Categorification of $\mathfrak{sl}_2$ and braid groups

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Abstract. We describe categorifications of $\mathfrak{sl}_2$ and braid groups. In a first part, we give a survey of the case of $\mathfrak{sl}_2$ (joint work with Joseph Chuang [**ChRou**]) and explain how it leads to the construction of derived equivalences. The second part points out the existence of "higher symmetries" in the examples of braid group actions on triangulated categories.

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1. Introduction

1.1. It is classical that various actions of Weyl groups or Lie algebras on vector spaces come from functors acting on abelian or triangulated categories of algebraic or geometric origin, whose Grothendieck group is that space. We want to explain that the natural transformations between these functors should satisfy certain "nice" algebraic relations, leading to a better control of the triangulated categories acted on. Namely, we believe there is a "canonical" categorification of a number of classical algebras or groups, in particular Kac-Moody algebras, Coxeter groups, braid groups (a monoidal category given by generators and relations). We describe here the case of the Lie algebra $\mathfrak{sl}_2$ and braid groups.

The idea of categorifications has been advocated by I. Frenkel (cf *e.g.* [**CrFr**]) and has already found deep applications in low dimensional topology in the work of Khovanov [**Khov**].

1.2. In a joint work with Joseph Chuang, we explain the setting for $\mathfrak{sl}_2$, which leads to a construction of some equivalences of derived categories of representations. Let us explain more precisely one of the main applications of $\mathfrak{sl}_2$-categorifications: let p be a prime number and $\bar{\mathbf{F}}_p$ an algebraic closure of $\mathbf{F}_p$. The following result is a simplified version of a theorem asserting that the derived category of a block of a symmetric group depends only on the defect group, up to equivalence (as conjectured by Broué).

THEOREM 1.1. *Let A and A' be two non-simple blocks of symmetric groups over $\bar{\mathbf{F}}_p$. Then, $D^b(A) \simeq D^b(B)$ if and only if A and B have same number of simple modules.*

Similar results hold for (cyclotomic) Hecke algebras, q-Schur algebras and group algebras of $\mathrm{GL}_n(\mathbf{F}_q)$. The methods apply also to category $\mathcal{O}$ for $\mathfrak{gl}_n(\mathbf{C})$ and to rational representations of $\mathrm{GL}_n(\bar{\mathbf{F}}_p)$.

The main idea is the following: we construct $\mathfrak{sl}_2$-categorifications on the relevant categories. The adjoint action of the Weyl group of $\mathfrak{sl}_2$ will give the derived equivalences.

An $\mathfrak{sl}_2$-categorification on an abelian category consists of the data of endofunctors E, F and natural transformations of functors X, T satisfying certain assumptions. In the examples of $\mathfrak{sl}_2$-categorifications described, X takes different incarnations (Jucys-Murphy elements for Hecke algebras, Casimir for $\mathfrak{gl}(V)$, new for $\mathrm{GL}_n(\mathbf{F}_q)$).

We expect a generalization of the theory to general Kac-Moody algebras. In that setting, one should obtain an action of the corresponding braid group. Note that this type of braid group action seems more general than the one considered in Part 2 of this paper. These categorifications should give an interpretation of the canonical bases and we also hope that vertex operators can be categorified.

1.3. Actions of braid groups on triangulated categories are quite widespread. They arise for instance in representation theory, for constructible sheaves on flag varieties and for coherent sheaves on Calabi-Yau varieties (cf [**RouZi**] and [**SeiTho**] for early occurrences). In Part 2, we suggest that not only the self-equivalences are important, but that the morphisms between them possess some interesting structure.

Let W be a Coxeter group and $\mathcal{C}$ a triangulated category. We consider gradually stronger actions of W or its braid group B_W :

(i) W acting on $K_0(\mathcal{C})$
(ii) a morphism from B_W to the group of isomorphism classes of invertible functors of $\mathcal{C}$
(iii) an action of B_W on $\mathcal{C}$.

We construct a strict monoidal category $\mathcal{B}_W$ categorifying (conjecturally) B_W and we propose an even stronger form of action :

(iv) a morphism of monoidal categories $\mathcal{B}_W \to \mathcal{H}om(\mathcal{C}, \mathcal{C})$

Section §8 is devoted to a construction of a self-equivalence of a triangulated category, generalizing various constructions in representation theory and algebraic geometry. This should be viewed as a categorification of an action of $\mathbf{Z}/2$.

In section §9, we construct a monoidal category categorifying (a quotient of) the braid group. It is a full subcategory of a homotopy category of complexes of bimodules over a polynomial algebra. The setting here is that of Soergel's bimodules.

Section §10 is devoted to the category $\mathcal{O}$ of a semi-simple complex Lie algebra. There are classical functors that induce an action of type (i). We show how to use the constructions of §9, via results of Soergel, to get a genuine action of the braid group and even the stronger type (iv).

The case of flag varieties is considered in §11. There again, there is a classical action up to isomorphism of the braid group on the derived category of constructible sheaves (type (ii)). Using a result of Deligne and checking some compatibilities for general kernel transforms (Appendix in §12), one gets a genuine action of the braid group (type (iii)). Now, using the link with modules over the cohomology ring, we get another proof of this and even the stronger (iv).

In a work in preparation we study homological vanishings and relation with the cohomology of Deligne-Lusztig varieties. We also expect that finding presentations by generators and relations will lead to a new proof of Beilinson-Bernstein's equivalence between category $\mathcal{O}$ and perverse sheaves on the flag variety (or rather of Soergel's version of the equivalence) and a new proof of Andersen-Jantzen-Soergel's proof of Lusztig's conjecture comparing representations in the quantum case and in characteristic p.

1.4. Part 1 is based on a series of lectures given at the Workshop ICRA XI, Queretaro, in August 2004. A few talks have been given in 1998–2000 on the main results of Part 2 (Freiburg, Paris, Yale, Luminy) and I apologize for the delay in putting them on paper.

I would like to thank I. Frenkel, M. Kashiwara, M. Khovanov, and W. Soergel for useful discussions, and the referee for pointing out a mistake in §8.1.1.

Part 1. $\mathfrak{sl}_2$-categorifications

This chapter surveys the main constructions and results of [**ChRou**].

2. $\mathfrak{sl}_2$-categorifications

2.1. We put

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \text{and } h = [e,f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that $\mathfrak{sl}_2(\mathbf{C}) = \mathbf{C}e \oplus \mathbf{C}f \oplus \mathbf{C}h$.

Fix k an algebraically closed field. Let $\mathcal{A}$ be a k-linear abelian category such that all objects have finite composition series (the theory has a counterpart for triangulated categories).

Definition 2.1. *A weak $\mathfrak{sl}_2$-categorification on $\mathcal{A}$ is the data of (E,F) an adjoint pair of exact functors $\mathcal{A} \to \mathcal{A}$ such that*

- *the actions of $[E]$ and $[F]$ on $V = \mathbf{C} \otimes K_0(\mathcal{A})$ give a locally finite representation of $\mathfrak{sl}_2$*

- *given $S \in \mathcal{A}$ simple, then $[S]$ is a weight vector*
- *the functor F is isomorphic to a left adjoint of E.*

The weight space decomposition $V = \bigoplus_{\lambda \in \mathbf{Z}} V_\lambda$ (where $V_\lambda = \ker(h - \lambda)$) gives a decomposition $\mathcal{A} = \bigoplus_\lambda \mathcal{A}_\lambda$ where $\mathcal{A}_\lambda = \{M \in \mathcal{A} \mid [M] \in V_\lambda\}$.

Example 2.2. The 3-dimensional irreducible representation

$$E \uparrow \downarrow F \qquad \begin{array}{c} k\text{-mod} \\ {\scriptstyle \mathrm{Res}} \uparrow \downarrow {\scriptstyle \mathrm{Ind}} \\ k[d]/d^2\text{-mod} \\ {\scriptstyle \mathrm{Ind}} \uparrow \downarrow {\scriptstyle \mathrm{Res}} \\ k\text{-mod} \end{array} \qquad \begin{array}{c} \mathbf{C} \\ {\scriptstyle 1} \uparrow \downarrow {\scriptstyle 2} \\ \mathbf{C} \\ {\scriptstyle 2} \uparrow \downarrow {\scriptstyle 1} \\ \mathbf{C} \end{array} \qquad e \uparrow \downarrow f$$

Example 2.3. Same representation

$$E \uparrow \downarrow F \qquad \begin{array}{c} k[d]/d^2\text{-mod} \\ {\scriptstyle \mathrm{Ind}} \uparrow \downarrow {\scriptstyle \mathrm{Res}} \\ k\text{-mod} \\ {\scriptstyle \mathrm{Res}} \uparrow \downarrow {\scriptstyle \mathrm{Ind}} \\ k[d]/d^2\text{-mod} \end{array} \qquad \begin{array}{c} \mathbf{C} \\ {\scriptstyle 2} \uparrow \downarrow {\scriptstyle 1} \\ \mathbf{C} \\ {\scriptstyle 1} \uparrow \downarrow {\scriptstyle 2} \\ \mathbf{C} \end{array} \qquad e \uparrow \downarrow f$$

The first example will lead to a genuine categorification, while the second one won't.

2.2.

Definition 2.4. *An $\mathfrak{sl}_2$-categorification on $\mathcal{A}$ is a weak categorification together with the data of $X \in \mathrm{End}(E)$, $T \in \mathrm{End}(E^2)$, $q \in k^\times$ and $a \in k$ (with $a \neq 0$ if $q \neq 1$) such that*

- $$\begin{array}{ccccc} & & EEE & & \\ & \swarrow^{ET} & & \searrow^{TE} & \\ EEE & & & & EEE \\ \downarrow{\scriptstyle TE} & & & & \downarrow{\scriptstyle ET} \\ EEE & & & & EEE \\ & \searrow_{ET} & & \swarrow_{TE} & \\ & & EEE & & \end{array} \quad \text{is commutative}$$
- $(T+1)(T-q) = 0$
- $T \circ (EX) \circ T = \begin{cases} q(XE) & \text{if } q \neq 1 \\ XE - T & \text{if } q = 1 \end{cases}$
- *$X - a$ is locally nilpotent.*

Remark 2.5. As soon as V contains a copy of a simple $\mathfrak{sl}_2$-module of dimension 3 or more, then a and q are determined by X and T.

Let us now state our two main Theorems.

THEOREM 2.6. *There is an equivalence* $\Theta : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A})$ *with* $[\Theta] = s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. *It restricts to equivalences* $D^b(\mathcal{A}_\lambda) \xrightarrow{\sim} D^b(\mathcal{A}_{-\lambda})$.
They come from equivalences $K^b(\mathcal{A}) \xrightarrow{\sim} K^b(\mathcal{A})$, *restricting to* $K^b(\mathcal{A}_\lambda) \xrightarrow{\sim} K^b(\mathcal{A}_{-\lambda})$.

Here, $K^b(\mathcal{A})$ denotes the homotopy category of bounded complexes of objects of $\mathcal{A}$.

THEOREM 2.7. *Let* $\lambda \geq 0$. *We have isomorphisms*

$$\sigma + \sum_{j=0}^{\lambda-1} (FX^j) \circ \eta \ : \ EF_{|\mathcal{A}_{-\lambda}} \oplus \mathrm{Id}_{\mathcal{A}_{-\lambda}}^{\bigoplus \lambda} \xrightarrow{\sim} FE_{|\mathcal{A}_{-\lambda}}$$

$$\sigma + \sum_{j=0}^{\lambda-1} \varepsilon \circ (X^j F) \ : \ EF_{|\mathcal{A}_\lambda} \xrightarrow{\sim} FE_{|\mathcal{A}_\lambda} \oplus \mathrm{Id}_{\mathcal{A}_\lambda}^{\bigoplus \lambda}$$

where σ *is the composition*

$$EF \xrightarrow{\eta EF} FEEF \xrightarrow{FTF} FEEF \xrightarrow{FE\varepsilon} FE.$$

EXAMPLE 2.8. We consider the case of Example 2.2. Put $q = 1$ and $a = 0$. Let X be the multiplication by d on $\mathrm{Res} : \mathcal{A}_0 \to \mathcal{A}_2$ and multiplication by $-d$ on $\mathrm{Ind} : \mathcal{A}_{-2} \to \mathcal{A}_0$. Let $T \in \mathrm{End}_k(k[d]/d^2)$ be the automorphism swapping 1 and d. This is an $\mathfrak{sl}_2$-categorification.

3. Affine Hecke algebras

3.1. For $q \neq 1$, let $H_n(q)$ be the affine Hecke algebra of type $\tilde{A}_{n-1}$. with generators $T_1, \ldots, T_{n-1}, X_1^{\pm 1}, \ldots, X_n^{\pm 1}$ and relations

$$\begin{aligned}
(T_i - q)(T_i + 1) &= 0 \\
T_i T_j &= T_j T_i \text{ if } |i-j| > 1 \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\
X_i X_j &= X_j X_i \\
X_i T_j &= T_j X_i \text{ if } i - j \neq 0, 1 \\
T_i X_i T_i &= q X_{i+1}.
\end{aligned}$$

Note that $H_n^f = k\langle T_1, \ldots, T_{n-1} \rangle$ is the Hecke algebra of $\mathfrak{S}_n$ and that $H_n = k[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \otimes H_n^f$.

For $q = 1$, define $H_n(1)$ as the degenerate affine Hecke algebra of type $\tilde{A}_{n-1}$, with generators $T_1, \ldots, T_{n-1}, X_1, \ldots, X_n$. The relations are the same as above, except the last relation which is replaced by $X_{i+1} T_i = T_i X_i + 1$.

We have $H_n(1) = k[X_1, \ldots, X_n] \otimes k\mathfrak{S}_n$.

3.2. An important feature is that the (degenerate) affine Hecke algebras arise naturally from an $\mathfrak{sl}_2$-categorification:

PROPOSITION 3.1. *The correspondence*

$$T_i \mapsto E^{n-i-1} T E^{i-1}, \ X_i \mapsto E^{n-i} X E^{i-1}$$

defines a morphism of algebras $H_n \to \mathrm{End}(E^n)$.

This will eventually enable us to control $\mathfrak{sl}_2$-categorifications. All H_n-modules considered will be locally nilpotent for $(X_1-a),\ldots,(X_n-a)$. As a consequence, their restriction to H_n^f will be free (this would fail for the affine Hecke algebra at $q=1$).

We define two one-dimensional representations of H_n^f:

- 1 is given by $T_i \mapsto q$
- sgn is given by $T_i \mapsto -1$.

