FROM STABLE EQUIVALENCES TO RICKARD EQUIVALENCES FOR BLOCKS WITH CYCLIC DEFECT

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1. Introduction

Let G and H be two finite groups, p a prime number. Let $\mathcal{O}$ be a complete discrete valuation ring with residue field k of characteristic p and with field of fractions K of characteristic 0, "big enough" for G and H. Let A and B be two blocks of G and H over $\mathcal{O}$.

Let M be a $(A\otimes B^\circ)$-module, projective as A-module and as B°-module, where B° denotes the opposite algebra of B. We denote by M^* the $(B\otimes A^\circ)$-module $\mathrm{Hom}_{\mathcal{O}}(M,\mathcal{O})$.

We say that M induces a *stable equivalence* between A and B if

$$M\otimes_B M^*\simeq A\oplus \text{projectives as } (A\otimes A^\circ)-\text{modules and}$$

$$M^*\otimes_A M\simeq B\oplus \text{projectives as } (B\otimes B^\circ)-\text{modules.}$$

Let C be a complex of $(A\otimes B^\circ)$-modules, all of which are projective as A-modules and as B°-modules.

Denoting by C^* the $\mathcal{O}$-dual of C, we say that C induces a *Rickard equivalence* between A and B if

$$C\otimes_B C^* \text{ is homotopy equivalent to } A \text{ as complexes of } (A\otimes A^\circ)-\text{modules and}$$

$$C^*\otimes_A C \text{ is homotopy equivalent to } B \text{ as complexes of } (B\otimes B^\circ)-\text{modules.}$$

By [Ri4, 5.5], from a complex C inducing a Rickard equivalence between A and B, one can construct a module M inducing a stable equivalence between A and B as follows : In the derived bounded category of $A\otimes B^\circ$, the complex C is isomorphic to a complex with only one term which is not projective as $(A\otimes B^\circ)$-module, V in degree $-n$ and then the n-th Heller translate (syzygy) $M=\Omega^n(V)$ induces a stable equivalence between A and B.

The main result of this note is a partial converse under very special assumptions (theorem 6). Since there are well-known situations where a module M induces a stable equivalence between two blocks (remark 9), for example when the Sylow p-subgroups of G are TI, H is the normalizer of a Sylow p-subgroup of G and A, B are principal blocks, it is tempting to try to construct a complex with two terms, M in degree 0 and a projective module in degree -1, inducing a Rickard equivalence between A and B. Using theorem 6, we prove that it is indeed possible when the Sylow p-subgroups of G are cyclic or when $G = A_5$ or $SL_2(8)$ and $p = 2$.

2. A CRITERION FOR DERIVED EQUIVALENCES BETWEEN BLOCKS

2.1. Some lemmas. Let A' be an $\mathcal{O}$-free $\mathcal{O}$-algebra, finitely generated as an $\mathcal{O}$-module.

If V is an A'-module, let P_V be an A'-module which is a projective cover of V. We will denote by $\mathrm{Rad}(V)$ the radical of V and by $\mathrm{hd}(V)$ the head $V/\mathrm{Rad}(V)$ of V, *i.e.*, its largest semi-simple quotient.

If M and N are two A'-modules, we say that M and N are *disjoint* if they have no non-zero isomorphic direct summands. If M and N are projective, they are disjoint if and only if $\mathrm{Hom}_{A'}(M, \mathrm{hd}(N)) = 0$ or equivalently, $\mathrm{Hom}_{A'}(N, \mathrm{hd}(M)) = 0$.

If X is an $\mathcal{O}$-module, we define $\bar{X} = X \otimes k$.

Lemma 1. *Let P, Q and R be three projective A'-modules and $\varphi : P \oplus Q \twoheadrightarrow R$ a surjective morphism. Assume that Q and R are disjoint. Then, the restriction $\varphi_{|P}$ of φ to P is surjective.*

Let U, V and W be three injective $\bar{A}'$-modules and $\varphi : W \hookrightarrow U \oplus V$ an injective morphism. Assume that V and W are disjoint. Then, denoting by p_U the projection of $U \oplus V$ onto U, the map $p_U\varphi$ is injective.

Proof. Let $h : R \to \mathrm{hd}(R)$ be the canonical projection. Since $h\varphi : P \oplus Q \twoheadrightarrow \mathrm{hd}(R)$ is surjective and $\mathrm{Hom}_{A'}(Q, \mathrm{hd}(R)) = 0$ by assumption, $h\varphi_{|P}$ is surjective. Hence, $\varphi(P) + \mathrm{Rad}(R) = R$ and by Nakayama's lemma, $\varphi(P) = R$. The second assertion follows immediately by duality since V and W are disjoint implies that V^* and W^* are disjoint. □

Lemma 2. *Let M be an $(A \otimes B^\circ)$-module, projective as A-module and as B°-module. A projective cover of M is*

$$\bigoplus_W P_{M \otimes_B W} \otimes P_W^*$$

where W runs over a complete set of representatives of isomorphism classes of simple B-modules. This module is isomorphic to

$$\bigoplus_V P_V \otimes P_{M^* \otimes_A V}^*$$

where V runs over a complete set of representatives of isomorphism classes of simple A-modules.

