CALOGERO-MOSER VERSUS KAZHDAN-LUSZTIG CELLS

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1. Introduction

In [KaLu], Kazhdan and Lusztig developed a combinatorial theory associated with Coxeter groups. They defined in particular partitions of the group in left and two-sided cells. For Weyl groups, these have a representation theoretic interpretation in terms of primitive ideals, and they play a key role in Lusztig's description of unipotent characters for finite groups of Lie type [Lu3]. Lusztig generalized this theory to Hecke algebras of Coxeter groups with unequal parameters [Lu2, Lu4].

We propose a definition of left cells and two-sided cells for complex reflection groups, based on ramification theory for Calogero-Moser spaces. These spaces have been defined via rational Cherednik algebras by Etingof and Ginzburg [EtGi]. We conjecture that these coincide with Kazhdan-Lusztig cells, for real reflection groups. Counterparts of families of irreducible characters have been studied by Gordon and Martino [GoMa], and we provide here a version of left cell representations. The Calogero-Moser cells are studied in detail in [BoRou].

2. Calogero-Moser spaces and cells

2.1. **Rational Cherednik algebras at $t = 0$.** Let us recall some constructions and results from [EtGi]. Let V be a finite-dimensional complex vector space and W a finite subgroup of $\mathrm{GL}(V)$. Let $\mathcal{S}$ be the set of reflections of W, *i.e.*, elements g such that $\ker(g-1)$ is a hyperplane. We assume that W is a reflection group, *i.e.*, it is generated by $\mathcal{S}$.

We denote by $\mathcal{S}/\sim$ the quotient of $\mathcal{S}$ by the conjugacy action of W and we let $\{\underline{c}_s\}_{s\in\mathcal{S}/\sim}$ be a set of indeterminates. We put $A = \mathbf{C}[\mathbf{C}^{\mathcal{S}/\sim}] = \mathbf{C}[\{\underline{c}_s\}_{s\in\mathcal{S}/\sim}]$. Given $s \in \mathcal{S}$, let $v_s \in V$ (resp. $\alpha_s \in V^*$) be an eigenvector for s associated to the non-trivial eigenvalue.

The 0-rational Cherednik algebra $\mathbf{H}$ is the quotient of $A \otimes T(V \oplus V^*) \rtimes W$ by the relations

$$[x, x'] = [\xi, \xi'] = 0$$

$$[\xi, x] = \sum_{s\in\mathcal{S}} \underline{c}_s \frac{\langle x, \alpha_s\rangle \cdot \langle v_s, \xi\rangle}{\langle v_s, \alpha_s\rangle} s \quad \text{for } x, x' \in V^* \text{ and } \xi, \xi' \in V.$$

We put $Q = Z(\mathbf{H})$ and $P = A \otimes S(V^*)^W \otimes S(V)^W \subset Q$. The ring Q is normal. It is a free P-module of rank $|W|$.

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2.2. **Galois closure.** Let $K = \mathrm{Frac}(P)$ and $L = \mathrm{Frac}(Q)$. Let M be a Galois closure of the extension L/K and R the integral closure of Q in M. Let $G = \mathrm{Gal}(M/K)$ and $H = \mathrm{Gal}(M/L)$. Let $\mathcal{P} = \mathrm{Spec}\, P = \mathbf{A}_{\mathbf{C}}^{\mathcal{S}/\sim} \times V/W \times V^*/W$, $\mathcal{Q} = \mathrm{Spec}\, Q$ the Calogero-Moser space, and $\mathcal{R} = \mathrm{Spec}\, R$.

We denote by $\pi : \mathcal{R} \to \mathcal{Q}$ the quotient by H, and by $\Upsilon : \mathcal{Q} \to \mathcal{P}$ and $\phi : \mathcal{P} \to \mathbf{A}_{\mathbf{C}}^{\mathcal{S}/\sim}$ the canonical maps. We put $p = \Upsilon\pi : \mathcal{R} \to \mathcal{P}$ the quotient by G.

2.3. **Ramification.** Let $\mathfrak{r} \in \mathcal{R}$ be a prime ideal of R. We denote by $D(\mathfrak{r}) \subset G$ its decomposition group and by $I(\mathfrak{r}) \subset D(\mathfrak{r})$ its inertia group.

