Let $X = \mathbf{A}^2 = \text{Spec } k[x, y]$. The line “ $x = 0$ ”, $L_1 = \text{Spec } k[y]$ is a closed subset of X and removing the origin, we get $Z_1 = \text{Spec } k[y, y^{-1}]$, a locally closed subset of X . Similarly, let $L_2 = \text{Spec } k[x]$ and $Z_2 = \text{Spec } k[x, x^{-1}]$. Let $Z = Z_1 \cup Z_2$. Then, $X - Z = (X - (L_1 \cup L_2)) \coprod \{0\}$ is a constructible subset of X , but it is not locally closed. It is not even an algebraic variety.

Define the Euler-Poincaré characteristic

$$\chi(X) = \sum_i (-1)^i \dim H_c^i(X, \mathbf{C}).$$

(this is the case $k = \mathbf{C}$. In general, one uses ℓ -adic cohomology).

Note that if Y is a locally closed subset of X , then $\chi(X) = \chi(Y) + \chi(X - Y)$. Consequently, one can extend by linearity the definition of Euler characteristic to constructible subsets of varieties.

Also, the Euler-Poincaré characteristic is multiplicative: $\chi(X \times X') = \chi(X) \times \chi(X')$.

Assume the variety X is defined over a finite subfield \mathbf{F}_q of k . The Lefschetz trace formula for etale cohomology shows that

$$\chi(X) = - \lim_{z \rightarrow \infty} \sum_{n=1}^{\infty} |X(\mathbf{F}_{q^n})| z^n.$$

Note that if there is a polynomial $P(t)$ such that $|X(\mathbf{F}_{q^n})| = P(q^n)$ for all $n \geq 1$, then $\chi(X) = P(1)$.

5.1.2. *Constructible functions.* A function $f : X \rightarrow \mathbf{Z}$ is *constructible* if it takes finitely many values and for every $z \in \mathbf{Z}$, then $f^{-1}(z)$ is a constructible subset of X . So, constructible functions are of the form $f = \sum_{\alpha} m_{\alpha} 1_{X_{\alpha}}$ where $X = \coprod_{\alpha} X_{\alpha}$ is a partition of X into constructible subsets and $m_{\alpha} \in \mathbf{Z}$.

We define

$$\int_X f = \sum_{\alpha} m_{\alpha} \chi(X_{\alpha}).$$

Let $\phi : X \rightarrow Y$ be a morphism of algebraic varieties. Given $g \in \text{CF}(Y)$, we define $\phi^*g \in \text{CF}(X)$ by $(\phi^*g)(x) = g(\phi(x))$.

Let f be a constructible function on X . We define

$$\int_{\phi} f \in \mathcal{F}(Y) : y \mapsto \int_{\phi^{-1}(y)} f.$$

We admit the following useful Theorem. It would be easy to check directly the property in our applications (and then get rid of the characteristic 0 assumption).

Theorem 5.2. *Assume k has characteristic 0. Given $\psi : Y \rightarrow Y'$, we have*

$$\int_{\psi} \int_{\phi} f = \int_{\psi \circ \phi} f.$$

Remark 5.3. If k has characteristic $p > 0$, the composition of the Artin-Schreier covering (the quotient by the free action of the additive group \mathbf{F}_p) $\mathbf{A}^1 \rightarrow \mathbf{A}^1$, $x \mapsto x^p - x$, with the projection to the point shows the previous Theorem is not true for fields of positive characteristic.

Assume k has characteristic 0 and consider a Cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{g_2} & X \\ f_2 \downarrow & & \downarrow f_1 \\ X' & \xrightarrow{g_1} & Y \end{array}$$

So, $Z = X' \times_Y X$. Then, we have

$$g_1^* \int_{f_1} = \int_{f_2} g_2^* : \text{CF}(X) \rightarrow \text{CF}(X').$$

5.2. Hall algebras via constructible functions.

5.2.1. *Quivers with relations.* Let Q be a quiver and k a commutative ring. A set R of relations for Q over k is a finite set of elements of $k(Q)_{\geq 2}$. We denote by $I = (R)$ the two-sided ideal of $k(Q)$ generated by R and we put $A = k(Q)/I$.

Exercice 5.1. Assume k is an algebraically closed field. Let A be a basic finite dimensional k -algebra (*i.e.*, all simple A -modules have dimension 1). Show that there is a quiver Q with relations R such that $A \simeq k(Q)/(R)$.

Given a dimension vector $d \in Q_0^{\mathbf{Z}_{\geq 0}}$, we denote by $\text{Rep}_d(Q, R)$ the closed subvariety of $\text{Rep}_d(Q) = \prod_a \mathbf{A}^{d(q(a)) \times d(p(a))}$ of representations of A .

5.2.2. *Preprojective algebras.* Let Q be a graph with no loop. Let \bar{Q} be the double quiver of Q : we have $\bar{Q}_0 = Q_0$, $\bar{Q}_1 = Q_1 \amalg \{\bar{a}\}_{a \in Q_1}$. Here, \bar{a} is an arrow with $p(\bar{a}) = q(a)$ and $q(\bar{a}) = p(a)$. We denote by I the ideal of relations generated by $\sum_{a \in Q_1} [\bar{a}, a]$. The preprojective algebra of Q is the algebra $P(Q) = k(Q)/I$. Note that given $i \in Q_0$, we have the relation

$$\sum_{a \in Q_1, p(a)=i} a\bar{a} = \sum_{a \in Q_1, q(a)=i} \bar{a}a.$$

Up to isomorphism, the algebra $P(Q)$ depends only on the graph underlying Q .

Theorem 5.4. *Assume Q is connected. The algebra $P(Q)$ is finite dimensional if and only if Q is a Dynkin quiver.*

Remark 5.5. Assume Q connected. It is known that $P(Q)$ has finite representation type if and only if Q is a Dynkin quiver of type A_n with $n \leq 4$.

Assume Q is a Dynkin quiver. Note that $P(Q)$ does not have finite global dimension, unless Q has type A_1 , in which case $P(Q) = k$. It can be shown that $P(Q)$ is actually selfinjective (*ie*, $P(Q)^*$ is projective as a left A -module).

Remark 5.6. Note that $k(Q)$ is a subalgebra of $P(Q)$. Assume Q is a Dynkin quiver. Then, as a $k(Q)$ -module, $P(Q)$ is isomorphic to the direct sum of all indecomposable $k(Q)$ -modules (taken up to isomorphism). In general, $P(Q)$ decomposes as the sum of the indecomposable preprojective modules.

Let $d \in \mathbf{Z}_{\geq 0}^{Q_0}$. We put $\Lambda_d = \text{Rep}_d(Q, I)$. The variety $\text{Rep}_d(\bar{Q})$ identifies with $T^* \text{Rep}_d(Q)$ and Λ_d is the closed subvariety $\mu^{-1}(0)$, where

$$\mu : T^* \text{Rep}_d(Q) \rightarrow \bigoplus_s \mathfrak{gl}(d(s)), \quad \{(\phi_a)_a, (\phi_{\bar{a}})_a\} \mapsto \sum_a [\phi_{\bar{a}}, \phi_a]$$

is the moment map for the action of G_d on $\text{Rep}_d(Q)$.

Remark 5.7. This is an instance of non-commutative geometry (Kontsevich-Soibelman): the preprojective algebra is defined via its representation space. Here, it appears as some sort of symplectic reduction.

5.2.3. *Lusztig's algebra of constructible functions.* From now on, we assume $k = \mathbf{C}$.

We denote by \mathcal{M}_α the subspace of $\text{CF}(\Lambda_\alpha)$ of functions that are constant on G_α -orbits.

We define the convolution $f * g$ of $f \in \mathcal{M}_\alpha$ by $g \in \mathcal{M}_\beta$ as

$$(f * g)(x) = \int_{U_x} f(x')g(x'')$$

where U_x is the variety of subspaces V of $k^{\alpha+\beta}$ such that V is stable under x and the dimension vector of V is β . Furthermore, $x' \in \Lambda_\alpha$ is an element such that $V(x') \simeq V(x)/V$ and $x'' \in \Lambda_\beta$ with $V(x'') \simeq V$.