Proof. Let V be an $\bar{A}$-module and W a $\bar{B}$-module. We have

$$\mathrm{Hom}_{\bar{B}^\circ}(\bar{M}, V \otimes W^*) \simeq \mathrm{Hom}_{\bar{B}^\circ}(\bar{M}, V \otimes W^*) \simeq \bar{M}^* \otimes_{\bar{B}^\circ} (V \otimes W^*)$$

since $\bar{M}$ is projective as $\bar{B}^\circ$-module. Hence,

$$\mathrm{Hom}_{\bar{B}^\circ}(\bar{M}, V \otimes W^*) \simeq (\bar{M} \otimes_{\bar{B}} W)^* \otimes V \simeq \mathrm{Hom}_k(\bar{M} \otimes_{\bar{B}} W, V)$$

and finally

$$\mathrm{Hom}_{\bar{A}\otimes\bar{B}^\circ}(\bar{M}, V\otimes W^*) \simeq \mathrm{Hom}_{\bar{A}}(\bar{M}\otimes_{\bar{B}} W, V).$$

Now, we have

$$\begin{aligned}\mathrm{hd}(M) &\simeq \bigoplus_{V,W} \dim \mathrm{Hom}_{A\otimes B^\circ}(M, V\otimes W^*)(V\otimes W^*)\\ &\simeq \bigoplus_{W}\left(\bigoplus_{V} \dim \mathrm{Hom}_A(M\otimes_B W, V)V\right)\otimes W^*\end{aligned}$$

where V (resp. W) runs over the simple A-modules (resp. B-modules) up to isomorphism, hence

$$\mathrm{hd}(M) \simeq \bigoplus_{W} \mathrm{hd}(M\otimes_B W)\otimes W^*,$$

so a projective cover of M is $\bigoplus_W P_{M\otimes_B W}\otimes P_W^*$.

To get the second description, replace A by B° and B by A° in the first description : a projective cover of M as $B^\circ\otimes(A^\circ)^\circ$-module is $\oplus_V P_{M\otimes_{A^\circ}V}\otimes P_{V^*}$ where V runs over the simple A°-modules. This module is isomorphic to $\oplus_V P_{M\otimes_{A^\circ}V^*}\otimes P_V$ where V runs over the simple A-modules, hence a projective cover of M as $(A\otimes B^\circ)$-module is $\oplus_V P_V\otimes P_{V^*\otimes_A M}$ where V runs over the simple A-modules. □

Lemma 3 (Linckelmann, [Li2, 6.8]**).** *Let M be an $(A\otimes B^\circ)$-module inducing a stable equivalence between A and B. Then, M has a unique non-projective direct summand, up to isomorphism.*

Proof. Let $M = M_1\oplus M_2$. Since $M^*\otimes_A M\simeq B\oplus$ projectives, we have $M^*\otimes_A M_1\oplus M^*\otimes_A M_2\simeq B\oplus$ projectives as $(B\otimes B^\circ)$-modules. As B is indecomposable as $(B\otimes B^\circ)$-module, there exists $i\in\{1,2\}$ such that $M^*\otimes_A M_i$ is projective as $(B\otimes B^\circ)$-module, so $M\otimes_B M^*\otimes_A M_i$ is projective as $(A\otimes B^\circ)$-module. Now, $(M\otimes_B M^*)\otimes_A M_i\simeq M_i\oplus$ projectives as $(A\otimes B^\circ)$-modules, hence M_i is projective as $(A\otimes B^\circ)$-module. □

Remark 4. A similar proof shows that a complex of $(A\otimes B^\circ)$-modules C inducing a Rickard equivalence between A and B has a unique non-homotopy equivalent to zero direct summand, up to isomorphism.

Lemma 5 (Linckelmann, [Li2, 6.3]**).** *Let M be an indecomposable $(A\otimes B^\circ)$-module inducing a stable equivalence between A and B. For any simple B-module V, the A-module $M\otimes_B V$ is indecomposable.*

Proof. (Linckelmann) Denote by $\mathrm{soc}(\bar{A})$ the largest semi-simple $\bar{A}$-submodule of $\bar{A}$. Recall that an $\bar{A}$-module V has no projective direct summand if and only if $\mathrm{soc}(\bar{A})V = 0$. We have $\mathrm{soc}(\bar{A}\otimes\bar{B}^\circ) = \mathrm{soc}(\bar{A})\otimes\mathrm{soc}(\bar{B}^\circ)$. Since M has no projective direct summand, $\mathrm{soc}(\bar{A}\otimes\bar{B}^\circ)M = 0$, hence $\mathrm{soc}(\bar{A})(M\otimes_B\mathrm{soc}(\bar{B})) = 0$, which means that $M\otimes_B\mathrm{soc}(\bar{B})$ has no projective direct summand. But, if V is a simple B-module, it is a direct summand of $\mathrm{soc}(\bar{B})$, so $M\otimes_B V$ has no projective direct summand : as M induces a stable equivalence, $M \otimes_B V$ has a unique indecomposable non projective direct summand and the lemma follows. $\square$