We have a decomposition into irreducible components

$$\mathcal{R} \times_{\mathcal{P}} \mathcal{Q} = \bigcup_{g \in G/H} \mathcal{O}_g, \text{ where } \mathcal{O}_g = \{(x, \pi(g^{-1}(x)))| x \in \mathcal{R}\}$$

inducing a decomposition into irreducible components

$$V(\mathfrak{r}) \times_{\mathcal{P}} \mathcal{Q} = \coprod_{g \in I(\mathfrak{r}) \backslash G/H} \mathcal{O}_g(\mathfrak{r}), \text{ where } \mathcal{O}_g(\mathfrak{r}) = \{(x, \pi(g^{-1}g'(x)))| x \in V(\mathfrak{r}),\ g' \in I(\mathfrak{r})\}.$$

2.4. **Undeformed case.** Let $\mathfrak{p}_0 = \phi^{-1}(0) = \sum_{s \in \mathcal{S}/\sim} P\, \underline{c}_s$. We have $P/\mathfrak{p}_0 = \mathbf{C}[V \oplus V^*]^{W \times W}$, $Q/\mathfrak{p}_0 Q = \mathbf{C}[V \oplus V^*]^{\Delta W}$, where $\Delta(W) = \{(w, w)| w \in W\} \subset W \times W$. A Galois closure of the extension of $\mathbf{C}(\mathfrak{p}_0 Q) = \mathbf{C}(V \oplus V^*)^{\Delta W}$ over $\mathbf{C}(\mathfrak{p}_0) = \mathbf{C}(V \oplus V^*)^{W \times W}$ is $\mathbf{C}(V \oplus V^*)^{\Delta Z(W)}$.

Let $\mathfrak{r}_0 \in \mathcal{R}$ above $\mathfrak{p}_0$. Since $\mathfrak{p}_0 Q$ is prime, we have $G = D(\mathfrak{r}_0)H = HD(\mathfrak{r}_0)$ and $I(\mathfrak{r}_0) = 1$. Fix an isomorphism $\iota : \mathbf{C}(\mathfrak{r}_0) \xrightarrow{\sim} \mathbf{C}(V \oplus V^*)^{\Delta Z(W)}$ extending the canonical isomorphism of $\mathbf{C}(\mathfrak{p}_0 Q)$ with $\mathbf{C}(V \oplus V^*)^{\Delta W}$.

The application ι induces an isomorphism $D(\mathfrak{r}_0) \xrightarrow{\sim} (W \times W)/\Delta Z(W)$, that restricts to an isomorphism $D(\mathfrak{r}_0) \cap H \xrightarrow{\sim} \Delta W/\Delta Z(W)$. This provides a bijection $G/H \xrightarrow{\sim} (W \times W)/\Delta W$. Composing with the inverse of the bijection $W \xrightarrow{\sim} (W \times W)/\Delta W,\ w \mapsto (w, 1)$, we obtain a bijection $G/H \xrightarrow{\sim} W$.

From now on, we identify the sets G/H and W through this bijection. Note that this bijection depends on the choices of $\mathfrak{r}_0$ and of ι. Since M is the Galois closure of L/K, we have $\bigcap_{g \in G} H^g = 1$, hence the left action of G on W induces an injection $G \subset \mathfrak{S}(W)$.

2.5. **Calogero-Moser cells.**

Definition 2.1. *Let $\mathfrak{r} \in \mathcal{R}$. The $\mathfrak{r}$-cells of W are the orbits of $I(\mathfrak{r})$ in its action on W.*

Let $c \in \mathbf{A}_{\mathbf{C}}^{\mathcal{S}/\sim}$. Choose $\mathfrak{r}_c \in \mathcal{R}$ with $\overline{p(\mathfrak{r}_c)} = \bar{c} \times 0 \times 0$. The $\mathfrak{r}_c$-cells are called the *two-sided Calogero-Moser c-cells* of W. Choose now $\mathfrak{r}_c^{\mathrm{left}} \in \mathcal{R}$ contained in $\mathfrak{r}_c$ with $\overline{p(\mathfrak{r}_c^{\mathrm{left}})} = \bar{c} \times V/W \times 0 \in \mathcal{P}$. The $\mathfrak{r}_c^{\mathrm{left}}$-cells are called the *left Calogero-Moser c-cells* of W. We have $I(\mathfrak{r}_c^{\mathrm{left}}) \subset I(\mathfrak{r}_c)$. Consequently, every left cell is contained in a unique two-sided cell.

