Weyl groups, affine Weyl groups and reflection groups

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1 Introduction

This paper is a survey of some of the basic results pertaining to reflection groups.

In §2, we start with the basic concepts and properties of Coxeter groups, such as the Exchange Lemma and in §4 we construct the geometric representation. Sections 3 and 5 are devoted to finite real reflection groups and finite Coxeter groups and §6 concerns Weyl groups, which are crystallographic reflection groups. Weyl groups give rise to affine Weyl groups, studied in §7. The Iwahori-Hecke algebra of a Coxeter group is introduced in §8, after a discussion on braid groups. Finite complex reflection groups are the subject of §9, where we describe the infinite families. Finally, we explain in §10 how the topology of the hyperplane complement allows to define braid groups and Iwahori-Hecke algebras for finite complex reflection groups.

This paper is expository : most proofs are to be found in [Bki] or [Hu] for §2-8 and in [BrMaRo] for §9-10.

2 Coxeter groups

Let W be a group and S a set of (distinct) generators of W of order 2. For $s, s' \in S$, we denote by $m_{s,s'} \in \{1, 2, \dots\} \cup \{\infty\}$ the order of the product ss'.

Definition 2.1 *The pair* (W, S) *is a* Coxeter system *if* W *has a presentation by generators and relations given by the set of generators* S *and the relations :*

$$s^2 = 1 \textit{ for } s \in S,$$

$$\underbrace{ss'ss'\cdots}_{m_{ss'} \textit{ terms}} = \underbrace{s'ss's\cdots}_{m_{ss'} \textit{ terms}} \textit{ for those } s, s' \in S \textit{ such that } m_{s,s'} \textit{ is finite.}$$

We then say also that W is a Coxeter group.

The relations $ss'ss'\cdots = s'ss's\cdots$ are called *braid relations.*

The *rank* of the system is the cardinality of S.

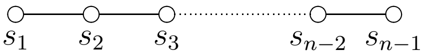
The *matrix* of the Coxeter system (W, S) is $(m_{s,s'})_{s,s'\in S}$; it has values in $\{1, 2, \ldots\} \cup \{\infty\}$.

This is a symmetric matrix with diagonal entries 1 and off-diagonal entries at least 2. A matrix with such properties is called a *Coxeter matrix.* We will see (Theorem 4.1) that every Coxeter matrix is the matrix of a Coxeter system (in a group given by generators and relations as in the definition, with $(m_{s,s'})$ an abstract Coxeter matrix, it isn't obvious that ss' will have order $m_{s,s'}$).

The *graph* associated with (W, S) is the graph with set of vertices S and edges $\{s, s'\}$ when $m_{s,s'} \geq 3$. Furthermore, the edge is then labelled by $m_{s,s'}$.

Some examples.

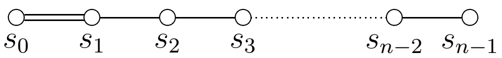
(i) The symmetric group $\mathfrak{S}_n = \mathfrak{S}(\{1, 2, \ldots, n\})$. Let $s_i = (i, i+1)$ and $S_{\mathfrak{S}_n} = \{s_1, \ldots, s_{n-1}\}$. Then, $(\mathfrak{S}_n, S_{\mathfrak{S}_n})$ is a Coxeter system (of type A_{n-1}) with graph



(the label is omitted when it is 3).

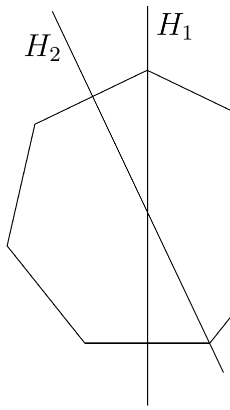
It has rank $n - 1$.

(ii) The hyperoctahedral group B_n, *i.e.*, the group of $n\times n$ monomial matrices with non-zero entries in $\{\pm 1\}$. It contains $\mathfrak{S}_n$, viewed as the group of permutation matrices, as a subgroup. Let $s_0 = \mathrm{diag}(-1, 1, ..., 1)$ and $S_{B_n} = \{s_0, s_1, \ldots, s_{n-1}\}$, with s_i, $i \geq 1$ as in (i). Then, (B_n, S_{B_n}) is a Coxeter system of rank n with graph



(where ○══○ stands for ○—4—○).