2.2. The criterion. We denote by $R_K(A)$ (resp. $R_K(B)$) the group of characters of $KA = K \otimes A$ (resp. KB).

Let us now state the main result :

Theorem 6. *Let M be an $(A \otimes B^\circ)$-module, projective as A-module and as B°-module. Let $\delta' : P' \twoheadrightarrow M$ be a projective cover of M. Let P be a direct summand of P', $\delta = \delta'_{|P}$ and $C = (0 \longrightarrow P \xrightarrow{\delta} M \longrightarrow 0)$ (M is in degree 0). Assume*

- (a_1) *$M^* \otimes_A M \simeq B \oplus Q$ where Q is a projective $(B \otimes B^\circ)$-module,*
- (a_2) *$M \otimes_B M^* \simeq A \oplus R$ where R is a projective $(A \otimes A^\circ)$-module,*
- (b_1) *$\mathrm{Res}_{B^\circ}^{A \otimes B^\circ} \bar{P}$ and $\mathrm{Res}_{B^\circ}^{A \otimes B^\circ} \bar{P}'/\bar{P}$ are disjoint,*
- (b_2) *$\mathrm{Res}_{A}^{A \otimes B^\circ} \bar{P}$ and $\mathrm{Res}_{A}^{A \otimes B^\circ} \bar{P}'/\bar{P}$ are disjoint,*
- (c) *KC induces an isometry between $R_K(A)$ and $R_K(B)$.*

Then, C induces a Rickard equivalence between A and B.

Proof. [1] Remark first that (b_1) implies that

$$(b') \qquad \mathrm{Res}_{B}^{B \otimes A^\circ} \bar{P}^* \text{ and } \mathrm{Res}_{B}^{B \otimes A^\circ} \left(\bar{P}'/\bar{P}\right)^* \text{ are disjoint.}$$

We have

$$C^* \otimes_A C = (0 \to M^* \otimes_A P \xrightarrow{(\delta^* \otimes id, id \otimes \delta)} P^* \otimes_A P \oplus M^* \otimes_A M \xrightarrow{\begin{pmatrix} id \otimes \delta \\ -\delta^* \otimes id \end{pmatrix}} P^* \otimes_A M \to 0).$$

Since KC induces an isometry between $R_K(A)$ and $R_K(B)$, the character of $K(C^* \otimes_A C)$ as $(B \otimes B^\circ)$-module is equal to the character of B. Hence,

$$K(M^* \otimes_A P \oplus P^* \otimes_A M) \simeq K(P^* \otimes_A P \oplus Q).$$

We know that P is a projective $(A \otimes B^\circ)$-module and $\mathrm{Res}_{B}^{B \otimes A^\circ} M^*$ is projective, so $M^* \otimes_A P$ is projective as $(B \otimes B^\circ)$-module. Similarly, $P^* \otimes_A M$, $P^* \otimes_A P$ and Q are projective $(B \otimes B^\circ)$-modules. Hence

$$M^* \otimes_A P \oplus P^* \otimes_A M \simeq P^* \otimes_A P \oplus Q, \text{ and}$$

$$\bar{M}^* \otimes_A \bar{P} \oplus \bar{P}^* \otimes_A \bar{M} \simeq \bar{P}^* \otimes_A \bar{P} \oplus \bar{Q}. \tag{1}$$

Let $\bar{Q} = \bar{Q}_1 \oplus \bar{Q}_2$ where

$$\mathrm{Res}_{B^\circ}^{B \otimes B^\circ} \bar{Q}_2 \text{ and } \mathrm{Res}_{B^\circ}^{A \otimes B^\circ} \bar{P} \text{ are disjoint,} \tag{2}$$

$$\mathrm{Res}_{B^\circ}^{B \otimes B^\circ} \bar{Q}_1 \text{ and } \mathrm{Res}_{B^\circ}^{A \otimes B^\circ} \bar{P}'/\bar{P} \text{ are disjoint.} \tag{3}$$

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[1] Using an unpublished result of J. Rickard, one can actually prove the theorem without the assumptions (a_2) and (b_2).

(Since the map $p_{\bar{Q}}(id \otimes \bar{\delta}') : \bar{M}^* \otimes_A \bar{P}' \to \bar{Q}$ is surjective, every indecomposable direct summand of $\mathrm{Res}_{B^\circ}^{B \otimes B^\circ} \bar{Q}$ is isomorphic to a direct summand of $\mathrm{Res}_{B^\circ}^{B \otimes B^\circ} \bar{P}'$, so $\bar{Q}_1$ and $\bar{Q}_2$ are unique up to isomorphism).