Given $\tau \in \{1, \mathrm{sgn}\}$, put $c_n^\tau = \sum_{w\in\mathfrak{S}_n} q^{-l(w)}\tau(T_w)T_w \in Z(H_n^f)$ and $E^{(n,\tau)} = E^n \cdot c_n^\tau$.

Proposition 3.2. *We have $E^n \simeq n! \cdot E^{(n,\tau)}$.*

We have obtained two different (though isomorphic) categorifications of the divided powers.

4. Categorifications on blocks of $\mathfrak{S}_n$

4.1. Assume $p = \mathrm{char}(k) > 0$. We view $k\mathfrak{S}_n$ as a quotient of $H_n(1)$ via the morphism of algebras given by $T_i \mapsto (i, i+1)$, $X_1 \mapsto 0$.
Let $a \in \mathbf{F}_p$. Given M a $k\mathfrak{S}_n$-module, we denote by $F_{a,n}(M)$ the generalized a-eigenspace of X_n. This is a $k\mathfrak{S}_{n-1}$-module. We have a decomposition $\mathrm{Res}^{k\mathfrak{S}_n}_{k\mathfrak{S}_{n-1}} = \bigoplus_{a\in k} F_{a,n}$. There is a corresponding decomposition $\mathrm{Ind}^{k\mathfrak{S}_n}_{k\mathfrak{S}_{n-1}} = \bigoplus_{a\in k} E_{a,n}$, where $E_{a,n}$ is left and right adjoint to $F_{a,n}$. We put $E_a = \bigoplus_{n\geq 1} E_{a,n}$ and $F_a = \bigoplus_{n\geq 1} F_{a,n}$.

Given $a \in \mathbf{F}_p$, the functors E_a and F_a give a weak $\mathfrak{sl}_2$-categorification on $\mathcal{A} = \bigoplus_{n\geq 0} k\mathfrak{S}_n$-mod.
We denote by X the endomorphism of E_a given on $E_{a,n}$ by right multiplication by X_n. We denote by T the endomorphism of E_a^2 given on $E_{a,n}E_{a,n-1}$ by right multiplication by $(n-1,n)$. This gives an $\mathfrak{sl}_2$-categorification on $\mathcal{A}$ (here, $q=1$).

4.2. Let us recall the action of the affine Lie algebra $\widehat{\mathfrak{sl}}_p$ and some of its properties [**LLT**].

Theorem 4.1 (Lascoux-Leclerc-Thibon). *-The functors E_a and F_a for $a \in \mathbf{F}_p$ give rise to an action of $\widehat{\mathfrak{sl}}_p$ on $\bigoplus_{n\geq 0} K_0(k\mathfrak{S}_n$-mod$)$.*
-The decomposition of $K_0(k\mathfrak{S}_n$-mod$)$ in blocks coincides with its decomposition in weight spaces.
-Two blocks of symmetric groups have the same p-weight if and only if they are in the same orbit under the adjoint action of the affine Weyl group.

Theorem 4.2. *Let A and B be two blocks of symmetric groups over k. The following are equivalent:*

1. *A and B have the same p-weights*
2. *A and B have isomorphic defect groups*
3. *$D^b(A) \simeq D^b(B)$.*

If we exclude the case where $p=2$ and A or B is a matrix algebra, we get an equivalence with

4. *A and B have the same number of simple modules.*

Hints of proof. Theorem 4.1 reduces the proof to the case of an elementary reflection s_a. Now, we have an $\mathfrak{sl}_2$-categorification and we get a derived equivalence by Theorem 2.6. □

We can now deduce that Broué's abelian defect conjecture [**Br**] holds for symmetric groups.

Corollary 4.3 (Broué's conjecture). *Let A be a block of a symmetric group over k. Assume the defect groups of A are abelian,* i.e., *the p-weight w is $< p$. Then, A is derived equivalent to $k[(\mathbf{Z}/p \rtimes \mathbf{Z}/(p-1))^w \rtimes \mathfrak{S}_w]$.*

Hints of proof. We know [**ChKe**] that given w, there is a block B of a symmetric group with p-weight w that is Morita equivalent to the principal block C of $k[(\mathfrak{S}_p)^w \rtimes \mathfrak{S}_w]$. Since the principal block of $k[\mathfrak{S}_p]$ is derived equivalent to $k[\mathbf{Z}/p \rtimes \mathbf{Z}/(p-1)]$ [**Ri1**], we deduce [**Mar**] that C is derived equivalent to $k[(\mathbf{Z}/p \rtimes \mathbf{Z}/(p-1))^w \rtimes \mathfrak{S}_w]$. Now, A is derived equivalent to B by Theorem 4.2. □

5. Minimal categorifications

5.1. Let $\mathfrak{m}_n$ be the ideal of polynomials generated by $X_1 - a, \ldots, X_n - a$ and let $\bar{H}_n = H_n/((\mathfrak{m}_n)^{\mathfrak{S}_n})$. Given $0 \le i \le n$, let $\bar{H}_{i,n}$ be the image of H_i in $\bar{H}_n$.

The algebra $\bar{H}_{i,n}$ is isomorphic to a matrix algebra over $H^*(\mathrm{Gr}(i,n), k)$, where $\mathrm{Gr}(i,n)$ is the Grassmannian variety of i-dimensional subspaces of $\mathbf{C}^n$.

The algebra $\bar{H}_{i,n}$ is symmetric and we have left and right adjoint functors

$$\bar{H}_{i,n}\text{-mod} \underset{\text{Res}}{\overset{\text{Ind}}{\rightleftarrows}} \bar{H}_{i+1,n}\text{-mod}$$

We get as in the symmetric group case an $\mathfrak{sl}_2$-categorification on $\mathcal{A}(n) = \bigoplus_{0 \le i \le n} \bar{H}_{i,n}\text{-mod}$. We obtain the irreducible representation of dimension $n+1$ of $\mathfrak{sl}_2$ on $K_0(\mathcal{A}(n))$. We will see below that $\mathcal{A}(n)$ gives a "minimal" categorification of that representation.

5.2. Let U be a simple object of $\mathcal{A}$ with $FU = 0$. Take n maximal such that $E^n(U) \neq 0$ (and put $h(U) = n$). Let $B_i = \bar{H}_{i,n}$. The morphism $H_i \to \mathrm{End}(E^i U)$ induces an isomorphism $B_i \xrightarrow{\sim} \mathrm{End}(E^i U)$.

There are commutative diagrams of functors

$$\begin{array}{ccc} B_{i+1}\text{-mod} & \xrightarrow{E^{i+1}U\otimes_{B_{i+1}}-} & \mathcal{A} \\ {\scriptstyle B_{i+1}\otimes_{B_i}-}\uparrow & & \uparrow{\scriptstyle E} \\ B_i\text{-mod} & \xrightarrow[E^{i}U\otimes_{B_{i}}-]{} & \mathcal{A} \end{array} \qquad \text{(easy)}$$

$$\begin{array}{ccc} B_{i+1}\text{-mod} & \xrightarrow{E^{i+1}U\otimes_{B_{i+1}}-} & \mathcal{A} \\ {\scriptstyle B_{i+1}\otimes_{B_{i+1}}-}\downarrow & & \downarrow{\scriptstyle F} \\ B_i\text{-mod} & \xrightarrow[E^{i}U\otimes_{B_{i}}-]{} & \mathcal{A} \end{array} \qquad \text{(key)}$$

Theorem 5.1. *The diagrams above give rise to a canonical morphism of $\mathfrak{sl}_2$-categorifications $R_U : \mathcal{A}(n) \to \mathcal{A}$.*

Let I_n be the set of isomorphism classes of simple objects U of $\mathcal{A}$ such that $FU = 0$ and $h(U) = n$ ("lowest weight simples").

COROLLARY 5.2. *We have a canonical morphism of $\mathfrak{sl}_2$-categorifications*

$$\sum_{n,U\in I_n} R_U : \bigoplus_{n,U\in I_n} \mathcal{A}(n) \to \mathcal{A}$$

It induces an isomorphism

$$\begin{array}{ccc} \bigoplus_{n,U\in I_n} \mathbf{Q}\otimes K_0(\mathcal{A}(n)\text{-proj}) & \xrightarrow{\sim} & \mathbf{Q}\otimes K_0(\mathcal{A}) \\ \sim\big\downarrow & & \\ \bigoplus_{n,U\in I_n} \mathbf{Q}\otimes K_0(\mathcal{A}(n)) & & \end{array}$$

HINTS OF PROOF OF THEOREM 2.7. To test isomorphism, it is enough to evaluate at E^iU for all U simple with $FU=0$. By Theorem 5.1, we deduce it is enough to consider the case $\mathcal{A}(n)$, where we do explicit calculations. □

6. Homotopy and derived equivalences

6.1. We have

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp(-f)\exp(e)\exp(-f).$$

Let $v \in V_{-\lambda}$. Then, $s(v) = \sum_r \frac{(-1)^r}{r!(\lambda+r)!} e^{\lambda+r} f^r(v)$.

There is no exact functor $\mathcal{A}\to\mathcal{A}$ giving rise to the action of s on $K_0(\mathcal{A})$: we need to use complexes. We define Θ_λ a complex of exact functors $\mathcal{A}_{-\lambda}\to\mathcal{A}_\lambda$, following Rickard [**Ri2**].

$$(\Theta_\lambda)^{-r} = \begin{cases} \left(E^{(\mathrm{sgn},\lambda+r)}F^{(1,r)}\right)_{|\mathcal{A}_{-\lambda}} & \text{if } r, \lambda+r \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The differential is defined by the commutative diagram:

$$\begin{array}{ccc} E^{(\mathrm{sgn},\lambda+r)}F^{(1,r)} & \hookrightarrow & E^{\lambda+r}F^r = E^{\lambda+r-1}EFF^{r-1} \\ \Big\downarrow{\scriptstyle d^{-r}} & & \Big\downarrow{\scriptstyle E^{\lambda+r-1}\varepsilon F^{r-1}} \\ E^{(\mathrm{sgn},\lambda+r-1)}F^{(1,r-1)} & \hookrightarrow & E^{\lambda+r-1}F^{r-1} \end{array}$$

Note that $d^{1-r}d^{-r}=0$. Indeed, $c_2^{\mathrm{sgn}}c_2^1 = 0$, hence $E^2F^2 \xrightarrow{c_2^{\mathrm{sgn}}c_2^1} E^2F^2 \xrightarrow{\varepsilon_2} \mathrm{Id}$ vanishes.

THEOREM 6.1. *The functor Θ_λ induces equivalences $K^b(\mathcal{A}_{-\lambda}) \xrightarrow{\sim} K^b(\mathcal{A}_\lambda)$ and $D^b(\mathcal{A}_{-\lambda}) \xrightarrow{\sim} D^b(\mathcal{A}_\lambda)$. The functor $\Theta = \bigoplus_\lambda \Theta_\lambda$ gives equivalences $K^b(\mathcal{A}) \xrightarrow{\sim} K^b(\mathcal{A})$ and $D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A})$. We have $[\Theta] = s$.*

HINTS OF PROOF. Consider a left adjoint $\Theta_\lambda^\vee$. To show that $p : \Theta_\lambda^\vee\Theta_\lambda \to \mathrm{Id}$ is an isomorphism in the derived category, it is enough to do so after evaluation at E^iU for U simple with $FU=0$. We reduce the problem to the case $\mathcal{A}(n)$ by Theorem 5.1, and this case is handled by Lemma 6.2 below.

To get the homotopy equivalence, note that given $M \in \mathcal{A}_{-\lambda}$, there is $N \in \mathcal{A}$ containing M as a direct summand, there is an $\mathfrak{sl}_2$-categorification on $\mathrm{End}(N)$-mod and a morphism of $\mathfrak{sl}_2$-categorifications $\mathrm{End}(N)$-mod $\to \mathcal{A}$. The derived equivalence for $\mathrm{End}_A(M)$-mod gives a homotopy isomorphism $p(M)$. □

LEMMA 6.2. *Let $n \geq 0$ and $\mathcal{A} = \mathcal{A}(n)$ be the minimal categorification. Fix $\lambda \geq 0$ and let $l = \frac{n-\lambda}{2}$. The homology of the complex of functors Θ_λ is concentrated in degree $-l$ and $H^{-l}\Theta_\lambda : \mathcal{A}_{-\lambda} \xrightarrow{\sim} \mathcal{A}_\lambda$ is an equivalence.*

6.2. There is a filtration of V, $0 = V\{-1\} \subset V\{0\} \subset \cdots$ where $V\{i\}$ is the sum of simple submodules of dimension $\leq i+1$ (recall that V is locally finite for $\mathfrak{sl}_2$, hence $V = \bigcup_{i\geq 0} V\{i\}$). Correspondingly, $\mathcal{A}\{i\} = \{M \in \mathcal{A} \mid [M] \in V\{i\}\}$. These are sub-$\mathfrak{sl}_2$-categorifications.

Thus, Θ restricts to $D^b(\mathcal{A}\{i\}) \xrightarrow{\sim} D^b(\mathcal{A}\{i\})$. Passing to quotients, we get a commutative diagram

$$\begin{array}{ccc}
\mathcal{A}\{i\}_{-\lambda}/\mathcal{A}\{i-1\}_{-\lambda} & \hookrightarrow & D^b(\mathcal{A}\{i\}_{-\lambda})/D^b_{\mathcal{A}\{i-1\}}(\mathcal{A}\{i\}_{-\lambda}) \\
\sim\downarrow & & \sim\downarrow \Theta_\lambda[(i-\lambda)/2] \\
\mathcal{A}\{i\}_{\lambda}/\mathcal{A}\{i-1\}_{\lambda} & \hookrightarrow & D^b(\mathcal{A}\{i\}_{\lambda})/D^b_{\mathcal{A}\{i-1\}}(\mathcal{A}\{i\}_{\lambda})
\end{array}$$

Θ_λ is a "perverse Morita equivalence".

7. Representations of $\mathfrak{gl}_n(\mathbf{C})$

Let $V = \mathbf{C}^n$ and $\mathfrak{g} = \mathfrak{gl}(V)$. Let M be a $\mathfrak{g}$-module : we have an action map $\mathfrak{g} \otimes M \to M$ giving by adjunction a map $X_M : V \otimes M \to V \otimes M$. This gives an endomorphism X of the functor $V \otimes -$. Denote by T the endomorphism of $V \otimes V \otimes -$ coming from the swap automorphism of $V \otimes V$.