(iii) The dihedral group $I_2(m)$: this is the symmetry group of a regular m-gon (*i.e.*, the subgroup of the group of isometries of the plane fixing the m-gon), $m \geq 2$.



Let H_1 be a line containing the center of the polygon and one of its vertices. Let H_2 be a line containing the center of the polygon and such that the angle between H_1 and H_2 is π/m. $I_2(m)$ is generated by the orthogonal reflections t_1 and t_2 with respect to H_1 and H_2. $(I_2(m), \{t_1, t_2\})$ is a Coxeter system with graph

$$\underset{t_1}{\circ}\overset{m}{\text{———}}\underset{t_2}{\circ}\ (m \geq 3) \quad \text{or} \quad \underset{t_1}{\circ} \qquad \underset{t_2}{\circ}\ (m = 2).$$

The group $I_2(m)$ has a decomposition $I_2(m) = \langle t_1 t_2 \rangle \rtimes \langle t_1 \rangle$. The subgroup $\langle t_1 t_2 \rangle$ is the subgroup of rotations, it has order m. The action of $\langle t_1 \rangle \simeq \{\pm 1\}$ on $\mathbf{Z}/m\mathbf{Z}$ in this decomposition is given by multiplication.

This suggests a construction for $m = \infty$: we denote by $\tilde{A}_1$ the group $\mathbf{Z} \rtimes \{\pm 1\}$, where $\{\pm 1\}$ acts by multiplication on $\mathbf{Z}$. Let $t_1 = (0, -1)$ and $t_2 = (-1, -1)$. Then, $(\tilde{A}_1, \{t_1, t_2\})$ is a Coxeter system with graph

$$\underset{t_1}{\circ}\overset{\infty}{\text{———}}\underset{t_2}{\circ}$$

The dihedral groups are the groups $I_2(m)$, $2 \leq m < \infty$ and $\tilde{A}_1$.

Note that every rank 2 Coxeter system is isomorphic to the Coxeter system of a dihedral group. In particular, the Coxeter systems for $\mathfrak{S}_3$ and $I_2(3)$ are isomorphic, as well as those for B_2 and $I_2(4)$.

The following theorem [Bki, Chap. IV, §1, Théorème 2] is an easy consequence of Theorem 4.1 below :

Theorem 2.2 *Let (W, S) be a Coxeter system, S' a subset of S and W' the subgroup of W generated by S'. Then, (W', S') is a Coxeter system with Coxeter matrix the submatrix of the Coxeter matrix of (W, S) given by S'.*

A Coxeter system is *irreducible* if its associated graph is connected. All systems in the previous examples are irreducible, except $I_2(2)$.

If S is the disjoint union of two subsets S_1 and S_2 and no vertex of S_1 is connected to a vertex of S_2, then $W = W_1 \times W_2$, where W_i is the subgroup of W generated by S_i.

Remark 1 *Note that for m odd, $I_2(2m) \simeq I_2(m) \times \mathfrak{S}_2$, but $(I_2(2m), \{s, s'\})$ is nevertheless irreducible for $m > 1$!*

Let $w \in W$. The *length* of w, $l(w)$, is the smallest integer m such that w is the product of m elements of S.

A decomposition $w = s_1 \cdots s_m$ with $s_1, \ldots, s_m \in S$ is *reduced* if $m = l(w)$.

Theorem 2.3 *Let $w = s_1 \cdots s_m$ with $s_1, \dots, s_m \in S$. Then, there is a subset $I = \{i_1 < i_2 < \cdots < i_k\}$ of $\{1, \dots, m\}$ with $k = l(w)$ elements such that $w = s_{i_1} \cdots s_{i_k}$.*

This theorem is a direct consequence of the exchange lemma [Bki, Chap. IV, §1, Proposition 4] :

Lemma 2.4 *Let $w = s_1 \cdots s_m$ be a reduced decomposition ($s_1, \dots, s_m \in S$). Let $s \in S$. