

# HIGHER REPRESENTATIONS OF LIE ALGEBRAS

RAPHAËL ROUQUIER

*Michaelmas 08, week 1*

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## 1. HECKE ALGEBRAS

### 1.1. Classical Hecke algebras.

1.1.1. *BGG-Demazure operators.* Given  $1 \leq i \leq n$ , we put  $s_i = (i, i + 1) \in \mathfrak{S}_n$ . We define an endomorphism of abelian groups  $\partial_i \in \text{End}_{\mathbf{Z}}(\mathbf{Z}[X_1, \dots, X_n])$  by

$$\partial_i(P) = \frac{P - s_i(P)}{X_{i+1} - X_i}.$$

The formula defines endomorphisms of various localizations, for example  $\mathbf{Z}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ .

Given  $w = s_{i_1} \cdots s_{i_r}$ , we put

$$\partial_w = \partial_{i_1} \cdots \partial_{i_r}.$$

This is independent of the choice of the reduced decomposition.

1.1.2. *Nil Hecke algebras.* Let  ${}^0H_n^f$  be the *nil Hecke algebra* of  $\text{GL}_n$ : this is the  $\mathbf{Z}$ -algebra generated by  $T_1, \dots, T_{n-1}$ , with relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \text{ if } |i - j| > 1 \text{ and } T_i^2 = 0.$$

Given  $w = s_{i_1} \cdots s_{i_r}$  a reduced decomposition of an element  $w \in \mathfrak{S}_n$ , we put  $T_w = T_{i_1} \cdots T_{i_r}$ . The nil Hecke algebra  ${}^0H_n$  is a graded algebra with  $\deg T_i = -2$  and  $t_0$  is homogeneous of degree  $n(n - 1)$ .

1.1.3. *Nil affine Hecke algebras.* Let  ${}^0H_n$  be the *nil affine Hecke algebra* of  $\mathrm{GL}_n$ :  ${}^0H_n = \mathbf{Z}[X_1, \dots, X_n] \otimes {}^0H_n^f$ , where  $\mathbf{Z}[X_1, \dots, X_n]$  and  ${}^0H_n^f$  are subalgebras and

$$T_i X_j = X_j T_i \text{ if } j - i \neq 0, 1, \quad T_i X_{i+1} - X_i T_i = 1 \text{ and } T_i X_i - X_{i+1} T_i = -1.$$

Given  $P \in \mathbf{Z}[X_1, \dots, X_n]$ , we have  $T_i P - s_i(P) T_i = P T_i - T_i s_i(P) = \partial_i(P)$ .

We have a faithful representation on  $\mathbf{Z}[X_1, \dots, X_n] = {}^0H_n \otimes_{{}^0H_n^f} \mathbf{Z}$  where

$$T_i(P) = \partial_i(P).$$

Let  $b_n = T_{w_0} X_1^{n-1} X_2^{n-2} \dots X_{n-1}$ . We have  $b_n^2 = b_n$  since  $\partial_{w_0}(X_1^{n-1} X_2^{n-2} \dots X_{n-1}) = 1$  and we have an isomorphism of  ${}^0H_n$ -modules

$$\mathbf{Z}[X_1, \dots, X_n] \xrightarrow{\sim} {}^0H_n b_n, \quad P \mapsto P b_n.$$

Since  $\{\partial_w(X_1^{n-1} \dots X_{n-1})_{w \in \mathfrak{S}_n}\}$  is a basis of  $\mathbf{Z}[X_1, \dots, X_n]$  over  $\mathbf{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n}$ , it follows that the multiplication map gives an isomorphism of  $({}^0H_n^f, \mathbf{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n})$ -bimodules

$${}^0H_n^f \otimes (\mathbf{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n} X_1^{n-1} \dots X_{n-1} b_n) \xrightarrow{\sim} {}^0H_n b_n.$$

**Proposition 1.1.** *The action of  ${}^0H_n$  on  $\mathbf{Z}[X_1, \dots, X_n]$  induces an isomorphism*

$${}^0H_n \xrightarrow{\sim} \mathrm{End}_{\mathbf{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n}}(\mathbf{Z}[X_1, \dots, X_n]).$$

*Since  $\mathbf{Z}[X_1, \dots, X_n]$  is a free  $\mathbf{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n}$ -module of rank  $n!$ , the algebra  ${}^0H_n$  is isomorphic to a  $(n! \times n!)$ -matrix algebra over  $\mathbf{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n}$ .*

*The restriction to  ${}^0H_n^f$  of any  ${}^0H_n$ -module is relatively  $\mathbf{Z}$ -projective.*

*Proof.* Since  $\mathbf{Z}[X_1, \dots, X_n]$  is a finitely generated projective  ${}^0H_n$ -module, the canonical map  ${}^0H_n \xrightarrow{\sim} \mathrm{End}_{\mathbf{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n}}(\mathbf{Z}[X_1, \dots, X_n])$  splits as a morphism of  $\mathbf{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n}$ -modules. The first two assertions of the proposition follow from the fact that  ${}^0H_n$  is a free  $\mathbf{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n}$ -module of rank  $(n!)^2$ .

The  $({}^0H_n^f, \mathbf{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n})$ -module  $\mathbf{Z}[X_1, \dots, X_n]$  is a direct summand of  ${}^0H_n$ . So, given  $M$  an  $\mathbf{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n}$ -module, then  $\mathbf{Z}[X_1, \dots, X_n] \otimes_{\mathbf{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n}} M$  is a direct summand of  ${}^0H_n^f \otimes_{\mathbf{Z}} (\mathbf{Z}[X_1, \dots, X_n] \otimes_{\mathbf{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n}} M)$  as an  ${}^0H_n^f$ -module. So, given  $N$  an  ${}^0H_n$ -module, then  $N$  is a direct summand of  ${}^0H_n^f \otimes_{\mathbf{Z}} N$  as an  ${}^0H_n^f$ -module and the proposition is proven.  $\square$

The nil affine Hecke algebra  ${}^0H_n$  is a graded algebra with  $\deg X_i = 2$  and  $\deg T_i = -2$  and  $t$  is homogeneous of degree 0. The nil affine Hecke algebra has also a bifiltration given by

$$F^{\leq(i,j)} ({}^0H_n) = \mathbf{Z}[X_1, \dots, X_n]_{\leq i} \otimes ({}^0H_n^f)_{\geq -j}.$$

Note that  $t(F^{<(n(n-1), n(n-1))}) = 0$ .

MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, 24-29 ST GILES', OXFORD, OX1 3LB, UK  
E-mail address: rouquier@maths.ox.ac.uk