AUTOMIZERS AS EXTENDED REFLECTION GROUPS

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1. Introduction

Let G be a finite group having an abelian Sylow p-subgroup P. Broué, Malle and Michel have shown that if G is a simple Chevalley group, then the automizer of P is an irreducible complex reflection group (for p not too small and different from the defining characteristic) [BrMaMi, BrMi].

The aim is this note is to show that a suitable version of this property holds for general finite groups.

We give a simple direct proof that the property above holds for simply connected simple algebraic groups G, provided p is not a torsion prime (Proposition 4.1): the automizer $E = N_G(P)/C_G(P)$ is a reflection group on $\Omega_1(P)$, the largest elementary abelian subgroup of P. This relies on the Lehrer-Springer theory [LeSp], that shows that certain subquotients of reflection groups are reflection groups.

On the other hand, we show that the presence of p-torsion in the Schur multiplier of a finite group G prevents the subgroup of E generated by reflections from being irreducible (Proposition 3.5).

This suggests considering covering groups of finite simple groups or equivalently finite simple groups G such that $H^2(G, \mathbf{F}_p) = 0$. We also need to allow p'-automorphisms and we now look for a description of the automizer as an extension of an irreducible reflection group W by a subgroup of $N_{\mathrm{GL}(\Omega_1(P))}(W)/W$.

We actually need a slight generalization: $\Omega_1(P)$ should be viewed in some cases as a vector space over a larger finite field (for example in the case of $\mathrm{PSL}_2(\mathbf{F}_{p^n})$) and we need to allow field automorphisms.

As an example, the automizer of an 11-Sylow subgroup in the Monster is the 2-dimensional complex reflection group G_{16}.

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2. Notation and definitions

Let p be a prime. Given P an abelian group, we denote by $\Omega_1(P)$ the subgroup of P of elements of order 1 or p, *i.e.*, the largest elementary abelian p-subgroup of P.

Let V be a free module of finite rank over a commutative algebra K. A *reflection* is an element $s \in \mathrm{GL}_K(V)$ of finite order such that $V/\ker(s-1)$ is a free K-module of rank 1 (note that we do not require $s^2 = 1$). A finite subgroup of $\mathrm{GL}_K(V)$ is a *reflection group* if it is generated by reflections.

3. Main result and remarks

Let p be a prime and H a simple group with an abelian Sylow p-subgroup P. Assume the p-part of the Schur multiplier of H is trivial, *i.e.* $H^2(H,\mathbf{F}_p)=0$. It is known from the classification of finite simple groups that $\mathrm{Out}(H)$ is solvable. Let $\tilde{H}\le \mathrm{Aut}(H)$ be a finite group containing H and such that $\tilde{H}/H$ is a Hall p'-subgroup of $\mathrm{Out}(H)$. Let $E=N_{\tilde{H}}(P)/C_{\tilde{H}}(P)$.

Theorem 3.1. *There is*

- *a finite field K*
- *an $\mathbf{F}_p$-subspace V of $\Omega_1(P)$ and an isomorphism of $\mathbf{F}_p$-vector spaces $K\otimes_{\mathbf{F}_p} V \xrightarrow{\sim} \Omega_1(P)$ endowing $\Omega_1(P)$ with a structure of K-vector space*
- *a subgroup N of $\mathrm{GL}_K\big(\Omega_1(P)\big)$ and*
- *a subgroup Γ of $\mathrm{Aut}(K)$*

such that $E=N\rtimes\Gamma$, as subgroups of $\mathrm{Aut}\big(\Omega_1(P)\big)$, and such that the normal subgroup W of N generated by reflections acts irreducibly on $\Omega_1(P)$.

The theorem will be proven in §4.

Remark 3.2. Gorenstein and Lyons have shown that $N_H(P)/C_H(P)$ acts irreducibly on $\Omega_1(P)$ viewed as a vector space over $\mathbf{F}_p$ and, as a consequence, P is homocyclic [GoLy, (12.1)]. Note nevertheless that the subgroup of $N_H(P)/C_H(P)$ generated by reflections might not be irreducible in its action on $\Omega_1(P)$. This happens for example in the case $H=\mathfrak{A}_{2p}$, $p>3$. In that case, the automizer is a subgroup of index 2 of $(\mathbf{Z}/(p-1))\wr\mathfrak{S}_2$ and its subgroup generated by reflections is contained in $(\mathbf{Z}/(p-1))^2$.