Consider now the functor $V \otimes - : \mathcal{O} \to \mathcal{O}$, where $\mathcal{O}$ is the BGG category of $\mathfrak{g}$-modules that are diagonalizable for the action of the diagonal matrices and locally finite for the action of the upper triangular matrices.

Fix $a \in \mathbf{C}$ and let E be the generalized a-eigenspace of X on $V \otimes -$. Similarly, considering $V^* \otimes -$, we get a functor F left and right adjoint to E. Together with the actions of X and T, this gives an $\mathfrak{sl}_2$-categorification on $\mathcal{O}$.

There is a similar construction for finite dimensional representations of $\mathrm{GL}_n(\bar{\mathbf{F}}_p)$. We deduce that blocks with the same $\tilde{A}_{n-1}$ stabilizer are derived equivalent (a conjecture of Rickard).

Part 2. Categorification of braid groups

8. Self-equivalences

We describe a categorification of the notion of reflection with respect to a subspace. The ambient space is K_0 of a triangulated category, the subspace is another triangulated category and the embeddings and projections are given by functors. We actually allow an automorphism of the "subspace" category, which allows to categorify the q-analog of a reflection.

We present here how a functor from a given triangulated category gives rise to a self-equivalence of that category, when the functor satisfies some conditions.

Then, we give three special “classical” cases. The first one concerns constructible sheaves on a $\mathbf{P}^1$-fibration (it occurs typically with flag varieties, cf §11). The second one deals with the case where the target category is the derived category of vector spaces, where we recover the theory of spherical objects and twist functors (it arises as counterparts of Dehn twists via mirror symmetry). The last application essentially concerns derived categories of finite dimensional algebras (it occurs in particular within rational representation theory, cf §10).

All functors between additive (resp. triangulated) categories are assumed to be additive (resp. triangulated).

Given an additive category $\mathcal{C}$, we denote by $K(\mathcal{C})$ the homotopy category of complexes of objects of $\mathcal{C}$.

Given an algebra A over a field k, we denote by A-mod the category of finitely generated left A-modules. We put $A^{\mathrm{en}} = A \otimes_k A^{\mathrm{opp}}$, where A^{opp} is the opposite algebra.

Given a graded algebra A, we denote by A-modgr the category of finitely generated graded A-modules.

8.1. A general construction.

8.1.1. This section can probably be skipped in a first reading.

We will be working here with *algebraic* triangulated categories (following Keller), a simple setting that provides functorial cones.

Let $\mathcal{E}$ be a Frobenius category (an exact category with enough projective and injective objects and where injective and projective objects coincide). Let $\mathrm{Comp}_{\mathrm{acyc}}(\mathcal{E}\text{-proj})$ be the category of acyclic complexes of projective objects of $\mathcal{E}$. Let Frob be the 2-category of Frobenius categories, with 1-arrows the exact functors that send projectives to projectives and 2-arrows the natural transformations of functors.

The construction $\mathcal{E} \mapsto \mathrm{Comp}_{\mathrm{acyc}}(\mathcal{E}\text{-proj})$ is an endo-2-functor of Frob. The 2-functor from $\mathcal{E}$ to the 2-category of triangulated categories that sends $\mathcal{E}$ to its stable category $\bar{\mathcal{E}}$ factors through the previous functor.

The important point is that the category $\mathrm{Comp}_{\mathrm{acyc}}(\mathcal{E}\text{-proj})$ has functorial cones. Given $F, G : \mathrm{Comp}_{\mathrm{acyc}}(\mathcal{E}\text{-proj}) \to \mathrm{Comp}_{\mathrm{acyc}}(\mathcal{E}'\text{-proj})$ and $\phi : F \to G$, then we have a well defined cone $C(\phi)$ of ϕ and we have morphisms $G \to C(\phi)$ and $C(\phi) \to F[1]$ such that $F \to G \to C(\phi) \to F[1]$ gives a distinguished triangle of functors from $\bar{\mathcal{E}}$ to $\bar{\mathcal{E}}'$.

Note that if $\phi_0 : F_0 \to G_0$ is a morphism of functors (exact, preserving projectives) between $\mathcal{E}$ and $\mathcal{E}'$, then we get via $\mathrm{Comp}_{\mathrm{acyc}}(-)$ a morphism of functors $\phi : F \to G$, with $F, G : \mathrm{Comp}_{\mathrm{acyc}}(\mathcal{E}\text{-proj}) \to \mathrm{Comp}_{\mathrm{acyc}}(\mathcal{E}'\text{-proj})$ the functors induced by F_0, G_0.

The category of functors (exact, preserving projectives) $\mathrm{Comp}_{\mathrm{acyc}}(\mathcal{E}\text{-proj}) \to \mathrm{Comp}_{\mathrm{acyc}}(\mathcal{E}'\text{-proj})$ is a Frobenius category. Let $\mathcal{C}$ be its stable category. We define the category $\mathrm{AlgTr}(\mathcal{E}, \mathcal{E}')$ to be the localization of $\mathcal{C}$ with respect to the morphisms ϕ such that $\bar{\phi}(X)$ is an isomorphism for all X in $\bar{\mathcal{E}}$, where $\bar{\phi}$ is the induced morphism at the level of stable categories. So, the objects of $\mathrm{AlgTr}(\mathcal{E}, \mathcal{E}')$ are the exact functors $\mathrm{Comp}_{\mathrm{acyc}}(\mathcal{E}\text{-proj}) \to \mathrm{Comp}_{\mathrm{acyc}}(\mathcal{E}'\text{-proj})$ and $\mathrm{Hom}_{\mathrm{AlgTr}(\mathcal{E},\mathcal{E}')}(F, G)$ is the image of $\mathrm{Hom}(F, G)$ in $\mathrm{Hom}_{\mathrm{Fun}(\bar{\mathcal{E}},\bar{\mathcal{E}}')}(\bar{F}, \bar{G})$.

This defines the 2-category of algebraic triangulated categories AlgTr, with objects the Frobenius categories $\mathcal{E}$. We have a 2-functor from AlgTr to the 2-category of triangulated categories obtained by sending $\mathcal{E}$ to $\bar{\mathcal{E}}$.

8.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be two algebraic triangulated categories, $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be two functors and Φ be a self-equivalence of $\mathcal{C}$. Let there be given also two adjoint pairs (F, G) and $(G, F\Phi)$. We have four morphisms (units and counits of the adjunctions)

$$\eta : 1_{\mathcal{D}} \to F\Phi G, \quad \varepsilon : GF\Phi \to 1_{\mathcal{C}}$$

$$\eta' : 1_{\mathcal{C}} \to GF, \quad \varepsilon' : FG \to 1_{\mathcal{D}}.$$

Let Υ be the cocone of ε' and Υ' be the cone of η : there are distinguished triangles of functors $\Upsilon \to FG \xrightarrow{\varepsilon'} 1_{\mathcal{D}} \rightsquigarrow$ and $1_{\mathcal{D}} \xrightarrow{\eta} F\Phi G \to \Upsilon' \rightsquigarrow$.

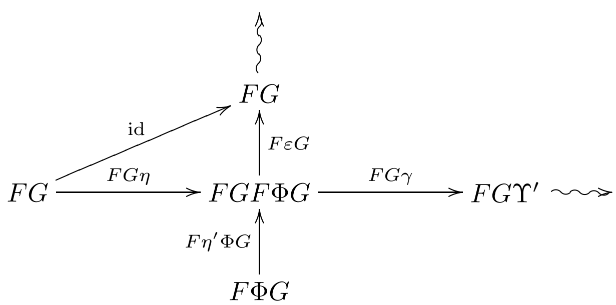
Assume

$$1_{\mathcal{C}} \xrightarrow{\eta'} GF \xrightarrow{\varepsilon\Phi^{-1}} \Phi^{-1} \overset{0}{\rightsquigarrow} \tag{1}$$

is a distinguished triangle (*i.e.*, there is an exact sequence $0 \to 1_{\mathcal{C}} \xrightarrow{\eta'} GF \xrightarrow{\varepsilon\Phi^{-1}} \Phi^{-1} \to 0$ in the additive category of functors).

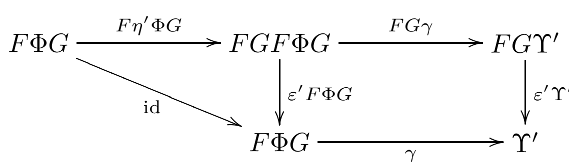
Proposition 8.1. *The functors Υ and Υ' are inverse self-equivalences of $\mathcal{D}$.*

Proof. Let γ be the map $F\Phi G \to \Upsilon'$ in the triangle above, *i.e.*, we have the distinguished triangle $1_{\mathcal{D}} \xrightarrow{\eta} F\Phi G \xrightarrow{\gamma} \Upsilon' \rightsquigarrow$. We have a commutative diagram with horizontal and vertical distinguished triangles



The octahedral axiom shows that $(FG\gamma) \circ (F\eta'\Phi G) : F\Phi G \xrightarrow{\sim} FG\Upsilon'$ is an isomorphism.

We have a commutative diagram



The distinguished triangle $\Upsilon\Upsilon' \to FG\Upsilon' \xrightarrow{\varepsilon'\Upsilon'} \Upsilon' \rightsquigarrow$ gives a distinguished triangle $\Upsilon\Upsilon' \to F\Phi G \xrightarrow{\gamma} \Upsilon' \rightsquigarrow$, hence $\Upsilon\Upsilon' \simeq 1_{\mathcal{D}}$.

The case of $\Upsilon'\Upsilon$ is similar — note that the triangle (1) shows that $GF \simeq \mathrm{id}_{\mathcal{D}} \oplus \Phi^{-1}$, hence Φ commutes with GF. $\square$

REMARK 8.2. One sees easily that $\eta F + F\Phi\eta' : F \oplus F\Phi \xrightarrow{\sim} F\Phi G F$ and $G\eta + \eta'\Phi G : G \oplus \Phi G \xrightarrow{\sim} GF\Phi G$ are isomorphisms. One can show that the requirement that $\eta FG + F\Phi\eta' G : FG \oplus F\Phi G \to F\Phi GFG$ and $FG\eta + F\eta'\Phi G : FG \oplus F\Phi G \to FGF\Phi G$ are isomorphisms, instead of the stronger requirement that (1) is a distinguished triangle, is enough to get Proposition 8.1.

8.1.3. Let us recall a version of Barr-Beck's Theorem ([**Mac**, §VI.7, exercice 7], [**De2**, §4.1]).

Let $\mathcal{C}$ be a category. A comonad is the data of a functor $H : \mathcal{C} \to \mathcal{C}$, of $c : H \to H^2$ and $\varepsilon : H \to \mathrm{id}$ such that $(cH) \circ c = (Hc) \circ c$ and $(\varepsilon H) \circ c = (H\varepsilon) \circ c$. Note that ε is determined by c.

A coaction of (H, c, ε) on an object M of $\mathcal{C}$ is the data of $\rho : M \to H(M)$ such that $\varepsilon(M) \circ \rho = \mathrm{id}_M$ and $c \circ \rho = F(\rho) \circ \rho$. The category (H, c, ε)-comod has objects the pairs (M, ρ) and a morphism $(M, \rho) \to (M', \rho')$ is a morphism $f : M \to M'$ such that $\rho' f = H(f)\rho$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian (resp. algebraic triangulated categories), $T : \mathcal{A} \to \mathcal{B}$ an exact functor (resp. a triangulated functor). Assume T has a right adjoint U. Put $H = TU$, denote by $\varepsilon : H \to \mathrm{id}_{\mathcal{B}}$ and $\eta : \mathrm{id}_{\mathcal{A}} \to UT$ the counit and unit of adjunctions and let $c = T\eta U : H \to H^2$.

We have a functor $\tilde{T} : \mathcal{A} \to (H, c, \varepsilon)$-comod given by $M \mapsto (TM, T\eta(M))$.

The following Theorem is an easy application of Barr-Beck's general result to abelian and triangulated categories.

THEOREM 8.3. *If T is faithful, then $\tilde{T} : \mathcal{A} \xrightarrow{\sim} (H, c, \varepsilon)$-*comod *is an equivalence.*

We deduce from this Theorem that the category $\mathcal{C}$, together with the functors F, G and the adjunctions, is determined by $\mathcal{D}$, $\Theta = FG$ and $c = F\eta' G : \Theta \to \Theta^2$. We view this as a categorical version of the "fixed points" construction.

REMARK 8.4. Let $V = K_0(\mathcal{D})$, $U = K_0(\mathcal{C})$, $f = [F] : U \to V$ and $g = [G] : V \to U$. Assume $[\Phi] = \mathrm{id}_U$. Then, $gf = 2\,\mathrm{id}_U$ and $\psi = -[\Upsilon] : x \mapsto x - fg(x) : V \to V$ is an involution. One recovers U (up to unique isomorphism) from ψ acting on V as V^{ψ}.

8.2. Applications.

8.2.1. We consider schemes of finite type type over an algebraic closure of a finite field $\mathbf{F}_q$ (the case of complex algebraic varieties is similar). Let $\pi : X \to Y$ be a smooth projective morphism already defined over $\mathbf{F}_q$. Assume the geometric fibers are projective lines.

Let Λ be a field of coefficients (=an extension of $\mathbf{Q}_\ell$ for $\ell \not| q$ a prime number). Put $\mathcal{C} = D^b(Y)$ and $\mathcal{D} = D^b(X)$ (bounded derived categories of constructible sheaves of Λ-vector spaces). Take $F = \pi^*$, $G = R\pi_*$ and $\Phi = ?(1)[2]$. We have a canonical isomorphism (projection formula) $? \otimes R\pi_*\Lambda_X \xrightarrow{\sim} R\pi_*\pi^*$. Via this isomorphism, η' becomes $\mathrm{id} \otimes \eta'(\Lambda_Y)$ and ε becomes $\mathrm{id} \otimes t$, where $t : R\pi_*\Lambda_X \to \Lambda_Y(-1)[-2]$ is the trace map (an isomorphism on $\mathcal{H}^2$).

So, the triangle (1) is obtained from the triangle

$$\Lambda_Y \xrightarrow{\eta'(\Lambda_Y)} R\pi_*\Lambda_X \xrightarrow{t} \Lambda_Y(-1)[-2] \rightsquigarrow$$

by applying $?\otimes$. This is indeed a distinguished triangle, for it is so at geometric fibers.

Let $\mathcal{L}$ be a relative ample sheaf for π and $c \in H^2(X, \Lambda(1))$ be its first Chern class. The hard Lefschetz Theorem states that the composition

$$\Lambda_Y \xrightarrow{\eta'(\Lambda_Y)} R\pi_*\Lambda_X \xrightarrow{c} R\pi_*\Lambda_X(1)[2] \xrightarrow{t(1)[2]} \Lambda_Y$$

is an isomorphism. It follows that the connecting map in the triangle above is zero.

Thus, we are in the setting of §8.1.2 and we get a self-equivalence of $D^b(X)$.

This can be also constructed as a kernel transform. Let $\alpha, \beta : X \times_Y X \to X$ be the first and second projections. Let $i : \Delta X \to X \times_Y X$ be the closed immersion of the diagonal and $j : Z \to X \times_Y X$ be the open immersion of the complement of ΔX. Denote by $\tilde{\varepsilon} : 1_{D^b(X\times_Y X)} \to i_*i^*$ and $\tilde{\eta} : Rj_!j^* \to 1_{D^b(X\times_Y X)}$ the adjunction morphisms. One checks easily that there is a commutative diagram where the rows are distinguished triangles

$$\begin{array}{ccccc} \Upsilon & \longrightarrow & \pi^*R\pi_* & \xrightarrow{\varepsilon'} & 1_{\mathcal{D}} \rightsquigarrow \\ & & \downarrow \wr & & \downarrow \wr \\ R\beta_*Rj_!j^*\alpha^* & \xrightarrow[R\beta_*\tilde{\eta}\alpha^*]{} & R\beta_*\alpha^* & \xrightarrow[R\beta_*\tilde{\varepsilon}\alpha^*]{} & R\beta_*i_*i^*\alpha^* \rightsquigarrow \end{array}$$

where the middle vertical map is the base change isomorphism.

Denote by $p, q : Z \to X$ the first and second projections. Then, $\Upsilon \simeq Rp_!q^*$ and $\Upsilon' \simeq Rp_*q^!$.

8.2.2. Assume we are in the setting of §8.1.2 with $\mathcal{C} = D^b(k\text{-mod})$ where k is a field and the categories and functors involved are k-linear. There is an integer n such that $\Phi = ?[n]$. Let $E = F(k)$. Then, $F \simeq E\otimes_k ?$ and $G \simeq R\operatorname{Hom}(E, ?)$. The morphism ε comes from $t : \operatorname{Hom}(E, E[n]) \to k$.

The morphism ε is the counit of an adjoint pair $(G, F\Phi)$ if and only if we have $\dim_k \bigoplus_i \operatorname{Hom}(E, M[i]) < \infty$ for all $M \in \mathcal{D}$ and

$$\operatorname{Hom}(E, M) \times \operatorname{Hom}(M, E[n]) \to k,\ (f, g) \mapsto t(gf)$$

is a perfect pairing for all $M \in \mathcal{D}$.

The triangle (1) is distinguished if and only if $0 \to k \cdot \mathrm{id} \to \bigoplus_i \operatorname{Hom}(E, E[i]) \xrightarrow{t} k \to 0$ is an exact sequence.

In other words, E is an n-spherical object and Υ, Υ' are the corresponding twist functors of Seidel and Thomas [**SeiTho**, §2b]. So, the framework above corresponds exactly to the twist functor theory when $\mathcal{C} \simeq D^b(k\text{-mod})$.

Remark 8.5. The case $\mathcal{C} = D^b(k^d\text{-mod})$ also leads to interesting examples.

Remark 8.6. It would be interesting to see if the construction of §8.1.2 can be used to construct automorphisms of derived categories of Calabi-Yau varieties corresponding, via Kontsevich's homological mirror symmetry conjecture, to graded symplectic automorphisms on the mirror associated to Lagrangian submanifolds more complicated than spheres.

8.2.3. Let us consider here two abelian categories $\mathcal{A}$ and $\mathcal{B}$ and $\tilde{F} : \mathcal{A} \to \mathcal{B}$, $\tilde{G} : \mathcal{B} \to \mathcal{A}$ and $\tilde{\Phi}$ a self-equivalence of $\mathcal{A}$. We assume we have two adjoint pairs $(\tilde{F}, \tilde{G})$ and $(\tilde{G}, \tilde{F}\tilde{\Phi})$. So, we have four morphisms (units and counits of the two adjunctions)

$$\tilde{\eta} : 1_{\mathcal{B}} \to \tilde{F}\tilde{\Phi}\tilde{G}, \quad \tilde{\varepsilon} : \tilde{G}\tilde{F}\tilde{\Phi} \to 1_{\mathcal{A}}$$
$$\tilde{\eta}' : 1_{\mathcal{A}} \to \tilde{G}\tilde{F}, \quad \tilde{\varepsilon}' : \tilde{F}\tilde{G} \to 1_{\mathcal{B}}.$$

Let $\tilde{\Upsilon}$ be the complex $0 \to \tilde{F}\tilde{G} \xrightarrow{\tilde{\varepsilon}'} 1_{\mathcal{B}} \to 0$ and $\tilde{\Upsilon}'$ the complex $0 \to 1_{\mathcal{B}} \xrightarrow{\tilde{\eta}} \tilde{F}\tilde{\Phi}\tilde{G} \to 0$ (with $\tilde{F}\tilde{G}$ and $\tilde{F}\tilde{\Phi}\tilde{G}$ in degree 0). We put $\mathcal{C} = K(\mathcal{A})$ and $\mathcal{D} = K(\mathcal{B})$ and we denote by F, G, etc... the extensions of $\tilde{F}$, $\tilde{G}$, etc... to $\mathcal{C}$ and $\mathcal{D}$.

Assume $\mathcal{B}$ is artinian and noetherian (every object is a finite extension of simple objects). If we have the equality $[\tilde{G}\tilde{F}] = [\mathrm{id}] + [\tilde{\Phi}^{-1}]$ as endomorphisms of $K_0(\mathcal{A})$ (or more generally, if $[\tilde{F}\tilde{\Phi}\tilde{G}\tilde{F}\tilde{G}] = [\tilde{F}\tilde{G}\tilde{F}\tilde{\Phi}\tilde{G}] = [\tilde{F}\tilde{G}] + [\tilde{F}\tilde{\Phi}\tilde{G}]$ in $\mathrm{End}(K_0(\mathcal{B}))$), then, the conclusion of Proposition 8.1 remains valid.

Let us justify this, following ideas of Rickard [**Ri3**, §3]. There is an adjoint pair (Υ', Υ), hence there is a map $u : \mathrm{id} \to \Upsilon\Upsilon'$ that doesn't vanish on a non-zero object of $\mathcal{B}$. One shows that $\Upsilon\Upsilon'$ is homotopy equivalent to a complex of functors with only one non-zero term, R, in degree 0 and R is an exact functor. The assumption on classes shows that $[R] = [\mathrm{id}]$. So, R sends a simple object to itself, for a simple object is characterized amongst objects of $\mathcal{B}$ by its class in $K_0(\mathcal{B})$. In particular, $u : \mathrm{id} \to R$ is an isomorphism on simple objects. So, u is an isomorphism.

9. The 2-braid group

9.1. Coxeter group action.

9.1.1. Let (W, S) be a Coxeter system (with S finite) and $V = \bigoplus_{s\in S} ke_s$ be the reflection representation of W over a field k. We assume the representation is faithful (this is always the case if the characteristic is 0). Given $s, t \in S$, we denote by m_{st} the order of st. We assume that $2m_{st}$ is invertible in k, for all $s, t \in S$ such that m_{st} is finite. We denote by $\{\alpha_s\}_{s\in S}$ the dual basis of $\{e_s\}_{s\in S}$ (so that $\ker(s - \mathrm{id}) = \ker \alpha_s$ for $s \in S$). Let B_W be the braid group of W. This is the group generated by $\mathbf{S} = \{\mathbf{s}\}_{s\in S}$ with relations

$$\underbrace{\mathbf{sts}\cdots}_{m_{st}\text{ terms}} \simeq \underbrace{\mathbf{tst}\cdots}_{m_{st}\text{ terms}}$$

for any $s, t \in S$ such that $m_{st} < \infty$

Let $A = k[V]$ be the algebra of polynomial functions on V. All A-modules considered in this section are graded.

We will sometimes identify an object M of $K^b(A^{\mathrm{en}}\text{-modgr})$ with the corresponding endofunctor $M \otimes_A -$ of $K^b(A\text{-modgr})$. In particular, we will sometimes omit the symbols $\otimes_A$ when taking tensor products of bimodules for the sake of clarity.

9.1.2. The action of W on V induces an action on A, hence on A-modgr and on $D^b(A\text{-modgr})$: the element $w \in W$ acts by $A_w \otimes_A -$ where A_w is the (A, A)-bimodule equal to A as a left A-module, with right action of $a \in A$ given by right multiplication by $w(a)$. We have an isomorphism of (A, A)-bimodules, $\mathrm{id} \otimes 1 : A_w \xrightarrow{\sim} k[\Delta_w]$, where $\Delta_w = \{(w(v), v)\}_{v\in V} \subset V \times V$.

We have a canonical isomorphism $A_w \otimes_A A_{w'} \xrightarrow{\sim} A_{ww'}$ given by multiplication. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ be sequences of elements of S such that $x_1 \cdots x_m = y_1 \cdots y_n = w$. We denote by $c_{x,y} : A_{x_1} \cdots A_{x_m} \xrightarrow{\sim} A_{y_1} \cdots A_{y_n}$ the isomorphism obtained by composing the multiplication map $A_{x_1} \cdots A_{x_m} \xrightarrow{\sim} A_w$ with the inverse of the multiplication map $A_{y_1} \cdots A_{y_n} \xrightarrow{\sim} A_w$.

9.2. Braid group action. Let us now construct a non-obvious lift of the action of W on $D^b(A\text{-modgr})$ to an action of B_W on $K^b(A\text{-modgr})$.

9.2.1. For $s \in S$, we define the complex of (A, A)-bimodules

$$F_s = F_{\mathbf{s}} = 0 \to A \otimes_{A^s} A \xrightarrow{\varepsilon'_s} A \to 0$$

where A is in degree 1 and ε'_s is the multiplication.

Since $A = A^s \oplus A^s \alpha_s$, the morphism of A^{en}-modules

$$A_s \to A \otimes_{A^s} A(1),\ a \mapsto a \otimes \alpha_s - a\alpha_s \otimes 1$$

induces an isomorphism

$$f_{\mathbf{s}} : A_s \xrightarrow{\sim} F_s(1) \text{ in } D^b(A^{\mathrm{en}}\text{-modgr}).$$

9.2.2. For $w \in W$, let $\Delta_{\le w} = \bigcup_{w' \le w} \Delta_{w'}$ and $D_w = k[\Delta_{\le w}]$. Note that $D_s = A \otimes_{A^s} A$ for $s \in S$.

Given $w' \le w$, we have a canonical quotient map $D_w \to D_{w'}$ given by restriction of functions. We have

$$\mathrm{Hom}(D_w, D_{w'}) = \begin{cases} k \cdot \mathrm{can} & \text{if } w' \le w \\ 0 & \text{otherwise} \end{cases}$$

9.2.3. In the next lemma, $0 \to L \to M \to 0$ denotes a complex with L in degree 0.

Lemma 9.1. *Assume W is a finite dihedral group,* i.e., $\dim V = 2$, $S = \{s, t\}$ *and $m_{st} < \infty$. Let $x \in W$ such that $tx > x$. Then,*

(i) $D_s(0 \to D_{tx} \xrightarrow{\mathrm{can}} D_x \to 0) \simeq (0 \to D_x(-1) \xrightarrow{\mathrm{id}} D_x(-1) \to 0) \oplus (0 \to D_{stx} \xrightarrow{\mathrm{can}} D_x \to 0)$.

(ii) $F_s D_x \simeq (0 \to D_x \xrightarrow{\mathrm{id}} D_x \to 0) \oplus (0 \to D_x(-1) \to 0 \to 0)$.

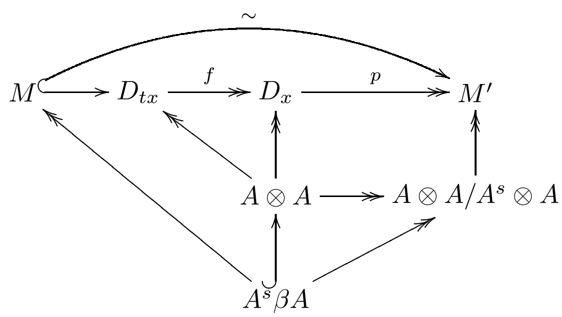
Proof. Let us recall some constructions and results of Soergel [**Soe4**, Lemma 4.5, Proposition 4.6 and their proofs]. Since $2m_{st}$ is invertible, then given u, u' two distinct reflections of W, we have $\ker(u + \mathrm{id}) \neq \ker(u' + \mathrm{id})$.

Let r be the reflection of W such that $rx < x$ and $rx \not< tx$. Then, $\Delta_x + \Delta_{rx}$ is a hyperplane of $V \times V$ and let $\beta \in V^* \times V^*$ be a linear form with kernel this hyperplane. Let M (resp. N) be the $(A^s \otimes A)$-submodule of D_{tx} generated by the image of the elements β (resp 1) of $A \otimes A$. Then, $D_{tx} = M \oplus N$, $M \simeq D_x^{(s,1)}(-1)$ and $N \simeq D_{stx}^{(s,1)}$ as $(A^s \otimes A)$-modules.

Let M' (resp. N') be the $(A^s \otimes A)$-submodule of D_x generated by $\alpha_s \otimes 1$ (resp. 1). Then, $D_x = M' \oplus N'$, $M' \simeq D_x^{(s,1)}(-1)$ and $N' = D_x^{(s,1)}$ as $(A^s \otimes A)$-modules. Denote by $p : D_x \to M'$ the projection.

Let us show now that $\beta \notin (V^*)^s \times V^*$. Equivalently, we need to show that $(\Delta_x + \Delta_{rx}) \cap (k\alpha_s \times 0) = 0$. This amounts to proving that $\mathrm{im}(\mathrm{id} - r) \neq k\alpha_s$. But this holds, since $r \neq s$.

Let us now come to our problem. Since $\beta \notin (V^*)^s \times V^*$, it follows that the image of β in $(A \otimes A)/(A^s \otimes A)$ is a generator as $(A^s \otimes A)$-module. Consequently, the restriction of $pf : D_{tx} \to M'$ to M is surjective, hence it is an isomorphism (we denote by $f : D_{tx} \to D_x$ the canonical map).