The map $p_{\bar{Q}_1}(id \otimes \bar{\delta}') : \bar{M}^* \otimes_A \bar{P}' \to \bar{Q}_1$ is surjective and using (3),

$$\mathrm{Res}_{B^\circ}^{B \otimes B^\circ} \bar{Q}_1 \text{ and } \mathrm{Res}_{B^\circ}^{B \otimes B^\circ} \left(\bar{M}^* \otimes_A \bar{P}'\right) / \left(\bar{M}^* \otimes_A \bar{P}\right) \text{ are disjoint,}$$

$$\text{hence } \bar{Q}_1 \text{ and } \left(\bar{M}^* \otimes_A \bar{P}'\right) / \left(\bar{M}^* \otimes_A \bar{P}\right) \text{ are disjoint}$$

and it follows from lemma 1 that the map

$$p_{\bar{Q}_1}(id \otimes \bar{\delta}) : \bar{M}^* \otimes_A \bar{P} \to \bar{Q}_1$$

is surjective.

From (1), $\bar{Q}$ is isomorphic to a direct summand of $\bar{M}^* \otimes_A \bar{P} \oplus \bar{P}^* \otimes_A \bar{M}$, hence $\bar{Q}_2$ is isomorphic to a direct summand of $\bar{P}^* \otimes_A \bar{M}$ using (2). By (b'),

$$\mathrm{Res}_{B}^{B \otimes B^\circ} \left(\bar{P}^* \otimes \bar{M}\right) \text{ and } \mathrm{Res}_{B}^{B \otimes B^\circ} \left(\bar{P}'^* \otimes \bar{M} / \bar{P}^* \otimes \bar{M}\right) \text{ are disjoint}$$

$$\text{hence } \mathrm{Res}_{B}^{B \otimes B^\circ} \bar{Q}_2 \text{ and } \mathrm{Res}_{B}^{B \otimes B^\circ} \left(\bar{P}'^* \otimes \bar{M} / \bar{P}^* \otimes \bar{M}\right) \text{ are disjoint.}$$

Now, since $(\bar{\delta}'^* \otimes id)_{|\bar{Q}_2} : \bar{Q}_2 \to \bar{P}'^* \otimes_A \bar{M}$ is injective, lemma 1 implies that

$$(\bar{\delta}^* \otimes id)_{|\bar{Q}_2} : \bar{Q}_2 \to \bar{P}^* \otimes_A \bar{M}$$

is injective.

Let $\bar{R}_2$ be a submodule of $\bar{P}^* \otimes_A \bar{M}$ such that $\bar{P}^* \otimes_A \bar{M} = \bar{R}_2 \oplus \mathrm{Im}(\bar{\delta}^* \otimes id)_{|\bar{Q}_2}$. We introduce

$$f_2 = p_{\bar{R}_2}(id \otimes \bar{\delta}) : \bar{P}^* \otimes_A \bar{P} \to \bar{R}_2$$

$$\text{and } f_2' = p_{\bar{R}_2}(id \otimes \bar{\delta}') : \bar{P}^* \otimes_A \bar{P}' \to \bar{R}_2$$

We have $\bar{P}^* \otimes_A \bar{M} \simeq \bar{Q}_2 \oplus \bar{R}_2$, so, by (1), as $\bar{Q}_1$ is isomorphic to a direct summand of $\bar{M}^* \otimes_A \bar{P}$, the module $\bar{R}_1$ is a direct summand of $\bar{P}^* \otimes_A \bar{P}$, hence, by (b_1),

$$\mathrm{Res}_{B^\circ}^{B \otimes B^\circ} \bar{R}_2 \text{ and } \mathrm{Res}_{B^\circ}^{B \otimes B^\circ} \left((\bar{P}^* \otimes_A \bar{P}')/(\bar{P}^* \otimes_A \bar{P})\right) \text{ are disjoint.}$$

Since f_2' is surjective, lemma 1 implies that f_2 is surjective.

It follows that the map $id \otimes \bar{\delta} - \bar{\delta}^* \otimes id$ is surjective. By Nakayama's lemma, the map $id \otimes \delta - \delta^* \otimes id$ is also surjective and hence splits. By duality, the map $\delta^* \otimes id + id \otimes \delta$ is injective and splits. Hence, the complex $C^* \otimes_A C$ is homotopy equivalent to B.

Similarly, the complex $C \otimes_B C^*$ is homotopy equivalent to A. Hence, the complex C induces a Rickard equivalence between A and B. □

2.3. An application. Let M be an indecomposable $(A\otimes B)^\circ$-module inducing a stable equivalence between A and B.

Assume that for every simple A-module V, the head of $M^*\otimes_A V$ is simple.

Theorem 7. *If there exists a direct summand P of*

$$\bigoplus_V P_V\otimes P^*_{M^*\otimes_A V}\simeq \bigoplus_W P_{M\otimes_B W}\otimes P^*_W$$

(V runs over the simple A-modules and W over the simple B-modules) such that $0\to P\xrightarrow{0} M\to 0$ induces an isometry between $R_K(KA)$ and $R_K(KB)$, then there is a complex $C=0\to P\to M\to 0$ inducing a Rickard equivalence between A and B.