We can take $K=\mathbf{F}_p$ in Theorem 3.1, except for

- $\mathrm{PSL}_2(p^n)$, $n>1$: $K=\mathbf{F}_{p^n}$
- J_1 and ${}^2G_2(q)$, $p=2$: $K=\mathbf{F}_8$.

In those cases, $V=\mathbf{F}_p$ and $P=\Omega_1(P)=K$.

Note that the theorem is trivial when P is cyclic: one takes $K=\mathbf{F}_p$ and $N=E=W\subset\mathbf{F}_p^\times$.

Using the classification of finite simple groups, we deduce a statement about finite groups with abelian Sylow p-subgroups.

Corollary 3.3. *Let G be a finite group with an abelian Sylow p-subgroup P. Let $H=O^{p'}(G/O_{p'}(G))$.*

Assume the p-part of the Schur multiplier of H is trivial. Then, there is a finite group X containing H as a normal subgroup of p'-index and

- *a product K of finite field extensions of $\mathbf{F}_p$*
- *an $\mathbf{F}_p$-subspace V of $\Omega_1(P)$ and an isomorphism of $\mathbf{F}_p$-vector spaces $K\otimes_{\mathbf{F}_p} V\xrightarrow{\sim}\Omega_1(P)$ endowing $\Omega_1(P)$ with a structure of a free K-module*
- *a subgroup N of $\mathrm{GL}_K\big(\Omega_1(P)\big)$ and*
- *a subgroup Γ of $\mathrm{Aut}(K)$*

such that $N_X(P)/C_X(P)=N\rtimes\Gamma$, as subgroups of $\mathrm{Aut}\big(\Omega_1(P)\big)$, and such that denoting by W the normal subgroup of N generated by reflections, we have $\Omega_1(P)^W=1$.

Proof. The case where H is simple is Theorem 3.1. In general, we have $O_p(H)=1$, *i.e.*, there is no non-trivial p-group as a direct factor of H, because $H^2(H,\mathbf{F}_p)=0$. The classification of finite simple groups shows that there are finite simple groups $H_1, \ldots, H_r$ such that $H = F^*(H) = H_1 \times \cdots \times H_r$. Now, we take $X = X_1 \times \cdots \times X_r$, where the X_i are associated with H_i. We put $K = K_1 \times \cdots \times K_r$, etc. Note that we ignore outer autorphisms coming from permutation of isomorphic components H_i. ☐

Following [GoLy, Proof of (12.1)], we give now the list of possible finite simple groups H and primes p such that Sylow p-subgroups of H are abelian non-cyclic and the p-part of the Schur multiplier of H is trivial. In the first case, instead of providing the group H, we provide a group G such that $H \le G/O_{p'}(G) \le \mathrm{Aut}(H)$ and $p \not| \, [G/O_{p'}(G) : H]$.

- $G = \mathbf{G}^F$ where $\mathbf{G}$ is a simply connected simple algebraic group and F is an endomorphism of $\mathbf{G}$, a power of which is a Frobenius endomorphism defining a rational structure over a finite field with q elements, $p \not| q$ and p is not a torsion prime for $\mathbf{G}$
- $H = \mathfrak{A}_n$ and $n < p^2$
- $H = \mathrm{PSL}_2(p^n)$
- $H = {}^2G_2(q)$, $p = 2$
- H is one of 14 sporadic groups (cf §4.5).

Assume $K = \mathbf{F}_p$. We have $V = \Omega_1(P)$ and $\Gamma = 1$. Furthermore, $N = E \subset N_{\mathrm{GL}(P)}(W)$. So, in this case, the theorem is equivalent to the statement that W acts irreducibly on P. As a consequence, in order to show that the theorem holds, it is enough to prove the statement with $\tilde{H}$ replaced by a group G as above.

Remark 3.4. The finite simple groups with an abelian Sylow p-subgroup such that the p-part of the Schur multiplier is non-trivial are the following (cf [Atl]):

- $H = M_{22}, ON, \mathfrak{A}_6, \mathfrak{A}_7$ and $p = 3$
- $H = \mathrm{PSL}_2(q)$, $q \equiv 3, 5 \pmod 8$ and $p = 2$
- $H = \mathrm{PSL}_3(q)$ and $3|q-1$ or $H = \mathrm{PSU}_3(q)$ and $3|q+1$ (here $p = 3$)

Note that the automizer of a Sylow 3-subgroup P in $\mathrm{Aut}(ON) = ON.2$ does not contain any reflection (when P is viewed as a vector space over $\mathbf{F}_3$). That automizer is not a subgroup of $\mathrm{GL}_2(9).2$ (extension by the Frobenius).