Finally, the multiplication map $A \otimes_{A^s} D_y^{(s,1)} \xrightarrow{\sim} D_y$ is an isomorphism for any $y \in W$ with $sy < y$. We have shown that the complex $A \otimes_{A^s} (0 \to D_{tx} \xrightarrow{\text{can}} D_x \to 0)$ is isomorphic to the direct sum of the complex $0 \to D_x(-1) \xrightarrow{\text{id}} D_x(-1) \to 0$ and a complex $D = 0 \to D_{stx} \xrightarrow{\phi} D_x \to 0$. Note that $\phi = r \cdot \text{can}$ for some $r \in k$ and we need to prove that $r \neq 0$. The complex $0 \to D_{tx} \xrightarrow{\text{can}} D_x \to 0$ has zero homology in degree 1, hence the same is true for D. It follows that $r \neq 0$.

Let us now prove the second assertion. The multiplication map $A \otimes_{A^s} D_x^{(s,1)} \to D_x$ is an isomorphism. Since $D_x = D_x^{(s,1)} \oplus M'$ and $M' \simeq D_x^{(s,1)}(-1)$, we obtain the second part of the Lemma. □

Proposition 9.2. *Take $s \neq t \in S$ with $m_{st} < \infty$. We have braid relations*

$$\underbrace{F_s F_t F_s \cdots}_{m_{st}\ terms} \simeq \underbrace{F_t F_s F_t \cdots}_{m_{st}\ terms}$$

in $K^b(A \otimes A)$.

Proof. We have a decomposition $V = V_1 \oplus V_2$ under the action of $\langle s, t\rangle$, with $V_1 = V^{\langle s,t\rangle}$. For the (A, A)-bimodules involved in the Proposition, the right and left actions of $k[V_1]$ are identical. So, we get the Proposition for V from the Proposition for V_2 by applying the functor $k[V_1^*] \otimes_k -$. It follows we can assume $\dim V = 2$. So, we assume W is finite dihedral with $S = \{s, t\}$. We put $s_+ = s$ and $s_- = t$.

Let $m = m_{st}$ and consider $i \leq m$ and $\varepsilon \in \{+, -\}$. Let $\sigma_i^\varepsilon = s_\varepsilon s_{-\varepsilon} s_\varepsilon \cdots$ (i terms) and $D_i^\varepsilon = D_{\sigma_i^\varepsilon}$. We put $D^\varepsilon = D_{s_\varepsilon}$. Consider the simplicial scheme over $V \times V$:

$$\Delta_1 \rightrightarrows \Delta_{\leq s_+} \coprod \Delta_{\leq s_-} \rightrightarrows \Delta_{\leq s_+ s_-} \coprod \Delta_{\leq s_- s_+} \rightrightarrows \cdots \rightrightarrows \Delta_{\leq \sigma_{i-1}^\varepsilon} \coprod \Delta_{\leq \sigma_{i-1}^{-\varepsilon}} \to \Delta_{\leq \sigma_i^\varepsilon}$$

where the maps are the inclusions.

We now define F_i^ε as the complex of (A, A)-bimodules coming from the structural complex of sheaves of this simplicial scheme :

$$0 \to D_i^\varepsilon \xrightarrow{\begin{pmatrix}+\\+\end{pmatrix}} D_{i-1}^\varepsilon \oplus D_{i-1}^{-\varepsilon} \xrightarrow{\begin{pmatrix}+ & -\\ + & -\end{pmatrix}} D_{i-2}^\varepsilon \oplus D_{i-2}^{-\varepsilon} \to \cdots \to D^+ \oplus D^- \xrightarrow{(+\ \ -)} D_1 \to 0$$

where the sign denotes the multiple of the canonical map considered (we put D_i^ε in degree 0). We have $H^r(F_i^\varepsilon) = 0$ for $r > 0$, since $\Delta_{\leq \sigma_r^+} \cap \Delta_{\leq \sigma_r^-} = \Delta_{\leq \sigma_{r-1}^+} \cup \Delta_{\leq \sigma_{r-1}^-}$

and we have an exact sequence

$$0 \to k[\Delta_{\leq \sigma_r^+} \cup \Delta_{\leq \sigma_r^-}] \xrightarrow{\binom{+}{+}} k[\Delta_{\leq \sigma_r^+}] \oplus k[\Delta_{\leq \sigma_r^-}] \xrightarrow{(+\quad -)} k[\Delta_{\leq \sigma_r^+} \cap \Delta_{\leq \sigma_r^-}] \to 0.$$

The complex F_1^ε is isomorphic to F_{s_ε}. We will now show by induction on i that $F_{s_\varepsilon} F_i^{-\varepsilon}$ is homotopy equivalent to F_{i+1}^ε for $\varepsilon = \pm$. This will prove the Proposition, since $F_m^+ \simeq F_m^-$.

Let us consider the complex $C = F_{s_\varepsilon} F_i^{-\varepsilon}$. This is the total complex of the double complex

$$\begin{array}{ccccccccc}
D^\varepsilon D_i^{-\varepsilon} & \longrightarrow & D^\varepsilon D_{i-1}^\varepsilon \oplus D^\varepsilon D_{i-1}^{-\varepsilon} & \longrightarrow & D^\varepsilon D_{i-2}^\varepsilon \oplus D^\varepsilon D_{i-2}^{-\varepsilon} & \longrightarrow & \cdots & \longrightarrow & D^\varepsilon D_1 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
D_i^{-\varepsilon} & \longrightarrow & D_{i-1}^\varepsilon \oplus D_{i-1}^{-\varepsilon} & \longrightarrow & D_{i-2}^\varepsilon \oplus D_{i-2}^{-\varepsilon} & \longrightarrow & \cdots & \longrightarrow & D_1
\end{array}$$

By Lemma 9.1, the complex $0 \to D^\varepsilon D_r^{-\varepsilon} \xrightarrow{\text{can}} D^\varepsilon D_{r-1}^\varepsilon \to 0$ is isomorphic to the direct sum of $0 \to D_{r-1}^\varepsilon(-1) \xrightarrow{\text{id}} D_{r-1}^\varepsilon(-1) \to 0$ and of $0 \to D_{r+1}^\varepsilon \xrightarrow{\text{can}} D_{r-1}^\varepsilon \to 0$. Also, the complex $0 \to D^\varepsilon D_r^\varepsilon \xrightarrow{\text{can}} D_r^\varepsilon \to 0$ is isomorphic to the direct sum of $0 \to D_r^\varepsilon \xrightarrow{\text{id}} D_r^\varepsilon \to 0$ and of $0 \to D_r^\varepsilon(-1) \to 0 \to 0$. It follows that C is homotopy equivalent to a complex

$$C' = 0 \to D_{i+1}^\varepsilon \to D_i^\varepsilon \oplus D_i^{-\varepsilon} \to \cdots \to D_1 \to 0,$$

where the maps remain to be determined. Since F_{s_ε} has non zero homology only in degree 0 and that homology is free as a right A-module, it follows that the homology of C vanishes in degrees > 0.

To conclude, we have to show that a complex X with the same terms as F_i^ε and with zero homology in degrees > 0 is actually isomorphic to F_i^ε. We have

$$X = 0 \to D_i^\varepsilon \xrightarrow{\begin{pmatrix} a_i \\ c_i \end{pmatrix}} D_{i-1}^\varepsilon \oplus D_{i-1}^{-\varepsilon} \xrightarrow{\begin{pmatrix} a_{i-1} & b_{i-1} \\ c_{i-1} & d_{i-1} \end{pmatrix}} D_{i-2}^\varepsilon \oplus D_{i-2}^{-\varepsilon} \to \cdots$$
$$\cdots \to D^+ \oplus D^- \xrightarrow{(c_1 \quad d_1)} D_1 \to 0$$

where the coefficients are in k and the maps are corresponding multiples of the canonical maps.

Take $r \leq i$ minimal such that there is an entry of $\begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix}$ that vanishes. Assume for example $c_r = 0$. Then, $a_{r-1} a_r = 0$, hence $a_r = 0$. We have $b_r c_{r+1} = d_r c_{r+1} = b_r d_{r+1} = d_r d_{r+1} = 0$. If $b_r = d_r = 0$, then X is the sum of the subcomplex with zero terms in degrees $\leq i - r$ and the subcomplex with zero terms in degrees $> i - r$. Otherwise, $c_{r+1} = d_{r+1} = 0$, hence X splits as the direct sum of the subcomplex $\cdots \to D_{r+1}^\varepsilon \oplus D_{r+1}^{-\varepsilon} \to D_r^\varepsilon \to 0$ and the subcomplex $0 \to D_r^{-\varepsilon} \to D_{r-1}^\varepsilon \oplus D_{r-1}^{-\varepsilon} \to \cdots$. Now, a morphism $D_r^{-\varepsilon} \to D_{r-1}^\varepsilon \oplus D_{r-1}^{-\varepsilon}$ is never injective, for the support of the left term is strictly larger than the support of the right term. Consequently, the complex X has non-zero homology in degree $i - r$, which is a contradiction. We have proven that none of the coefficients a_r, b_r, c_r, d_r can be zero.

Let Z be the closed subvariety of the affine space of coefficients a_r, b_r, c_r, d_r that define a complex (*i.e.*, $\begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix} \begin{pmatrix} a_{r+1} & b_{r+1} \\ c_{r+1} & d_{r+1} \end{pmatrix} = 0$) and let Z^0 be its open subset corresponding to non-zero coefficients. We have an isomorphism $Z^0 \xrightarrow{\sim} (\mathbf{G}_m)^{2i-1}$, $h : (a_r, b_r, c_r, d_r)_r \mapsto (a_r, c_r)_r$. The action of $(\mathbf{G}_m)^{2i}$ on the terms of the complex induce an action on Z. The corresponding action on $Z^0 \simeq (\mathbf{G}_m)^{2i-1}$ has a unique orbit. It follows that X is isomorphic to F_i^ε. □

9.2.4. Let us define the complex of A^{en}-modules

$$F_{\mathbf{s}^{-1}} = 0 \to A \xrightarrow{\eta_s} A \otimes_{A^s} A(1) \to 0$$

where A is in degree -1 and $\eta_s(a) = a\alpha_s \otimes 1 + a \otimes \alpha_s$.

Lemma 9.3. *The complexes $F_{\mathbf{s}}$ and $F_{\mathbf{s}^{-1}}$ are inverse to each other in the monoidal category $K^b(A^{\mathrm{en}}\text{-modgr})$.*

Proof. Let $\mathcal{C} = K^b(A^s\text{-modgr})$ and $\mathcal{D} = K^b(A\text{-modgr})$. Let $F = A\otimes_{A^s}?$, $G = A\otimes_A?$ and $\Phi = ?(1)$. The morphisms of (A^s, A^s)-bimodules

$$\varepsilon_s : A(1) \to A^s, 1 \mapsto 0 \text{ and } \alpha_s \mapsto 1 \quad \text{and} \quad \eta_s' : A^s \to A, 1 \mapsto 1$$

together with η_s and ε_s' previously defined give rise to adjoint pairs (F, G) and $(G, F\Phi)$.

We have a split exact sequence of (A^s, A^s)-bimodules

$$0 \to A^s \xrightarrow{\eta_s'} A \xrightarrow{\varepsilon_s} A^s \to 0,$$

hence we deduce the Lemma from Proposition 8.1. □

By Proposition 9.2 and Lemma 9.3, we have already obtained an action "up to isomorphism" of B_W on $K^b(A)$:

Proposition 9.4. *The map $s \mapsto F_{\mathbf{s}}$ extends to a morphism from B_W to the group of isomorphism classes of invertible objects of $K^b(A^{\mathrm{en}}\text{-modgr})$.*

9.3. Rigidification. The key point here is that the rigidification of the braid relations at the homotopy category level is equivalent to the one at the derived category level, where the problem is trivial, since we have a genuine action of W.

9.3.1. Consider the morphism of A^{en}-modules $A \otimes_{A^s} A \to A_s$ that sends $1 \otimes 1$ to 1. It induces a quasi-isomorphism $F_{\mathbf{s}^{-1}}(-1) \xrightarrow{\sim} A_s$. We denote its inverse (a morphism in $D^b(A^{\mathrm{en}}\text{-modgr})$) by $f_{\mathbf{s}^{-1}}$.

Now, let $v \in B_W$ and $v = t_1 \cdots t_m = u_1 \cdots u_n$ be two decompositions in elements of $\mathbf{S} \cup \mathbf{S}^{-1}$. By Proposition 9.4, the invertible objects $F_{t_1} \cdots F_{t_m}$ and $F_{u_1} \cdots F_{u_n}$ of $K^b(A^{\mathrm{en}}\text{-modgr})$ are isomorphic, hence

$$\mathrm{Hom}_\square(F_{t_1} \cdots F_{t_m}, F_{u_1} \cdots F_{u_n}) \simeq \mathrm{End}_\square(A) = k,$$

where $\square \in \{K^b(A^{\mathrm{en}}\text{-modgr}), D^b(A^{\mathrm{en}}\text{-modgr})\}$. It follows that the canonical morphism

$$\mathrm{Hom}_{K^b(A^{\mathrm{en}}\text{-modgr})}(F_{t_1} \cdots F_{t_m}, F_{u_1} \cdots F_{u_n}) \to \\ \to \mathrm{Hom}_{D^b(A^{\mathrm{en}}\text{-modgr})}(F_{t_1} \cdots F_{t_m}, F_{u_1} \cdots F_{u_n})$$

is an isomorphism.

So, we have a unique isomorphism

$$\gamma_{t,u} \in \mathrm{Hom}_{K^b(A^{\mathrm{en}}\text{-modgr})}(F_{t_1} \cdots F_{t_m}, F_{u_1} \cdots F_{u_n})$$

such that the induced element in $\mathrm{Hom}_{D^b(A^{\mathrm{en}}\text{-modgr})}(F_{t_1}\cdots F_{t_m}, F_{u_1}\cdots F_{u_n})$ corresponds to

$$c_{(t_1,\ldots,t_m),(u_1,\ldots,u_n)} : A_{t_1}\cdots A_{t_m} \xrightarrow{\sim} A_{u_1}\cdots A_{u_n}$$

via the quasi-isomorphisms $f_{t_1}\cdots f_{t_m}$ and $f_{u_1}\cdots f_{u_n}$.