Proof. The modules P and M being projective as A-modules and as B°-modules, the isometry induced by $0\to P\xrightarrow{0} M\to 0$ is perfect [Br3, 1.2] and it follows that the algebras A and B have the same number s of isomorphism classes of simple modules [Br3, 1.5].

If V is a simple A-module, the modules P_V and $P^*_{M^*\otimes_A V}$ are indecomposable. Hence, a projective cover of M is a sum of s indecomposable $(A\otimes B^\circ)$-modules, which are mutually non-isomorphic when restricted to A or when restricted to B°. Hence, if P is a direct summand of P' then $\mathrm{Res}^{A\otimes B^\circ}_{B^\circ} P$ and $\mathrm{Res}^{A\otimes B^\circ}_{B^\circ} P'/P$ are disjoint and $\mathrm{Res}^{A\otimes B^\circ}_{A} P$ and $\mathrm{Res}^{A\otimes B^\circ}_{A} P'/P$ are disjoint. Now, theorem 6 gives the conclusion. □

Let us denote by $\mathrm{CF}(G,K)$ the space of class functions $G\to K$, by $\mathrm{CF}(A,K)$ the subspace generated by $R_K(A)$. We denote by $\mathrm{CF}_p(G,K)$ (resp. $\mathrm{CF}_{p'}(G,K)$) the subspace of $\mathrm{CF}(G,K)$ consisting of class functions which vanish on p-regular (resp. p-singular) elements and $\mathrm{CF}_p(A,K)$ (resp. $\mathrm{CF}_{p'}(A,K)$) the intersection $\mathrm{CF}_p(G,K)\cap\mathrm{CF}(A,K)$ (resp. $\mathrm{CF}_{p'}(G,K)\cap\mathrm{CF}(A,K)$).

As the next lemma shows, in the situation of theorem 7, if the map induced by $0\to P\xrightarrow{0} M\to 0$ is an isometry on a subspace of $\mathrm{CF}(A,K)$ which contains a complement of $\mathrm{CF}_p(A,K)$, then it is an isometry :

Lemma 8. *Let P_1, P_2 be two projective $(A\otimes B^\circ)$-modules and $C=0\to P_1\xrightarrow{0} M\oplus P_2\to 0$. Let I be the map between $R_K(A)$ and $R_K(B)$ induced by C. Let X be a subspace of $\mathrm{CF}(A,K)$ such that $\mathrm{CF}(A,K)=X+\mathrm{CF}_p(A,K)$.*

If the restriction of I to X is an isometry, then I is an isometry.

Proof. Let $f,g\in\mathrm{CF}(A,K)$. We decompose f and g as $f=f_p+f_{p'}$ and $g=g_p+g_{p'}$ where $f_p,g_p\in\mathrm{CF}_p(A,K)$ and $f_{p'},g_{p'}\in\mathrm{CF}_{p'}(A,K)$. Since I is perfect [Br3, 1.2], $I(f_p),I(g_p)\in\mathrm{CF}_p(B,K)$ and $I(f_{p'}),I(g_{p'})\in\mathrm{CF}_{p'}(B,K)$. Hence, the scalar product of $I(f)$ and $I(g)$ is $<I(f),I(g)>=<I(f_p),I(g_p)>+<I(f_{p'}),I(g_{p'})>$.

Furthemore, the restriction of I to $\mathrm{CF}_p(A,K)$ is an isometry because M induces a stable equivalence between A and B and as P_1 and P_2 are projective, the map induced by M between $R_K(A)$ and $R_K(B)$ is equal to I on $\mathrm{CF}_p(A,K)$ [Br2, 5.3]. It follows that $<I(f_p),I(g_p)>=<f_p,g_p>$ and we have now to prove that

$< I(f_{p'}), I(g_{p'}) >=< f_{p'}, g_{p'} >$. But, as $\mathrm{CF}(A,K) = X + \mathrm{CF}_p(A,K)$, we can decompose $f_{p'}$ and $g_{p'}$ as $f_{p'} = f_1 + f_2$ and $g_{p'} = g_1 + g_2$ where $f_1, g_1 \in X$ and $f_2, g_2 \in \mathrm{CF}_p(A,K)$. Now, $<f_1, g_1>=<f_{p'}, g_{p'}> - <f_2, g_2>$ and $<I(f_1), I(g_1)>=< I(f_{p'}), I(g_{p'})> - <I(f_2), I(g_2)>$. Finally, we know that $<I(f_1), I(g_1)>=<f_1, g_1>$ and $<I(f_2), I(g_2)>=<f_2, g_2>$, hence $<I(f_{p'}), I(g_{p'})>=<f_{p'}, g_{p'}>$. □

Remark 9. Stable equivalences induced by bimodules arise for example in the following situation [Br2, 6.4] :

Assume that H is a subgroup of G with index prime to p and e, f are the units of A and B. Following Broué, let us assume that for every non trivial p-subgroup P of H, we have $N_G(P) = N_H(P)O_{p'}C_G(P)$. Then, the $(A \otimes B^\circ)$-module $e\mathcal{O}Gf$ induces a stable equivalence between A and B. Let M be an indecomposable non-projective direct summand of $e\mathcal{O}Gf$; by lemma 3, such a module is unique up to isomorphism ; we have $e\mathcal{O}Gf = M \oplus$ projectives, so M induces a stable equivalence between A and B.