The map sending $w \in W$ to $\pi(w^{-1}(\mathfrak{r}_c))$ induces a bijection from the set of two-sided cells to $\Upsilon^{-1}(c \times 0 \times 0)$.

2.6. **Families and cell multiplicities.** Let E be an irreducible representation of $\mathbf{C}[W]$. We extend it to a representation of $S(V) \rtimes W$ by letting V act by 0. Let

$$\Delta(E) = e \cdot \mathrm{Ind}_{S(V)\rtimes W}^{\mathbf{H}}(A \otimes_{\mathbf{C}} E), \text{ where } e = \frac{1}{|W|}\sum_{w\in W} w,$$

be the spherical Verma module associated with E. It is a Q-module.

Let $c \in \mathbf{A}_{\mathbf{C}}^{\mathcal{S}/\sim}$ and let $\Delta^{\mathrm{left}}(E) = (R/\mathfrak{r}_c^{\mathrm{left}}) \otimes_P \Delta(E)$.

Definition 2.2. *Given Γ a left cell, we define the cell multiplicity $m_\Gamma(E)$ of E as the multiplicity of $\Delta^{\mathrm{left}}(E)$ at the component $\mathcal{O}_\Gamma(\mathfrak{r}_c^{\mathrm{left}})$.*

Note that $\sum_\Gamma m_\Gamma(E) \cdot [\mathcal{O}_\Gamma(\mathfrak{r}_c^{\mathrm{left}})]$ is the support cycle of $\Delta^{\mathrm{left}}(E)$.

There is a unique two-sided cell Λ containing all left cells Γ such that $m_\Gamma(E) \neq 0$. Its image in $\mathcal{Q}$ is the unique $\mathfrak{q} \in \Upsilon^{-1}(c \times 0 \times 0)$ such that $(Q/\mathfrak{q}) \otimes_Q \Delta(E) \neq 0$. The corresponding map $\mathrm{Irr}(W) \to \Upsilon^{-1}(c \times 0 \times 0)$ is surjective, and its fibers are the *Calogero-Moser families* of $\mathrm{Irr}(W)$, as defined by Gordon [Go1].

2.7. **Dimension 1.** Let V be a one-dimensional complex vector space, let $d \geq 2$ and let W be the group of d-th roots of unity acting on V. Let $\zeta = \exp(2i\pi/d)$, let $s = \zeta \in W$ and $\underline{c}_i = \underline{c}_{s^i}$ for $1 \leq i \leq d-1$. We have $A = \mathbf{C}[\underline{c}_1, \ldots, \underline{c}_{d-1}]$ and

$$\mathbf{H} = A\langle x, \xi, s | sxs^{-1} = \zeta^{-1}x,\ s\xi s^{-1} = \zeta\xi \text{ and } [\xi, x] = \sum_{i=1}^{d-1} \underline{c}_i s^i\rangle.$$

Let $\mathrm{eu} = \xi x - \sum_{i=1}^{d-1}(1-\zeta^i)^{-1}\underline{c}_i s^i$. We have $P = A[x^d, \xi^d]$ and $Q = A[x^d, \xi^d, \mathrm{eu}]$. Define $\underline{\kappa}_1, \ldots, \underline{\kappa}_d = \underline{\kappa}_0$ by $\underline{\kappa}_1 + \cdots + \underline{\kappa}_d = 0$ and $\sum_{i=1}^{d-1} \underline{c}_i s^i = \sum_{i=0}^{d-1}(\underline{\kappa}_i - \underline{\kappa}_{i+1})\varepsilon_i$, where $\varepsilon_i = \frac{1}{d}\sum_{j=0}^{d-1}\zeta^{ij}s^j$. We have $A = \mathbf{C}[\underline{\kappa}_1, \ldots, \underline{\kappa}_d]/(\underline{\kappa}_1 + \cdots + \underline{\kappa}_d)$.