We now define G_v as the limit of the functors $F_{t_1}\cdots F_{t_m}$, where $t=(t_1,\ldots,t_m)$ runs over the decompositions of v in $\mathbf{S}\cup\mathbf{S}^{-1}$, with the transitive system of isomorphisms $\gamma_{t,u}$.

There are unique isomorphisms $m_{v,v'} : G_vG_{v'} \xrightarrow{\sim} G_{vv'}$ for $v,v'\in B_W$ and $m_1 : G_1 \xrightarrow{\sim} A$ in $K^b(A^{\mathrm{en}}\text{-modgr})$ that are compatible with the isomorphisms $c_{t,u}$, in $D^b(A^{\mathrm{en}}\text{-modgr})$. So, we get the following result :

THEOREM 9.5. *The family $(G_v, m_{v,v'}, m_1)$ defines an action of B_W on the triangulated category $K^b(A\text{-modgr})$.*

This means we have a monoidal functor from

- the strict monoidal category with set of objects B_W, with only arrows the identity maps and with tensor product given by multiplication
- to the strict monoidal category of endofunctors of $K^b(A\text{-modgr})$.

REMARK 9.6. Using tensor products on the right, one obtains a right action of B_W on $K^b(A\text{-modgr})$. This action commutes trivially with the left action of B_W, so, we have an action of $B_W\times B_W^{\mathrm{opp}}$ on $K^b(A\text{-modgr})$.

9.3.2. We denote by $\mathcal{B}_W$ the full subcategory of $K^b(A^{\mathrm{en}}\text{-modgr})$ with objects the G_v for $v\in B_W$. The product $G_v\boxtimes G_{v'}=G_{vv'}$ provides $\mathcal{B}_W$ with the structure of a strict monoidal category. Define G_v^* as $G_{v^{-1}}$.

We have obtained our "categorification" of the braid group :

THEOREM 9.7. *The category $\mathcal{B}_W$ is a strict rigid monoidal category. Its "decategorification" is a quotient of B_W.*

CONJECTURE 9.8. *The decategorification of $\mathcal{B}_W$ is equal to B_W.*

REMARK 9.9. One can show that the conjecture is true in type A_n, as a consequence of [**KhovSei**, Corollary 1.2].

9.3.3. Let $C=A/(A\cdot A_+^W)$ be the coinvariant algebra. Then, we get by restriction of functors an action of B_W on $K^b(C\text{-modgr})$ and on $K^b(C\text{-mod})$. We get as well monoidal functors from $\mathcal{B}_W$ to the category of self-equivalences of $K^b(C\text{-modgr})$ or $K^b(C\text{-mod})$. Note that we get also right actions, and this gives a monoidal functor from $\mathcal{B}_W\times\mathcal{B}_W^{\mathrm{opp}}$ to the category of self-equivalences of $K^b(C\text{-modgr})$ or $K^b(C\text{-mod})$.

REMARK 9.10. Let $\mathcal{C}$ be the smallest full subcategory of $(A\otimes A)$-modgr containing the objects $A\otimes_{A^s}A$ and closed under finite direct sums, direct summands and tensor products. This is a monoidal subcategory of $(A\otimes A)$-modgr which is a categorification of the Hecke algebra of W, according to Soergel [**Soe2**]. The quotient $\bar{\mathcal{C}}$ of $\mathcal{C}$ by the smallest additive tensor ideal subcategory containing the $A\otimes_{A^{\langle s,t\rangle}}A$, where $s,t\in S$ and $m_{st}\neq 2,\infty$, is a categorification of the Temperley-Lieb quotient of the Hecke algebra.

When W has type A_n, an action of $\bar{\mathcal{C}}$ on an algebraic triangulated category is the same as the datum of an A_n-configuration of spherical objects [**SeiTho**, §2.c].

10. Principal block of a semi-simple complex Lie algebra

10.1. Review of category $\mathcal{O}$.

10.1.1. Let $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{h} \subseteq \mathfrak{b}$ a Cartan and a Borel subalgebra. Let $\mathcal{O}$ be the Bernstein-Gelfand-Gelfand category of finitely generated $\mathfrak{g}$-modules which are diagonalizable for $\mathfrak{h}$ and locally finite for $\mathfrak{b}$. Denote by Z the center of the enveloping algebra U of $\mathfrak{g}$. Let $P \subset \mathfrak{h}^*$ be the weight lattice, $Q \subset \mathfrak{h}^*$ be the root lattice, R (resp. R^+) be the set of roots (resp. positive roots) and Π the set of simple roots.

10.1.2. We have a decomposition $\mathcal{O} = \bigoplus_\theta \mathcal{O}_\theta$, where $\mathcal{O}_\theta$ is the subcategory of modules with central character θ. Let D be a duality on $\mathcal{O}$ that fixes simple modules (up to isomorphism).

Let $\Delta(\chi) = U \otimes_{U(\mathfrak{b})} \mathbf{C}_\chi$ be the Verma module associated to $\chi \in \mathfrak{h}^*$. It has a unique simple quotient $L(\chi)$. We denote a projective cover of $L(\chi)$ by $P(\chi)$. We put $\nabla(\chi) = D\Delta(\chi)$.

Consider the dot action of W on $\mathfrak{h}^*$, $w \cdot \lambda = w(\lambda+\rho) - \rho$ (we denote by $\dot{W}$ the group W acting via the dot action on $\mathfrak{h}^*$), where ρ is the half-sum of the positive roots.

Given $\lambda \in \mathfrak{h}^*$, let $\xi(\lambda)$ be the character by which Z acts on $L(\lambda)$ and $\mathfrak{m}_\lambda$ be its kernel, an element of $\operatorname{Specm} Z$, the maximal spectrum of Z. The morphism $\mathfrak{h}^* \to \operatorname{Specm} Z$, $\lambda \mapsto \mathfrak{m}_\lambda$ induces an isomorphism $\mathfrak{h}^*/\dot{W} \xrightarrow{\sim} \operatorname{Specm} Z$, *i.e.*, an isomorphism of algebras $h : Z \xrightarrow{\sim} A^{\dot{W}}$ where $A = \mathbf{C}[\mathfrak{h}^*]$. The simple objects in $\mathcal{O}_\theta$ are those $L(\lambda)$ with $\xi(\lambda) = \theta$.

10.1.3. Consider B the set of intersections of orbits of $\dot{W}$ and of Q on $\mathfrak{h}^*$. For $d \in B$, we denote by $\mathcal{O}_d$ (or by $\mathcal{O}_\mu$ for a $\mu \in d$) the thick subcategory of $\mathcal{O}$ generated by the $L(\lambda)$ for $\lambda \in d$. Then, $\mathcal{O} = \bigoplus_{d\in B} \mathcal{O}_d$ is the decomposition of $\mathcal{O}$ into blocks.

Let $\Lambda \in \mathfrak{h}^*/P$ and $\lambda \in \Lambda$. We have a root system $R_\Lambda = \{\alpha \in R | \langle \lambda, \alpha^\vee\rangle \in \mathbf{Z}\}$ with set of simple roots $\Pi_\Lambda \subset R^+$, Weyl group $W_\Lambda = \{w \in W | w(\lambda) - \lambda \in Q\}$ and set of simple reflections S_Λ (they depend only on Λ). Note that $R_\Lambda = R$ if and only if $\Lambda = P$. We define

$$\Lambda^+ = \{\lambda \in \Lambda | \langle \lambda + \rho, \alpha^\vee\rangle \geq 0 \text{ for all } \alpha \in \Pi_\lambda\}$$
$$\Lambda^{++} = \{\lambda \in \Lambda | \langle \lambda + \rho, \alpha^\vee\rangle > 0 \text{ for all } \alpha \in \Pi_\lambda\}.$$

Then, Λ^+ is a fundamental domain for the action of $\dot{W}_\Lambda$ on Λ. The module $L(\lambda)$ is finite dimensional if and only if $\lambda \in P^{++}$.

10.1.4. We define a translation functor between $\mathcal{O}_d$ and $\mathcal{O}_{d'}$ when $d, d' \in B$ are in the same P-orbit. Take $\Lambda \in \mathfrak{h}^*/P$ and $\lambda, \mu \in \Lambda^+$. Let ν be the only element in $W(\mu - \lambda) \cap \Lambda^{++}$. Then, we define $T_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu, M \mapsto pr_\mu(M \otimes L(\nu))$ where $pr_\mu : \mathcal{O} \to \mathcal{O}_\mu$ is the projection functor. Since $-w_0\nu \in \Lambda^{++}$ and $L(\nu)^* \simeq L(-w_0\nu)$, it follows that the functors T_λ^μ and T_μ^λ are left and right adjoint to each other.

Let $d \in B$ containing 0. The corresponding block $\mathcal{O}_0 = \mathcal{O}_d$ is the principal block of $\mathcal{O}$. Note that $d = W \cdot 0$ is a regular W-orbit and we put $L(w) = L(w \cdot 0)$, etc...

For $s \in S$, we fix $\mu \in P^+$ with stabilizer $\{1, s\}$ in $\dot{W}$. We put $T^s = T_0^\mu$ and $T_s = T_\mu^0$ and $\Theta_s = T_s T^s : \mathcal{O}_0 \to \mathcal{O}_0$.

10.1.5. Let $F_{\mathbf{s}} = F_s$ be the complex of functors on $\mathcal{O}_0$ given by $0 \to \Theta_s \xrightarrow{\varepsilon'_s} \mathrm{id} \to 0$ where ε'_s is the counit of adjunction (id is in degree 1).

Let $F_{\mathbf{s}^{-1}} = 0 \to \mathrm{id} \xrightarrow{\eta_s} \Theta_s \to 0$, where η_s is the unit of the other adjunction. Then, Rickard [**Ri3**, Proposition 2.2] proved that $F_{\mathbf{s}}$ and $F_{\mathbf{s}^{-1}}$ are inverse self-equivalences of $K^b(\mathcal{O}_0)$ (this follows from §8.2.3 by the classical character calculation $[T^sT_s] = 2[\mathrm{id}]$).

It is easy and classical that the F_s induce an action of W on $K_0(\mathcal{O}_0)$ (the reflection $s \in S$ acts as $[F_s]$). This realizes the regular representation of W. A permutation basis for this action is provided by $\{[\Delta(w)]\}_{w\in W}$.

It seems difficult to check directly that the F_s satisfy the braid relations. Using the equivalence between $\mathcal{O}_0$ and perverse sheaves on the flag variety, this can be deduced from §11.

10.2. Link with bimodules.

10.2.1. We start by recalling results of Soergel [**Soe1, Soe2, Soe3**] relating the category $\mathcal{O}$ to modules over the coinvariant algebra.

Let $\Lambda \in \mathfrak{h}^*/P$. We denote by $C_\Lambda = A/(A \cdot A_+^{W_\Lambda})$ the coinvariant algebra of (W_Λ, S_Λ) and $p_\Lambda : A \to C_\Lambda$ the canonical surjection. Let $\lambda \in \Lambda^+$. We denote by $t_\lambda : A \to A$ the translation by λ, given by $f \mapsto (z \mapsto f(z+\lambda))$ We have Soergel's Endomorphismensatz [**Soe1**, Endomorphismensatz 7] :

Theorem 10.1. *The image of the composite morphism* $Z \xrightarrow{h} A^{\dot W} \hookrightarrow A \xrightarrow{t_\lambda} A \xrightarrow{p_\Lambda} C_\Lambda$ *is* $C_\Lambda^{W_\lambda}$ *and the canonical morphism* $Z \to \mathrm{End}(P(w_0\cdot\lambda))$ *factors through this morphism* $Z \to C_\Lambda^{W_\lambda}$*. The induced morphism* $\sigma_\lambda : C_\Lambda^{W_\lambda} \xrightarrow{\sim} \mathrm{End}(P(w_0\cdot\lambda))$ *is an isomorphism.*

Let us now recall Soergel's Struktursatz [**Soe1**, Struktursatz 9] :

Theorem 10.2. *The functor* $\mathrm{Hom}(P(w_0\cdot\lambda), -) : \mathcal{O}_\lambda\text{-proj} \to C_\Lambda^{W_\lambda}\text{-mod}$ *is fully faithful.*

Let $\mu \in \Lambda$ be regular (*i.e.*, with trivial stabilizer in W_Λ).

There is an isomorphism $\phi : T_\lambda^\mu P(w_0\cdot\lambda) \xrightarrow{\sim} P(w_0\cdot\mu)$. Any such isomorphism ϕ induces a commutative diagram [**Soe1**, Bemerkung p.431]

$$\begin{array}{ccc}
C_\Lambda^{W_\lambda} & \xrightarrow{\text{inclusion}} & C_\Lambda \\
\downarrow{\scriptstyle \sigma_\lambda} & & \downarrow{\scriptstyle \sigma_\mu} \\
\mathrm{End}(P(w_0\cdot\lambda)) & \xrightarrow{\phi_* T_\lambda^0} & \mathrm{End}(P(w_0\cdot\mu))
\end{array}$$

This gives us an isomorphism, via the adjunction $(T_\lambda^\mu, T_\mu^\lambda)$:

$$\mathrm{Res}_{C_\Lambda^{W_\lambda}}^{C_\Lambda} \mathrm{Hom}(P(w_0\cdot\mu), ?) \xrightarrow{\sim} \mathrm{Hom}(T_\lambda^\mu P(w_0\cdot\lambda), ?) \xrightarrow{\sim} \mathrm{Hom}(P(w_0\cdot\lambda), T_\mu^\lambda(?))$$

between functors $\mathcal{O}_\mu \to C_\Lambda^{W_\lambda}$-mod. So, we have a commutative diagram, with fully faithful horizontal functors

$$\begin{array}{ccc}
\mathcal{O}_\mu\text{-proj} & \xrightarrow{\mathrm{Hom}(P(w_0\cdot\mu),?)} & C_\Lambda\text{-mod} \\
\downarrow{\scriptstyle T_\mu^\lambda} & & \downarrow{\scriptstyle \mathrm{Res}} \\
\mathcal{O}_\lambda\text{-proj} & \xrightarrow{\mathrm{Hom}(P(w_0\cdot\lambda),?)} & C_\Lambda^{W_\lambda}\text{-mod}
\end{array}$$

10.2.2. From the last commutative diagram, we deduce

PROPOSITION 10.3. *There is a commutative diagram with fully faithful horizontal arrows*

$$\begin{array}{ccc} K^b(\mathcal{O}_0\text{-proj}) & \xrightarrow{\operatorname{Hom}(P(w_0),-)} & K^b(C\text{-mod}) \\ \downarrow{\scriptstyle F_s} & & \downarrow{\scriptstyle F_s} \\ K^b(\mathcal{O}_0\text{-proj}) & \xrightarrow{\operatorname{Hom}(P(w_0),-)} & K^b(C\text{-mod}) \end{array}$$

So, we deduce from Theorem 9.5 the following : given $v \in B_W$ and $v = t_1 \cdots t_m = u_1 \cdots u_n$ two decompositions in elements of $\mathbf{S} \cup \mathbf{S}^{-1}$, there is an isomorphism $F_{t_1} \cdots F_{t_m} \xrightarrow{\sim} F_{u_1} \cdots F_{u_n}$ between functors on $D^b(\mathcal{O}_0)$ coming by restriction from the isomorphism between functors on $K^b(A\text{-modgr})$. These form a transitive system of isomorphisms, *i.e.*

THEOREM 10.4. *The functors F_s induce an action of B_W on $D^b(\mathcal{O}_0)$.*

More precisely,

THEOREM 10.5. *There is a monoidal functor from $\mathcal{B}_W$ to the category of self-equivalences of $D^b(\mathcal{O}_0)$ sending G_s to F_s.*

REMARK 10.6. One has a similar statement for the deformed category $\mathcal{O}$.