A more formal way to proceed is to introduce a variety Z of quadruples (x, V, ζ, ξ) , where $x \in \Lambda_{\alpha+\beta}$, V is as above, $\zeta : k^{\alpha+\beta}/V \xrightarrow{\sim} k^\alpha$ and $\xi : V \xrightarrow{\sim} k^\beta$. We denote by $\pi_1 : Z \rightarrow \Lambda_\alpha$ and $\pi_2 : Z \rightarrow \Lambda_\beta$ the projections. Let U be the variety of pairs (x, V) where $x \in \Lambda_{\alpha+\beta}$ and V is as above. Let $\pi : Z \rightarrow U$ and $\pi_3 : U \rightarrow \Lambda_{\alpha+\beta}$ be the projections. We consider the function $\pi_1^* f \cdot \pi_2^* g$ on Z . There is a unique constructible function h on U with $\pi^* h = \pi_1^* f \cdot \pi_2^* g$ and we put $f * g = \int_{\pi_3} h$.

$$\begin{array}{ccccc}
 & & Z & \xrightarrow{\pi} & U \\
 & \swarrow \pi_1 & & \searrow \pi_2 & \searrow \pi_3 \\
 & \Lambda_\alpha & & \Lambda_\beta & \Lambda_{\alpha+\beta}
 \end{array}$$

This defines an associative product on $\tilde{\mathcal{M}} = \bigoplus_\alpha \tilde{\mathcal{M}}_\alpha$.

Given $i \in S$, we denote by $\mathbf{1}_i \in \mathcal{M}_i$ the constructible function that takes value 1 on the one-element variety Λ_i . Let \mathcal{M} be the subalgebra of $\tilde{\mathcal{M}}$ generated by the functions $\mathbf{1}_i$.

5.2.4. *Rank 2.* Let us start with the one vertex quiver Q . Given $d \geq 0$, the variety $\Lambda_d = \text{Rep}_d(Q)$ is a point. Then, $1_{\Lambda_d} * 1_{\Lambda_{d'}} = \chi(\text{Gr}(d+d', d')) 1_{\Lambda_{d+d'}}$ where $\text{Gr}(d+d', d')$ is the Grassmanian variety of d' -dimensional subspaces of $k^{d+d'}$. Hence, $1_{\Lambda_d} * 1_{\Lambda_{d'}} = \binom{d+d'}{d'} 1_{\Lambda_{d+d'}}$.

Let us now consider the quiver $1 \longrightarrow 2$ of type A_2 (cf Example 3.8). Let us describe some of the varieties Λ_α and their decomposition into G_α -orbits.

- The varieties Λ_{α_1} and Λ_{α_2} are a point.
- The variety $\Lambda_{\alpha_1+\alpha_2}$ identifies with the union of the two coordinate hyperplanes $a\bar{a} = 0$ in \mathbf{A}^2 . There are three orbits of $k^\times \times k^\times$: L corresponding to $a \neq 0$ and $\bar{a} = 0$, \bar{L} corresponding to $a = 0$ and $\bar{a} \neq 0$ and L_0 corresponding to $a = \bar{a} = 0$.

$$k \begin{array}{c} \xrightarrow{a} \\ \xleftarrow{\bar{a}} \end{array} k$$

- The variety $\Lambda_{2\alpha_1+\alpha_2}$ identifies with the closed subvariety of \mathbf{A}^4 with equation $a_i \bar{a}_j = 0$ for all $i, j \in \{1, 2\}$. There are three orbits of $\text{GL}_2(k) \times k^\times$: M corresponding to $(a_1, a_2) \neq (0, 0)$ and $\bar{a}_1 = \bar{a}_2 = 0$, \bar{M} corresponding to $a_1 = a_2 = 0$ and $(\bar{a}_1, \bar{a}_2) \neq 0$ and M_0 corresponding to $a_1 = a_2 = \bar{a}_1 = \bar{a}_2 = 0$.

$$k^2 \begin{array}{c} \xrightarrow{(a_1, a_2)} \\ \xleftarrow{\begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \end{pmatrix}} \end{array} k$$

We have $\mathbf{1}_2 * \mathbf{1}_1 = 1_{\bar{L}} + 1_{L_0}$, $\mathbf{1}_1 * \mathbf{1}_2 = 1_L + 1_{L_0}$, $1_L * \mathbf{1}_1 = 1_M$, $1_{\bar{L}} * \mathbf{1}_1 = 1_{\bar{M}}$, $1_{L_0} * \mathbf{1}_1 = 1_{\bar{M}} + 2 \cdot 1_{M_0}$, $\mathbf{1}_1 * 1_L = 1_M$ and $\mathbf{1}_1 * 1_{L_0} = 1_M + 2 \cdot 1_{M_0}$. It follows that

$$\mathbf{1}_1^2 \mathbf{1}_2 - 2(\mathbf{1}_1 \mathbf{1}_2 \mathbf{1}_1) + \mathbf{1}_2 \mathbf{1}_1^2 = 0.$$

So, $\mathbf{1}_1$ and $\mathbf{1}_2$ satisfy the “ \mathfrak{sl}_3 -relations”.

5.2.5. *Hall algebra via constructible functions.*

Theorem 5.8. *The correspondence $E_i \mapsto \mathbf{1}_i$ extends uniquely to an isomorphism of algebras $U(\mathfrak{n}^+) \xrightarrow{\sim} \mathcal{M}$.*

Proof. In order to prove the Theorem, we will relate \mathcal{M} to an algebra of constructible functions on representations of Q . This algebra will be related to the $q = 1$ specialization of the Hall algebra, which we know is isomorphic to $U(\mathfrak{n}^+)$.

Let us first show that $E_i \mapsto \mathbf{1}_i$ extends to a morphism of algebras $U(\mathfrak{n}^+) \xrightarrow{\sim} \mathcal{M}$. This reduces to rank 2 and follows from the calculations in §5.2.4.

Let \mathcal{N}_α be the space of G_α -invariant constructible functions on $\text{Rep}_\alpha(Q)$. We define a convolution as in §5.2.3 on $\mathcal{N} = \bigoplus \mathcal{N}_\alpha$. Given M a representation of Q , we denote by $\delta_M \in \mathcal{N}$ the function that takes value 1 on representations isomorphic to M and 0 otherwise. Given M and N two representations of Q , we have

$$\delta_M * \delta_N = \sum_L \chi(\mathcal{F}_{M,N}^L) \delta_L,$$

where L runs over isomorphism classes of representations of Q and $\mathcal{F}_{M,N}^L$ is the variety of submodules N' of L with $N' \simeq N$ and $L/N' \simeq M$. A proof similar to that of Theorem 4.6 shows that we have an isomorphism

$$U(\mathfrak{n}^+) \xrightarrow{\sim} \mathcal{N}, \quad E_i \mapsto \mathbf{1}_i.$$

The morphism of varieties $\text{Rep}_\alpha(Q) \rightarrow \Lambda_\alpha$ induces a morphism of $\tilde{\mathcal{M}}_\alpha \rightarrow \mathcal{N}_\alpha$ and this provides a morphism of algebras $\mathcal{M} \rightarrow \tilde{\mathcal{M}} \rightarrow \mathcal{N}$. The Theorem follows. \square

Remark 5.9. It is possible to avoid having to translate the counting arguments into Euler characteristic ones to prove Theorem 5.8 following the proof of Theorem 4.6.

By Theorem 4.11, there is a polynomial P depending on the dimension vectors of M , N and L (but independent of k) such that if k is a finite field with q elements, then, $\mathcal{F}_{M,N}^L$ has $P(q)$ points over k , hence $\chi(\mathcal{F}_{M,N}^L) = P(1)$ for k algebraically closed of characteristic p . On the other hand, the variety $\mathcal{F}_{M,N}^L$ is obtained from a variety defined over \mathbf{Z} (independent of k), hence if k is an algebraically closed field of characteristic 0, then $\chi(\mathcal{F}_{M,N}^L) = P(1)$ as well. It follows now from Theorem 4.6 that we have an isomorphism $U(\mathfrak{n}^+) \xrightarrow{\sim} \mathcal{N}$, $E_i \mapsto \mathbf{1}_i$.

Exercice 5.2. Check the associativity of convolution.

Exercice 5.3. Check that the morphism $\tilde{\mathcal{M}} \rightarrow \mathcal{N}$ in the proof of Theorem 5.8 is compatible with convolution.

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