The normalization of the Galois closure is described as follows. There is an isomorphism of A-algebras

$$A[X, Y, Z]/\big(XY - \prod_{i=1}^{d}(Z - \underline{\kappa}_i)\big) \xrightarrow{\sim} Q,\ X \mapsto x^d,\ Y \mapsto \xi^d \text{ and } Z \mapsto \mathrm{eu}.$$

We have an isomorphism of A-algebras

$$A[X, Y, \lambda_1, \ldots, \lambda_d]/\big(e_1(\lambda) = e_1(\underline{\kappa}), \ldots, e_{d-1}(\lambda) = e_{d-1}(\underline{\kappa}), e_d(\lambda) = e_d(\underline{\kappa}) + (-1)^{d+1}XY\big) \xrightarrow{\sim} R$$

where $Z = \lambda_d$ and where e_i denotes the $i{-}th$ elementary symmetric function. We have $G = \mathfrak{S}_d$, acting by permuting the λ_i's, and $H = \mathfrak{S}_{d-1}$.

Let $\mathfrak{p}_0 = (\underline{\kappa}_1, \ldots, \underline{\kappa}_d) \in \mathrm{Spec}\, P$ and $\mathfrak{r}_0 = (\underline{\kappa}_1, \ldots, \underline{\kappa}_d, \lambda_1 - \zeta\lambda_d, \ldots, \lambda_{d-1} - \zeta^{d-1}\lambda_d) \in \mathrm{Spec}\, R$. We have $D(\mathfrak{r}_0) = \langle(1, 2, \ldots, d)\rangle \subset \mathfrak{S}_d$ and $\mathbf{C}(\mathfrak{r}_0) = \mathbf{C}(X, Y, \lambda_d = \sqrt[d]{XY}) = \mathbf{C}(X, Y, Z = \sqrt[d]{XY})$. The composite bijection $D(\mathfrak{r}_0) \xrightarrow{\sim} G/H \xrightarrow{\sim} W$ is an isomorphism of groups given by $(1, \ldots, d) \mapsto s$.

Fix $c \in \mathbf{C}^{d-1}$ and let $\kappa_1, \ldots, \kappa_d \in \mathbf{C}$ corresponding to c. Consider $\mathfrak{r} = \mathfrak{r}_c$ or $\mathfrak{r}_c^{\mathrm{left}}$ as in §2.5. Then $I(\mathfrak{r})$ is the subgroup of $\mathfrak{S}_d$ stabilizing $(\kappa_1, \ldots, \kappa_d)$. The left c-cells coincide with the two-sided c-cells and two elements s^i and s^j are in the same cell if and only if $\kappa_i = \kappa_j$. Finally, the multiplicity $m_\Gamma(\det^j)$ is 1 if $s^j \in \Gamma$ and 0 otherwise.

3. Coxeter groups

3.1. **Kazhdan-Lusztig cells.** Following Kazhdan-Lusztig [KaLu] and Lusztig [Lu2, Lu4], let us recall the construction of cells.

We assume here V is the complexification of a real vector space $V_{\mathbf{R}}$ acted on by W. We choose a connected component C of $V_{\mathbf{R}} - \bigcup_{s\in\mathcal{S}} \ker(s-1)$ and we denote by S the set of $s \in \mathcal{S}$ such that $\ker(s-1) \cap \bar{C}$ has codimension 1 in $\bar{C}$. This makes (W, S) into a Coxeter group, and we denote by l the length function.

Let Γ be a totally ordered free abelian group and let $L : W \to \Gamma$ be a weight function, *i.e.*, a function such that $L(ww') = L(w) + L(w')$ if $l(ww') = l(w) + l(w')$. We denote by v^γ the element of the group algebra $\mathbf{Z}[\Gamma]$ corresponding to $\gamma \in \Gamma$.

We denote by H the Hecke algebra of W: this is the $\mathbf{Z}[\Gamma]$-algebra generated by elements T_s with $s \in S$ subject to the relations

$$(T_s - v^{L(s)})(T_s + v^{-L(s)}) = 0 \text{ and } \underbrace{T_sT_tT_s\cdots}_{m_{st} \text{ terms}} = \underbrace{T_tT_sT_t\cdots}_{m_{st} \text{ terms}} \quad \text{for } s,t \in S \text{ with } m_{st} \neq \infty$$

where m_{st} is the order of st. Given $w \in W$, we put $T_w = T_{s_1} \cdots T_{s_n}$, where $w = s_1 \cdots s_n$ is a reduced decomposition.

Let i be the ring involution of H given by $i(v^\gamma) = v^{-\gamma}$ for $\gamma \in \Gamma$ and $i(T_s) = T_s^{-1}$. We denote by $\{C_w\}_{w\in W}$ the Kazhdan-Lusztig basis of H. It is uniquely defined by the properties that $i(C_w) = C_w$ and $C_w - T_w \in \bigoplus_{w'\in W} \mathbf{Z}[\Gamma_{<0}]T_{w'}$.

We introduce the partial order $\prec_L$ on W. It is the transitive closure of the relation given by $w' \prec_L w$ if there is $s \in S$ such that the coefficient of $C_{w'}$ in the decomposition of C_sC_w in the Kazhdan-Lusztig basis is non-zero. We define $w \sim_L w'$ to be the corresponding equivalence relation: $w \sim_L w'$ if and only if $w \prec_L w'$ and $w' \prec_L w$. The equivalence classes are the left cells. We define $\prec_{LR}$ as the partial order generated by $w \prec_{LR} w'$ if $w \prec_L w'$ or $w^{-1} \prec_L w'^{-1}$. As above, we define an associated equivalence relation $\sim_{LR}$. Its equivalence classes are the two-sided cells.

When $\Gamma = \mathbf{Z}$, $L = l$, and W is a Weyl group, a definition of left cells based on primitive ideals in enveloping algebras was proposed by Joseph [Jo]: let $\mathfrak{g}$ be a complex semi-simple Lie algebra with Weyl group W. Let ρ be the half-sum of the positive roots. Given $w \in W$, let I_w be the annihilator in $U(\mathfrak{g})$ of the simple module with highest weight $-w(\rho) - \rho$. Then, w and w' are in the same left cell if and only if $I_w = I_{w'}$.

3.2. **Representations and families.** Let Γ be a left cell. Let $W_{\leq\Gamma}$ (resp. $W_{<\Gamma}$) be the set of $w \in W$ such that there is $w' \in \Gamma$ with $w \prec_L w'$ (resp. $w \prec_L w'$ and $w \notin \Gamma$). The left cell representation of W over $\mathbf{C}$ associated with Γ [KaLu, Lu4] is the unique representation, up to isomorphism, that deforms into the left H-module

$$\Big(\bigoplus_{w\in W_{\leq\Gamma}} \mathbf{Z}[\Gamma]C_w\Big)/\Big(\bigoplus_{w\in W_{<\Gamma}} \mathbf{Z}[\Gamma]C_w\Big).$$

Lusztig [Lu1, Lu4] has defined the set of constructible characters of W inductively as the smallest set of characters with the following properties: it contains the trivial character, it is stable under tensoring by the sign representation and it is stable under J-induction from a parabolic subgroup. Lusztig's families are the equivalences classes of irreducible characters of W for the relation generated by $\chi \sim \chi'$ if χ and χ' occur in the same constructible character. Lusztig has determined constructible characters and families for all W and all parameters.

Lusztig has shown for equal parameters, and conjectured in general, that the set of left cell characters coincides with the set of constructible characters.

3.3. **A conjecture.** Let $c \in \mathbf{R}^{\mathcal{S}/\sim}$. Let Γ be the subgroup of $\mathbf{R}$ generated by $\mathbf{Z}$ and $\{c_s\}_{s\in\mathcal{S}}$. We endow it with the natural order on $\mathbf{R}$. Let $L : W \to \Gamma$ be the weight function determined by $L(s) = c_s$ if $s \in S$.

The following conjecture is due to Gordon and Martino [GoMa]. A similar conjecture has been proposed independently by the second author[1]. It is known to hold for types A_n, B_n, D_n and $I_2(n)$ [Go2, GoMa, Be, Ma1, Ma2].

Conjecture 3.1. *The Calogero-Moser families of irreducible characters of W coincide with the Lusztig families.*

We propose now a conjecture involving partitions of elements of W, via ramification. The part dealing with left cell characters could be stated in a weaker way, using Q and not R, and thus not needing the choice of prime ideals, by involving constructible characters.

Conjecture 3.2. *There is a choice of $\mathfrak{r}_c^{\mathrm{left}} \subset \mathfrak{r}_c$ such that*

- *the Calogero-Moser two-sided cells (resp. left cells) coincide with the Kazhdan-Lusztig two-sided cells (resp. left cells)*
- *the representations $\sum_{E\in\mathrm{Irr}(W)} m_\Gamma(E)E$, where Γ is a Calogero-Moser left cell, coincide with the left cell representations of Kazhdan-Lusztig.*

Various particular cases and general results supporting Conjecture 3.2 are provided in [BoRou]. In particular, the conjecture holds for $W = B_2$, for all choices of parameters.

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