Note that we deduce from §9.3.3 that there is also a right action of B_W on $D^b(\mathcal{O}_0)$. We leave it to the reader to check that this corresponds to the actions using Zuckerman functors [**MazStr**], or equivalently, Arkhipov functors [**KhomMaz**].

In the graded setting (mixed perverse sheaves for example), note that the left and right actions of B_W should be swapped by the self-Koszul duality equivalence, cf [**BerFreKhov**] (and [**BeiGi**, Conjecture 5.18] for an analog in the equivariant case).

Various constructions have been given of weak actions of braid groups on $D^b(\mathcal{O}_0)$, cf [**AnStr, Ar, KhomMaz, MazStr, Str**].

11. Flag varieties

11.1. Classical results. Let G be a semi-simple complex algebraic group with Weyl group W.

Let $\mathbf{W} = \{\mathbf{w}\}_{w \in W}$. The braid group B_W of W is isomorphic to the group with set of generators $\mathbf{W}$ and relations $\mathbf{w}\mathbf{w}' = \mathbf{w}''$ when $ww' = w''$ and $l(w'') = l(w) + l(w')$.

Let $\mathcal{B}$ be the flag variety of G. We decompose

$$\mathcal{B} \times \mathcal{B} = \coprod_{w \in W} \mathcal{O}(w)$$

into orbits for the diagonal G-action. Consider the first and second projections

$$\begin{array}{ccccc} & & \mathcal{O}(w) & & \\ & \swarrow{\scriptstyle p_w} & & \searrow{\scriptstyle q_w} & \\ \mathcal{B} & & & & \mathcal{B} \end{array}$$

Then, we have a functor

$$F_{\mathbf{w}} = R(p_w)_!(q_w)^* : D^b(\mathcal{B}) \to D^b(\mathcal{B})$$

where $D^b(\mathcal{B})$ is the derived category of bounded complexes of constructible sheaves of $\mathbf{C}$-vector spaces over $\mathcal{B}$.

First and last projections induce an isomorphism

$$\mathcal{O}(w) \times_{\mathcal{B}} \mathcal{O}(w') \xrightarrow{\sim} \mathcal{O}(ww') \text{ when } l(ww') = l(w) + l(w').$$

This induces an isomorphism (cf §12.2)

$$\gamma_{w,w'} : F_{\mathbf{w}} F_{\mathbf{w}'} \xrightarrow{\sim} F_{\mathbf{ww}'} \text{ when } l(ww') = l(w) + l(w').$$

For $s \in S$, then $F_{\mathbf{s}}$ is obtained as in §8.2.1 for the canonical morphism $\pi_s : \mathcal{B} \to \mathcal{P}_s$, where $\mathcal{P}_s$ is the variety of parabolic subgroups of type s. So, $F_{\mathbf{s}}$ is invertible, with inverse $F_{\mathbf{s}^{-1}} = R(p_s)_*(q_s)^!$. It follows that $F_{\mathbf{w}}$ is invertible for $w \in W$, with inverse $F_{\mathbf{w}^{-1}} = R(p_w)_*(q_w)^!$, hence we get a morphism from B_W to the group of isomorphism classes of invertible functors on $D^b(\mathcal{B})$.

11.2. Genuine braid group action. We have a commutative diagram

$$\begin{array}{ccc} F_xF_yF_z & \xrightarrow{\gamma_{x,y}F_z} & F_{xy}F_z \\ {\scriptstyle F_x\gamma_{y,z}}\downarrow & & \downarrow{\scriptstyle \gamma_{xy,z}} \\ F_xF_{yz} & \xrightarrow[\gamma_{x,yz}]{} & F_{xyz} \end{array}$$

for $x, y, z \in \mathbf{W}$ such that $l(x) + l(y) + l(z) = l(xyz)$, by Theorem 12.2.

Let $b \in B_W$ and $b = t_1 \cdots t_m = u_1 \cdots u_n$ with $u_i \in \mathbf{W} \cup \mathbf{W}^{-1}$. Applying braid relations and the corresponding isomorphisms γ, we get various isomorphisms $F_{t_1} \cdots F_{t_m} \xrightarrow{\sim} F_{u_1} \cdots F_{u_n}$. By Deligne [**De3**], they are all equal. Let us denote by $\gamma_{t.,u.}$ their common value.

We now define

$$\tilde{F}_b = \lim_{(t_1,\cdots,t_n)} F_{t_1} \cdots F_{t_n}$$

where $(t_1, \cdots, t_n)$ runs over the set of sequences of elements of $\mathbf{W} \cup \mathbf{W}^{-1}$ such that $b = t_1 \cdots t_n$ and where we are using the transitive system of isomorphisms $\gamma_{t.,s.}$.

We have now the following result

THEOREM 11.1. *The assignment $b \mapsto \tilde{F}_b$ defines an action of B_W on $D^b(\mathcal{B})$.*

REMARK 11.2. Deligne [**De3**] defines a variety $\mathcal{O}_b$ with two morphisms $p_b, q_b : \mathcal{O}_b \to \mathcal{B}$ for any $b \in B_W^+$. Then, the action of b on $\mathcal{D}^b(\mathcal{B})$ is given by $p_{b!}q_b^*$.

11.3. Link with bimodules. The results in this section are based on [**Soe1, BeiGiSoe**].

11.3.1. Fix a Borel subgroup B of G. We consider the setting of §9 with $k = \mathbf{C}$ and V^* the complexified character group of B. In this section, we will consider the algebra A with double grading, *i.e.*, V^* is in degree 2.

Let $C \xrightarrow{\sim} H^*(\mathcal{B}, \mathbf{C})$ be the Borel isomorphism (send a character of B to the Chern class of the corresponding line bundle) and denote by β its inverse.

Let I be a subset of S, W_I the subgroup of W generated by I, W^I be the set of minimal right coset representatives of W/W_I and P_I the parabolic subgroup of G of type I containing B. Put $\mathcal{P}_I = G/P_I$. Denote by $\pi_I : \mathcal{B} \to \mathcal{P}_I$ the canonical morphism. The map $\pi_I^* : \bigoplus_i \operatorname{Hom}(\mathbf{C}_{\mathcal{P}_I}, \mathbf{C}_{\mathcal{P}_I}[i]) \to \bigoplus_i \operatorname{Hom}(\mathbf{C}_{\mathcal{B}}, \mathbf{C}_{\mathcal{B}}[i])$ induces, via β, an isomorphism $\beta_I : \bigoplus_i \operatorname{Hom}(\mathbf{C}_{\mathcal{P}_I}, \mathbf{C}_{\mathcal{P}_I}[i]) \xrightarrow{\sim} C^{W_I}$.

11.3.2. Consider the full subcategory $D^b_\sigma(\mathcal{P}_I)$ of $D^b(\mathcal{P}_I)$ of complexes whose cohomology sheaves are smooth along B-orbits. Given $w \in W^I$, let $\mathcal{L}_w$ be the perverse sheaf corresponding to the intersection cohomology complex of $\overline{BwP_I/P_I}$. Let $\mathcal{L}_I = \bigoplus_{w\in W^I} \mathcal{L}_w$. The dg-algebra $R\operatorname{End}(\mathcal{L}_I)$ is formal and let $R_I = \bigoplus_i \operatorname{Hom}(\mathcal{L}_I, \mathcal{L}_I[i])$. We have an equivalence $\mathcal{L}_I\otimes ?$ from the category R_I-dgperf of perfect differential graded R_I-modules to $D^b_\sigma(\mathcal{P}_I)$.

The functor $\bigoplus_i \operatorname{Hom}(\mathbf{C}_{\mathcal{P}_I}, ?[i]) : D^b_\sigma(\mathcal{P}_I) \to C^{W_I}$-modgr restricts to a fully faithful functor on the full subcategory containing the $\mathcal{L}_I[i]$. So, we get a fully faithful functor R_I-dgperf $\to K(C^{W_I}$-dgmod), hence a fully faithful functor $\mathbf{H}_I : D^b_\sigma(\mathcal{P}_I) \to K(C^{W_I}$-dgmod), where we denote by $K(C^{W_I}$-dgmod) the homotopy category of differential graded C^{W_I}-modules.

As in §10.2, we get a commutative diagram

$$\begin{array}{ccc} D^b_\sigma(\mathcal{B}) & \xrightarrow{\mathbf{H}} & K(C\text{-dgmod}) \\ {\scriptstyle R\pi_{I*}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}} \\ D^b_\sigma(\mathcal{P}_I) & \xrightarrow[\mathbf{H}_I]{} & K(C^{W_I}\text{-dgmod}) \end{array}$$

and we deduce

Proposition 11.3. *Let $s \in S$. There is a commutative diagram with fully faithful horizontal arrows*

$$\begin{array}{ccc} D^b_\sigma(\mathcal{B}) & \xrightarrow{\mathbf{H}} & K(C\text{-dgmod}) \\ {\scriptstyle F_s}\downarrow & & \downarrow{\scriptstyle F_s} \\ D^b_\sigma(\mathcal{B}) & \xrightarrow[\mathbf{H}]{} & K(C\text{-dgmod}) \end{array}$$

In particular, we get a monoidal functor from $\mathcal{B}_W$ to the category of self-equivalences of $D^b_\sigma(\mathcal{B})$.

Remark 11.4. We believe the monoidal functor above is the restriction of a functor with values in $D^b(\mathcal{B})$.

12. Appendix : associativity of kernel transforms

12.1. Classical isomorphisms.

12.1.1. We consider here

- schemes of finite type over a field of characteristic $p \geq 0$ and the derived category of constructible sheaves of Λ-modules, where Λ is a torsion ring with torsion prime to p or Λ is a $\mathbf{Q}_\ell$-algebra, for l prime to p

  or
- locally compact topological spaces of finite soft c-dimension and the derived category of constructible sheaves of $\mathbf{C}$-vector spaces.

We will quote results pertaining to either of the two settings above, depending on the convenience of references. The maps involved will be concatenations of canonical isomorphisms.

We denote a derived functor with the same notation as the original functor : we write $\otimes$ for $\otimes^{\mathbf{L}}$, $f_!$ for $Rf_!$, etc...

12.1.2. Let $f : Y \to X$ and $g : Z \to Y$ be two morphisms. There are canonical isomorphisms [**KaScha**, 2.6.6 and 2.3.9]

$$(fg)_! \xrightarrow{\sim} f_! g_! \quad \text{and} \quad (fg)^* \xrightarrow{\sim} g^* f^*.$$

These isomorphisms satisfy a cocycle property (cf [**De1**, Théorème 5.1.8] for the case $(-)_!$) :

Lemma 12.1. *Consider $X_3 \xrightarrow{w} X_2 \xrightarrow{v} X_1 \xrightarrow{u} X_0$. Then, the following diagrams are commutative*

$$\begin{array}{ccc} w^*v^*u^* & \longrightarrow & w^*(uv)^* \\ \downarrow & & \downarrow \\ (vw)^*u^* & \longrightarrow & (uvw)^* \end{array} \qquad \begin{array}{ccc} u_!v_!w_! & \longrightarrow & (uv)_!w_! \\ \downarrow & & \downarrow \\ u_!(vw)_! & \longrightarrow & (uvw)_! \end{array}$$

We will take the liberty to identify the functors v^*u^* and $(uv)^*$ through the canonical isomorphism.

There are canonical isomorphisms [**KaScha**, 2.6.18]

$$f^*(-_1 \otimes -_2) \xrightarrow{\sim} (f^* -_1) \otimes (f^* -_2) \quad \text{and} \quad (-_1 \otimes -_2) \otimes -_3 \xrightarrow{\sim} -_1 \otimes (-_2 \otimes -_3)$$

We identify the bifunctors $f^*(-_1 \otimes -_2)$ and $(f^*-_1) \otimes (f^*-_2)$ through the canonical isomorphism. Given $A_i \in D^b(X)$, $i \in \{1, 2, 3\}$, we identify $(A_1 \otimes A_2) \otimes A_3$ with $A_1 \otimes (A_2 \otimes A_3)$ and we denote this object by $A_1 \otimes A_2 \otimes A_3$.

Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

be a cartesian square. Then, there is the canonical base change isomorphism [**KaScha**, 2.6.20] :

$$g^* f_! \xrightarrow{\sim} f'_! g'^*.$$

We have a canonical isomorphism [**KaScha**, 2.6.19]

$$-_1 \otimes (f_! -_2) \xrightarrow{\sim} f_!(f^* -_1 \otimes -_2).$$

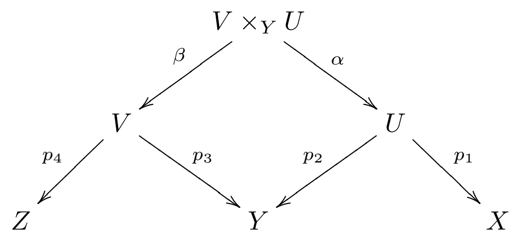
12.2. Kernel transforms.

12.2.1. Let us define a 2-category $\mathcal{K}$.

- The 0-arrows are the varieties.
- 1-arrows : $\mathrm{Hom}(X, Y)$ is the family of (K, U) where U is a variety over $Y \times X$ and $K \in D^b(U)$.
- 2-arrows : $\mathrm{Hom}((K, U), (K', U'))$ is the set of (ϕ, f) where $f : U \xrightarrow{\sim} U'$ is an isomorphism of $(Y \times X)$-varieties and $\phi : K \xrightarrow{\sim} f^*K'$.