Example 1. Let $G = SL_2(4) = A_5$ and $H = A_4 = 2^2 \rtimes 3$ a Borel subgroup, $p = 2$. The principal block ekG of G has three simple modules : k, S_1 and S_2 of dimension 2. The module $\mathrm{Res}_H^G(S_1)$ is a non-split extension of V_2 by V_1, where V_1 and V_2 are the two non-trivial non-isomorphic simple kH-modules and $\mathrm{Res}_H^G(S_2)$ is a non-split extension of V_1 by V_2. An immediate character calculation shows that

$$0 \to P_{S_1} \otimes P_{V_1}^* \oplus P_{S_2} \otimes P_{V_2}^* \xrightarrow{0} e\mathcal{O}G \to 0$$

induces an isometry between $R_K(eKG)$ and $R_K(KH)$. Hence, by remark 9 and theorem 7, there exists a complex $0 \to P_{S_1} \otimes P_{V_1} \oplus P_{S_2} \otimes P_{V_2} \to e\mathcal{O}G \to 0$ inducing a Rickard equivalence between the principal blocks of G and H, result due to J. Rickard [Ri3].

Example 2. Let $G = SL_2(8)$ and $H = 2^3 \rtimes 7$ a Borel subgroup, $p = 2$. Then, theorem 7 applies also to construct a complex inducing a Rickard equivalence between the principal blocks of G and H : The $(A \otimes B^\circ)$-bimodule $e\mathcal{O}G$ is indecomposable. We leave to the reader to check that a projective cover of this module is :

$$P_1 \otimes Q_1^* \oplus P_{2_1} \otimes Q_{2_1}^* \oplus P_{2_2} \otimes Q_{2_2}^* \oplus P_{2_3} \otimes Q_{2_3}^* \oplus P_{4_1} \otimes Q_{4_1}^* \oplus P_{4_2} \otimes Q_{4_2}^* \oplus P_{4_3} \otimes Q_{4_3}^*$$

(where P_1 (resp. Q_1) is a projective cover of the trivial A-module (resp. B-module), P_{2_1}, P_{2_2} and P_{2_3} (resp. P_{4_1}, P_{4_2} and P_{4_3}) are projective covers of the three non-isomorphic 2-dimensional (resp. 4-dimensional) simple A-modules and $Q_{2_1}, Q_{2_2}, Q_{2_3}, Q_{4_1}, Q_{4_2}, Q_{4_3}$ are projective covers of the six non-isomorphic non-trivial simple B-modules) and that the complex

$$0 \to \oplus P_{4_1} \otimes Q_{4_1}^* \oplus P_{4_2} \otimes Q_{4_2}^* \oplus P_{4_3} \otimes Q_{4_3}^* \xrightarrow{0} e\mathcal{O}G \to 0$$

induces an isometry between $R_K(A)$ and $R_K(B)$, so that by remark 9 and theorem 7, there exists a complex $0 \to \oplus P_{4_1} \otimes Q_{4_1}^* \oplus P_{4_2} \otimes Q_{4_2}^* \oplus P_{4_3} \otimes Q_{4_3}^* \to e\mathcal{O}G \to 0$ inducing a Rickard equivalence between the principal blocks of G and H.

3. APPLICATION TO PRINCIPAL BLOCKS WITH CYCLIC DEFECT

Let G be a finite group with a cyclic Sylow p-subgroup P and let $H = N_G(P)$. As before, $A = \mathcal{O}Ge$ and $B = \mathcal{O}Hf$ are the principal blocks of G and H, where e and f are primitive idempotents of the centers of $\mathcal{O}G$ and $\mathcal{O}H$.

The functor $e \operatorname{Ind}_H^G$ induces a stable equivalence between A and B with inverse functor $f \operatorname{Res}_H^G$ (remark 9).

As conjectured by J. Rickard (cf [Ri2]), a slight modification of these functors leads to a derived equivalence, and this proves in particular the conjecture of Broué and Rickard on abelian defect, for principal blocks with cyclic defect (cf [Br1]) :

Theorem 10. *There exists a projective* $(A \otimes B^\circ)$*-module* Y *and a map* $\phi : Y \to e\mathcal{O}Gf$ *such that, if* $C = 0 \longrightarrow Y \xrightarrow{\phi} e\mathcal{O}Gf \longrightarrow 0$*, then* C *induces a Rickard equivalence between* A *and* B*. In particular,* C *is a Rickard tilting complex of* p*-permutation modules.*

Note that the fact that A and B are derived-equivalent was already known by the work of Rickard and Linckelman (cf [Ri1] and [Li1]).