Note that the presence of p-torsion in the Schur multiplier is an obstruction to the irreducibility of the subgroup of the automizer generated by reflections on $\Omega_1(P)$, viewed as a vector space over $\mathbf{F}_p$.

Proposition 3.5. *Let G be a finite group with an abelian Sylow p-subgroup P. Let $E = N_G(P)/C_G(P)$ and let W be the subgroup of E generated by reflections on $\Omega_1(P)$, viewed as an $\mathbf{F}_p$-vector space. Assume $p > 2$.*

If $H^2(G, \mathbf{F}_p) \neq 0$, then $\Omega_1(P)^W \neq 0$.

Proof. Let $V = \Omega_1(P)^*$. We have $H^2(G, \mathbf{F}_p) \simeq H^2(N_G(P), \mathbf{F}_p) \simeq H^2(P, \mathbf{F}_p)^E$. On the other hand, we have an isomorphism of $\mathbf{F}_pE$-modules $H^2(P, \mathbf{F}_p) \xrightarrow{\sim} V \oplus \Lambda^2(V)$, so $H^2(G, \mathbf{F}_p) \simeq V^E \oplus \Lambda^2(V)^E \subset V^W \oplus \Lambda^2(V)^W$. By Solomon's Theorem [So], we have $\Lambda^2(V)^W \simeq \Lambda^2(V^W)$. The result follows when P is not cyclic. If P is cyclic, then $W = E$, so $H^2(G, \mathbf{F}_p) \simeq V^W$ and the result follows as well. ☐

Remark 3.6. Let W be a reflection group on a complex vector space L, with minimal field of definition K. The subgroup of the outer automorphism group of W of elements fixing the set of reflections has always a decomposition as a semi-direct product $(N_{\mathrm{GL}(L)}(W)/W) \rtimes \mathrm{Gal}(K/\mathbf{Q})$ as shown by Marin and Michel [MaMi].

Remark 3.7. It would be interesting to investigate if there is a version of Theorem 3.1 for non-principal blocks with abelian defect groups.

In a work in progress, we study automizers of maximal elementary abelian p-subgroups in covering groups of simple groups.

4. Proof of Theorem 3.1

We run through the list of groups H (or G) as described above.

4.1. **Chevalley groups.** Let $\mathbf{G}$ be a connected and simply connected reductive algebraic group over an algebraic closure k of a finite field and endowed with an endomorphism F, a power of which is a Frobenius endomorphism. Let $G = \mathbf{G}^F$. Assume p is invertible in k and p is not a torsion prime for $\mathbf{G}$.

4.1.1. *Abelian p-subgroups.* Since p is not a torsion prime for $\mathbf{G}$, every abelian p-subgroup Q of G is contained in an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ and $\mathbf{L} = C_{\mathbf{G}}(Q)$ is a Levi subgroup ([SpSt, Corollary 5.10 and Theorem 5.8] and [GeHi, Proposition 2.1]). Furthermore, $N_{\mathbf{G}}(Q) = N_G(Q)C_{\mathbf{G}}(Q)$ [SpSt, Corollary 5.10], hence the canonical map is an isomorphism $N_G(Q)/C_G(Q) \xrightarrow{\sim} N_{\mathbf{G}}(Q)/C_{\mathbf{G}}(Q)$.

Let $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, $X = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m)$ and $Y = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T})$. If $\mathbf{G}$ is simple, then the action of W on $\mathbf{C} \otimes_{\mathbf{Z}} X$ is irreducible.

We have a canonical map $N_W(Q) \to N_{\mathbf{G}}(Q)/\mathbf{T}$. Since $\mathbf{L} \subset N_{\mathbf{G}}(Q) \subset N_{\mathbf{G}}(\mathbf{L})$, we obtain an isomorphism

$$N_W(Q)/C_W(Q) \xrightarrow{\sim} N_{\mathbf{G}}(Q)/C_{\mathbf{G}}(Q).$$

Given L an abelian group, we denote by L_{p^∞} the subgroup of p-elements of L. Let $\mu = k^\times$. We have an isomorphism

$$\mathbf{T}_{p^\infty} \xrightarrow{\sim} \mathrm{Hom}(X, \mu_{p^\infty}),\ t \mapsto (\chi \mapsto \chi(t)).$$

This provides an isomorphism

$$\mathbf{T}_{p^\infty} \xrightarrow{\sim} Y \otimes_{\mathbf{Z}} \mu_{p^\infty}.$$

These isomorphisms are equivariant for the actions of W and F.