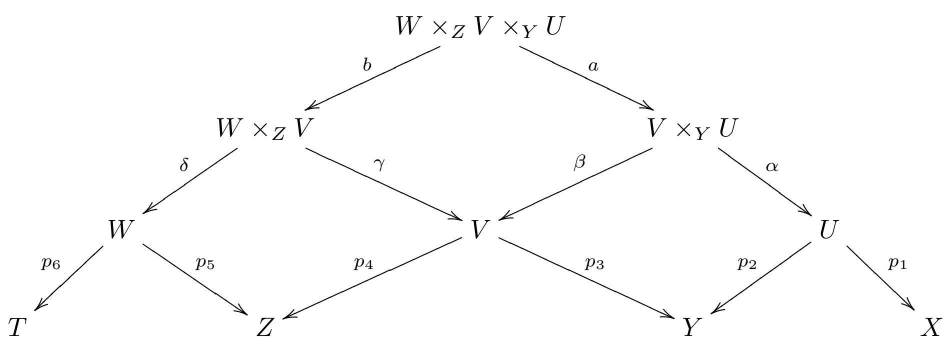
We define the composition of 1-arrows. Consider the following diagram where the square is cartesian

(2)



Let $K \in D^b(U)$ and $L \in D^b(V)$. We put $L \boxtimes K = \beta^* L \otimes \alpha^* K$. The composition $(L,V)(K,U)$ is defined to be $(L \boxtimes K, V \times_Y U)$.

Let us consider now the diagram with all squares cartesian



and take $M \in D^b(W)$. We have

$$(M \boxtimes L) \boxtimes K = b^*(\delta^* M \otimes \gamma^* L) \otimes (\alpha a)^* K \xrightarrow{\sim} (\delta b)^* M \otimes a^*(\beta^* L \otimes \alpha^* K) = M \boxtimes (L \boxtimes K).$$

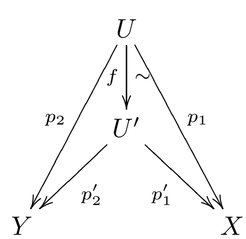
This provides the associativity isomorphisms for $\mathcal{K}$. With our conventions, we will write $M \boxtimes L \boxtimes K$ for the objects in the isomorphism above. It is straightforward to check that $\mathcal{K}$ is indeed a 2-category.

12.2.2. We put $\Phi_K = \Phi_K^{p_2,p_1} = p_{2!}(K \otimes p_1^* -) : D^b(X) \to D^b(Y)$.

Let $c_{L,K} : \Phi_L \Phi_K \xrightarrow{\sim} \Phi_{L \boxtimes K}$ be defined as the composition

$$\begin{aligned} p_{4!}(L \otimes p_3^* p_{2!}(K \otimes p_1^* -)) &\to p_{4!}(L \otimes \beta_! \alpha^*(K \otimes p_1^* -)) \\ &\to p_{4!}\beta_!(\beta^* L \otimes \alpha^*(K \otimes p_1^* -)) \\ &\to (p_4\beta)_!((\beta^* L \otimes \alpha^* K) \otimes \alpha^* p_1^* -) \\ &\to (p_4\beta)_!((\beta^* L \otimes \alpha^* K) \otimes (p_1\alpha)^* -). \end{aligned}$$

Let $(\phi, f) \in \mathrm{Hom}_{\mathcal{K}}((K,U),(K',U'))$. We have a commutative diagram



and we define $\Phi(\phi, f)$ as the composition

$$p_{2!}(K \otimes p_1^* -) \xrightarrow{\sim} p'_{2!} f_!(f^* K' \otimes f^* p_1^{'*} -) \xrightarrow{\sim} p'_{2!}(K' \otimes p_1^{'*} -).$$

THEOREM 12.2. *Φ is a 2-functor from $\mathcal{K}$ to the 2-category of triangulated categories.*

We have $c_{M \boxtimes L, K} \circ (c_{M,L} \Phi_K) = c_{M, L \boxtimes K} \circ (\Phi_M c_{L,K})$, i.e., the following diagram commutes :

$$\begin{array}{ccc} \Phi_M \Phi_L \Phi_K & \longrightarrow & \Phi_{M \boxtimes L} \Phi_K \\ \downarrow & & \downarrow \\ \Phi_M \Phi_{L \boxtimes K} & \longrightarrow & \Phi_{M \boxtimes L \boxtimes K} \end{array}$$

12.2.3. The next two Lemmas deal with composition of base change isomorphisms.

For the first Lemma, see [**De1**, Lemme 5.2.5] :

LEMMA 12.3. *Let*

$$\begin{array}{ccccc} X_2 & \xrightarrow{f_2} & X_1 & \xrightarrow{f_1} & X \\ {\scriptstyle h_2}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle h}\downarrow \\ S_2 & \xrightarrow[g_2]{} & S_1 & \xrightarrow[g_1]{} & S \end{array}$$

be a diagram with all squares cartesian. Then, the following diagram commutes

$$\begin{array}{ccccc} (g_1 g_2)^* h_! & & \longrightarrow & & h_{2!}(f_1 f_2)^* \\ \downarrow & & & & \uparrow \\ g_2^* g_1^* h_! & \longrightarrow & g_2^* h_{1!} f_1^* & \longrightarrow & h_{2!} f_2^* f_1^* \end{array}$$

The second Lemma is [**De1**, Lemme 5.2.4] :

LEMMA 12.4. *Let*

$$\begin{array}{ccc} X'_2 & \xrightarrow{g_2} & X_2 \\ {\scriptstyle f'_2}\downarrow & & \downarrow{\scriptstyle f_2} \\ X'_1 & \xrightarrow{g_1} & X_1 \\ {\scriptstyle f'_1}\downarrow & & \downarrow{\scriptstyle f_1} \\ S' & \xrightarrow[g]{} & S \end{array}$$

be a diagram with all squares cartesian. Let $A \in D^b(S')$. Then, the following diagram commutes

$$\begin{array}{ccccc} A \otimes g^*(f_1 f_2)_! - & \longrightarrow & A \otimes (f'_1 f'_2)_! g_2^* - & \longrightarrow & (f'_1 f'_2)_!((f'_1 f'_2)^* A \otimes g_2^* -) \\ \downarrow & & & & \uparrow \\ A \otimes g^* f_{1!} f_{2!} - & & & & f'_{1!} f'_{2!}(f_2'^{*} f_1'^{*} A \otimes g_2^* -) \\ \downarrow & & & & \uparrow \\ A \otimes f'_{1!} g_1^* f_{2!} - & \longrightarrow & f'_{1!}(f_1'^{*} A \otimes g_1^* f_{2!} -) & \longrightarrow & f'_{1!}(f_1'^{*} A \otimes f'_{2!} g_2^* -) \end{array}$$

Lemma 12.5. *Let* $f : Y \to X$ *and* $A, B \in D^b(X)$ *and* $C \in D^b(Y)$. *Then, the following diagram commutes*

$$\begin{array}{ccc} A \otimes B \otimes f_! C & \longrightarrow & f_!(f^*(A \otimes B) \otimes C) \\ \big\downarrow & & \big\downarrow \\ A \otimes f_!(f^* B \otimes C) & \longrightarrow & f_!(f^* A \otimes f^* B \otimes C) \end{array}$$

Proof. The corresponding statement for $f_!$ replaced by f_* is easy, the key point is that the composition $f^* \xrightarrow{f^*\eta} f^* f_* f^* \xrightarrow{\varepsilon f^*} f^*$ is the identity of f^*, where η and ε are the unit and counit of the adjoint pair (f^*, f_*). The Lemma follows easily from this (in the algebraic case, we have only to check in addition the trivial case where f is an open immersion thanks to the transitivity of Lemma 12.4, whereas in the topological case we use the embedding $f_! C \subset f_* C$ for C injective). □

Lemma 12.6. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

be a cartesian square. Let $A \in D^b(S)$ *and* $B \in D^b(X)$. *Then, the following diagram commutes*

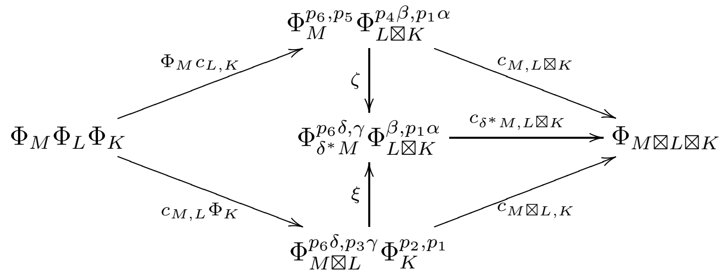
$$\begin{array}{ccccccc} g^* A \otimes g^* f_! B & \longrightarrow & g^* A \otimes f'_! g'^* B & \longrightarrow & f'_!(f'^* g^* A \otimes g'^* B) & \longrightarrow & f'_!(g'^* f^* A \otimes g'^* B) \\ \big\downarrow & & & & & & \big\downarrow \\ g^*(A \otimes f_! B) & & \longrightarrow & & g^* f_!(f^* A \otimes B) & \longrightarrow & f'_! g'^*(f^* A \otimes B) \end{array}$$

Proof. As in the previous Lemma, one reduces to proving the analog of the Lemma with $?_!$ replaced by $?_*$. This follows then from the easily checked commutativity of the two diagrams

$$\begin{array}{ccc} f'^* g^* f_* & \longrightarrow & f'^* f'_* g'^* \\ \big\downarrow & & \big\downarrow \\ g'^* f^* f_* & \longrightarrow & g'^* \end{array} \qquad\qquad \begin{array}{ccc} g^* & \longrightarrow & g^* f_* f^* \\ \big\downarrow & & \big\downarrow \\ f'_* f'^* g^* & \longrightarrow & f'_* g'^* f^* \end{array}$$

where we have used the units and counits of the adjoint pairs (f^*, f_*) and (f'^*, f'_*). □

Proof of the Theorem. We will show the commutativity of the following diagram



where ζ is the composition

$$\begin{aligned}
&p_{6!}(M\otimes p_5^*(p_4\beta)_!(\beta^*L\otimes\alpha^*K\otimes(p_1\alpha)^*-))\to\\
&\qquad\to p_{6!}\,(M\otimes p_5^*p_{4!}\beta_!(\beta^*L\otimes\alpha^*K\otimes(p_1\alpha)^*-))\\
&\qquad\to p_{6!}\,(M\otimes\delta_!\gamma^*\beta_!(\beta^*L\otimes\alpha^*K\otimes(p_1\alpha)^*-))\\
&\qquad\to p_{6!}\delta_!\,(\delta^*M\otimes\gamma^*\beta_!(\beta^*L\otimes\alpha^*K\otimes(p_1\alpha)^*-))\\
&\qquad\to (p_6\delta)_!\,(\delta^*M\otimes\gamma^*\beta_!(\beta^*L\otimes\alpha^*K\otimes(p_1\alpha)^*-))
\end{aligned}$$

and ξ the composition

$$\begin{aligned}
&(p_6\delta)_!(\delta^*M\otimes\gamma^*L\otimes(p_3\gamma)^*p_{2!}(K\otimes p_1^*-))\to\\
&\qquad\to (p_6\delta)_!\,(\delta^*M\otimes\gamma^*L\otimes\gamma^*p_3^*p_{2!}(K\otimes p_1^*-))\\
&\qquad\to (p_6\delta)_!\,(\delta^*M\otimes\gamma^*(L\otimes p_3^*p_{2!}(K\otimes p_1^*-)))\\
&\qquad\to (p_6\delta)_!\,(\delta^*M\otimes\gamma^*(L\otimes\beta_!\alpha^*(K\otimes p_1^*-)))\\
&\qquad\to (p_6\delta)_!\,(\delta^*M\otimes\gamma^*(L\otimes\beta_!(\alpha^*K\otimes\alpha^*p_1^*-)))\\
&\qquad\to (p_6\delta)_!\,(\delta^*M\otimes\gamma^*(L\otimes\beta_!(\alpha^*K\otimes(p_1\alpha)^*-)))\\
&\qquad\to (p_6\delta)_!\,(\delta^*M\otimes\gamma^*\beta_!(\beta^*L\otimes\alpha^*K\otimes(p_1\alpha)^*-))
\end{aligned}$$

Let u and v be the compositions

$$u: p_{6!}(M\otimes p_5^*p_{4!}-)\to p_{6!}(M\otimes\delta_!\gamma^*-)\to p_{6!}\delta_!(\delta^*M\otimes\gamma^*-)\to(p_6\delta)_!(\delta^*M\otimes\gamma^*-)$$

and

$$\begin{aligned}
v: L\otimes p_3^*p_{2!}(K\otimes p_1^*-)\to L\otimes\beta_!\alpha^*(K\otimes p_1^*-)\to\beta_!(\beta^*L\otimes\alpha^*(K\otimes p_1^*-))\to\\
\to\beta_!(\beta^*L\otimes\alpha^*K\otimes(p_1\alpha)^*-).
\end{aligned}$$

Then, one has trivially

$$\zeta(\Phi_M c_{L,K}) = u(L\otimes p_3^*p_{2!}(K\otimes p_1^*-))\circ p_{6!}(M\otimes p_5^*p_{4!}v) = \xi(c_{M,L}\Phi_K).$$

The equality $c_{M,L\boxtimes K} = c_{\delta^*M,L\boxtimes K}\zeta$ follows from Lemma 12.4 applied to $g = p_5$, $g_1 = \gamma$, $g_2 = a$, $f_1 = p_4$, $f_2 = \beta$, $f_1' = \delta$, $f_2' = b$ and $A = M$ and from Lemma 12.1 applied to $u = p_6$, $v = \delta$ and $w = b$.

The equality $c_{M\boxtimes L,K} = c_{\delta^*M,L\boxtimes K}\xi$ follows from Lemma 12.3 applied to $f_1 = \alpha$, $f_2 = a$, $g_1 = p_3$, $g_2 = \gamma$, $h = p_2$, $h_1 = \beta$ and $h_2 = b$, from Lemma 12.5 applied to

$f = b$, $A = \delta^* M$, $B = \gamma^* L$ and $C = (\alpha a)^*(K \otimes p_1^* -)$ and from Lemma 12.6 applied to $f = \beta$, $g = \gamma$, $f' = b$, $g' = a$, $A = L$ and $B = \alpha^*(K \otimes p_1^* -)$. $\square$

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