3.1. Construction of C.

Let us quote some classical results about A (cf [Gr]).

The set of irreducible characters of KA is $\operatorname{Irr}(A) = \{\chi_1, \ldots, \chi_e\} \cup \{\chi_\lambda\}_{\lambda \in \Lambda}$ where $\chi_1, \ldots, \chi_e$ are the non-exceptional characters and the $\chi_\lambda, \lambda \in \Lambda$, are the exceptional characters. (In the case there is only one exceptional character, one can choose it different from the character 1_G.)

Define $\chi_{e+1} = \sum_{\lambda \in \Lambda} \chi_\lambda$ and $\Gamma = \{\chi_1, \ldots, \chi_{e+1}\}$.

The Brauer tree $\mathcal{T}_A$ is then defined as follows :

- the set of its vertices is Γ,
- two vertices v and v' are incident if and only if $v + v'$ is the character of a projective indecomposable A-module. We denote by $\{v, v'\}$ the corresponding edge.

The vertex χ_{e+1} is called the exceptional vertex of $\mathcal{T}_A$. Every character of a projective indecomposable A-module is an edge of $\mathcal{T}_A$ and we have a bijection beween the set of edges of $\mathcal{T}_A$ and the set of characters of projective indecomposable A-modules. If v and v' are two vertices of $\mathcal{T}_A$, we denote by $d(v, v')$ the distance between v and v'.

There is a "walk" on $\mathcal{T}_A$ starting from 1_G, the trivial character of G, *i.e.*, a sequence $v_0 = 1_G, v_1, \ldots, v_{2e}$ of vertices of $\mathcal{T}_A$ such that v_i is incident with v_{i+1} for $0 \leq i \leq 2e-1$, with the following properties :

- Each edge is traversed twice, *i.e.*, denoting by l_i the edge $\{v_i, v_{i+1}\}$, then for every edge l of $\mathcal{T}_A$, there exists i and j two distinct integers, $0 \leq i, j \leq 2e-1$, such that $l = l_i = l_j$;

- denote by P_i a projective indecomposable module with character l_i. Then, we have a minimal projective resolution of the A-module $\mathcal{O}$, periodic of period $2e$:

$$\cdots \to P_0 \to P_{2e-1} \to \cdots \to P_1 \to P_0 \to \mathcal{O} \to 0. \tag{4}$$

We have $v_{2e} = v_0$. Given three vertices $v, v', v"$ of $\mathcal{T}_A$, we have $d(v,v') + d(v', v") \equiv d(v, v") \pmod 2$, hence $d(v_i, v_0) \equiv i \pmod 2$. Suppose $l_i = l_j$. Since $\mathcal{T}_A$ is a tree, we have $v_i = v_{j+1}$ and $v_j = v_{i+1}$, hence $i \equiv j+1 \pmod 2$. It follows that $\{l_{2i}\}_{0\leq i\leq e-1}$ is the set of all edges of $\mathcal{T}_A$.

If X is an A-module (resp. a B-module) and i an integer, we define $\Omega^i_A(X)$ (resp. $\Omega^i_B(X)$) to be the i-th Heller translate of X.

The character of $\Omega^i_A\mathcal{O}$ is v_i.

The block B has a similar description which is a particular case of the previous one :

The Brauer tree of B, $\mathcal{T}_B$, is a star whose center is the exceptional vertex, *i.e.*, every edge of $\mathcal{T}_B$ is of the form $\{w, w'\}$ where w' is the exceptional vertex. There is a walk $w_0 = 1_H, w_1, \ldots, w_{2e}$ on $\mathcal{T}_B$ such that :

- Every edge is traversed twice ;
- denote by Q_i a projective indecomposable module with character $w_i + w_{i+1}$. Then, we have a minimal projective resolution of the B-module $\mathcal{O}$, periodic of period $2e$:

$$\cdots \to Q_0 \to Q_{2e-1} \to \cdots \to Q_1 \to Q_0 \to \mathcal{O} \to 0.$$

Note that for any i, $0 \leq i \leq e-1$, w_{2i+1} is the exceptional vertex and $\{w_{2i}\}_{0\leq i\leq e-1}$ is the set of all non-exceptional characters of KB. The module $\Omega^{2i}_B\mathcal{O}$ remains irreducible modulo p and its character is w_{2i}.