4.1.2. *Abelian Sylow p-subgroups.* Assume now $P = Q$ is an abelian Sylow p-subgroup of G. Let $V = Y \otimes_{\mathbf{Z}} \mathbf{F}_p$. We have $V^F \simeq \Omega_1(P)$.

Proposition 4.1. *The group $N_W(P)/C_W(P)$ is a reflection group on $\Omega_1(P)$. If $\mathbf{G}$ is simple, then this reflection group is irreducible.*

Proof. Note that $N_W(P)/C_W(P)$ is a p'-group, since P is an abelian Sylow p-subgroup of G and $N_W(P)/C_W(P) \simeq N_G(P)/C_G(P)$. So, the canonical map is an isomorphism

$$N_W(P)/C_W(P) \xrightarrow{\sim} N_W(\Omega_1(P))/C_W(\Omega_1(P)).$$

The proposition follows now from the next lemma by Lehrer-Springer theory [LeSp] extended to positive characteristic [Rou]. □

Lemma 4.2. *We have $\dim V^F \geq \dim V^{wF}$ for all $w \in W$.*

Proof. Let $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$. By Lang's Lemma, there is $x \in \mathbf{G}$ such that $\dot{w} = x^{-1}F(x)$. Given $t \in \mathbf{T}$, we have $F(xtx^{-1}) = x\dot{w}F(t)\dot{w}^{-1}$. So, $x\mathbf{T}x^{-1}$ is F-stable and the isomorphism

$$\mathbf{T} \xrightarrow{\sim} x\mathbf{T}x^{-1},\ t \mapsto xtx^{-1}$$

transfers the action of wF on the left to the action of F on the right. So,

$$V^{wF} \simeq \big(Y(x\mathbf{T}x^{-1}) \otimes \mathbf{F}_p\big)^F \simeq \Omega_1\big((x\mathbf{T}x^{-1})^F\big).$$

The rank of that elementary abelian p-subgroup of G is at most the rank of P and we are done. □

4.2. **Alternating groups.** Let $G = \mathfrak{S}_n$, $n > 7$. Put $n = pr + s$ with $0 \leq s \leq p-1$ and $r < p$. We have $P \simeq (\mathbf{Z}/p)^r$. We put $K = \mathbf{F}_p$, $N = W = \mathbf{F}_p^\times \wr \mathfrak{S}_r$.

Remark 4.3. Note that when $n = 5$ and $p = 2$ or $n = 6, 7$ and $p = 3$, the p-part of the Schur multiplier is not trivial but the description above is still valid. Note though that when $n = 6$ and $p = 3$, then G contains $\mathfrak{S}_6$ as a subgroup of index 2. We have $K = \mathbf{F}_3$, $P = K^2$, $N = E$, W is a Weyl group of type B_2 and $[N : W] = 2$.

4.3. PSL_2**.** Assume $H = \mathrm{PSL}_2(K)$ for a finite field K of characteristic p. We have $W = N = K^\times$ and $\Gamma = \mathrm{Gal}(K/\mathbf{F}_p)$.

4.4. ${}^2G_2(q)$**.** Assume $H = {}^2G_2(q)$ and $p = 2$. We have $K = \mathbf{F}_8$, $W = N = K^\times$ and $\Gamma = \mathrm{Gal}(K/\mathbf{F}_2)$.

4.5. **Sporadic groups.** We refer to [BrMaRou] for the diagrams for complex reflection groups. For sporadic groups, we have $P = \Omega_1(P)$.

$\tilde{H}$	K	$\dim_K(P)$	W	N/W	Γ	diagram of W
J_1	$\mathbf{F}_8$	1	$\mathbf{F}_8^\times$	1	$\mathrm{Gal}(\mathbf{F}_8/\mathbf{F}_2)$	⑦
$M_{11}, M_{23}, HS.2$	$\mathbf{F}_3$	2	B_2	2	1	○══○
$J_2.2, Suz.2$	$\mathbf{F}_5$	2	G_2	2	1	○≡≡○
$He.2, Fi_{22}.2, Fi_{23}, Fi_{24}$	$\mathbf{F}_5$	2	G_8	1	1	④──④
Co_1	$\mathbf{F}_7$	2	G_5	1	1	③══③
Th, BM	$\mathbf{F}_7$	2	G_5	2	1	③══③
M	$\mathbf{F}_{11}$	2	G_{16}	1	1	⑤──⑤

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