Since $e\mathcal{O}Gf$ induces a stable equivalence between A and B, we have $e\mathcal{O}Gf = M \oplus U$ as $(A \otimes B^\circ)$-modules, where M is indecomposable – and then $\bar{M}$ is also indecomposable since M is a p-permutation module – and U is projective (cf lemma 3). We still have

$$M \otimes_B M^* \simeq A \oplus \text{projectives} \quad \text{and} \quad M^* \otimes_A M \simeq B \oplus \text{projectives}.$$

Since M induces a stable equivalence between A and B, tensoring by M commutes with Heller translates, up to projectives, hence $M \otimes_B \Omega^{2i}_B\mathcal{O} \simeq \Omega^{2i}_A\mathcal{O} \oplus$ projectives. Since $\bar{M}$ is indecomposable and $\Omega^{2i}_B k$ is simple, $\bar{M} \otimes_{\bar{B}} \Omega^{2i}_B k$ is indecomposable (cf lemma 5), so that

$$M \otimes_B \Omega^{2i}_B\mathcal{O} \simeq \Omega^{2i}_A\mathcal{O}. \tag{5}$$

Now, since a projective cover of $\Omega_A^{2i}\mathcal{O}$ is P_{2i} and a projective cover of $\Omega_B^{2i}\mathcal{O}$ is Q_{2i}, it follows from lemma 2 that a projective cover of M is :

$$\bigoplus_{0\leq i\leq e-1} P_{2i}\otimes Q_{2i}^* \xrightarrow{\psi} M.$$

For $l=\{v',v"\}$ an edge and v a vertex of $\mathcal{T}_A$, define $\delta(l,v)=\inf(d(v',v),d(v",v))$. Let x be an integer, $0\leq x\leq 2e$, such that v_x is the exceptional vertex of $\mathcal{T}_A$.

Let

$$X=\bigoplus_{\delta(l_{2i},v_x)\equiv x \pmod 2} P_{2i}\otimes Q_{2i}^*$$

and ϕ be the restriction of ψ to X. We then define D to be $0\longrightarrow X\xrightarrow{\phi} M\longrightarrow 0$ (where M is in degree 0).

3.2. Proof of theorem 10.

Let i and j be two integers, $0\leq i,j\leq e-1$. We have $Q_{2j}^*\otimes_B\Omega_B^{2i}\mathcal{O}\simeq \mathrm{Hom}_B(Q_{2j},\Omega_B^{2i}\mathcal{O})$. Since Q_{2j} is a projective cover of $\Omega_B^{2i}\mathcal{O}$ if and only if $i=j$, we have

$$Q_{2j}^*\otimes_B\Omega_B^{2i}\mathcal{O}\simeq\begin{cases}\mathcal{O} & \text{if } i=j,\\ 0 & \text{otherwise.}\end{cases}$$

Hence, we have

$$X\otimes_B\Omega_B^{2i}\mathcal{O}\simeq\begin{cases}P_{2i} & \text{if } \delta(l_{2i},v_x)\equiv x \pmod 2,\\ 0 & \text{otherwise.}\end{cases}$$

It follows from (5) that

$$D\otimes_B\Omega_B^{2i}\mathcal{O}\simeq\begin{cases}0\to 0\to\Omega_A^{2i}\mathcal{O}\to 0 & \text{if } \delta(l_{2i},v_x)\not\equiv x \pmod 2,\\ 0\to P_{2i}\to\Omega_A^{2i}\mathcal{O}\to 0 & \text{if } \delta(l_{2i},v_x)\equiv x \pmod 2\end{cases}$$

where in both cases, $\Omega_A^{2i}\mathcal{O}$ is in degree 0. Let I be the map between the group of characters of B, $R_K(B)$, and the ring of characters of A, $R_K(A)$, induced by D. By (4), we have :

$$I(w_{2i})=\begin{cases}v_{2i} & \text{if } \delta(l_{2i},v_x)\not\equiv x \pmod 2,\\ -v_{2i+1} & \text{if } \delta(l_{2i},v_x)\equiv x \pmod 2.\end{cases}$$

Lemma 11. *The restriction of the map I to the submodule of $R_K(B)$ with basis $\{w_0,w_2,\dots,w_{2(e-1)}\}$ is an isometry.*

Proof. We have $\delta(l_{2i},v_x)\equiv x \pmod 2$ if and only if $\delta(l_{2i},v_x)=d(v_{2i},v_x)$, since $d(v_{2i+1},v_x)\equiv x+1 \pmod 2$. Hence, $\delta(l_{2i},v_x)\equiv x \pmod 2$ if and only if $d(v_{2i},v_x)<d(v_{2i+1},v_x)$. So, $I(w_{2i})$ is, up to sign, the furthest vertex of l_{2i} from v_x. Since $\mathcal{T}_A$ is a tree, the vertices corresponding to $I(w_{2i})$ and $I(w_{2j})$ are equal if

and only if $w_{2i} = w_{2j}$. Note furthermore that $I(w_{2i})$ is, up to sign, an irreducible character. Hence, the lemma follows. □

Corollaire 12. *The map I is an isometry.*

Proof. Indeed, we have $CF(B,K) = K < w_0, w_2, \ldots, w_{2(e-1)} > \oplus CF_p(B,K)$ and the result is given by lemma 8 and lemma 11. □

The following is now a direct consequence of theorem 7 :

Theorem 13. *The complex D induces a Rickard equivalence between A and B.*

We obtain the exact formulation of theorem 10 by replacing D by $0 \longrightarrow X \oplus U \xrightarrow{\delta+id} M \oplus U \longrightarrow 0$, which is homotopy equivalent to D.

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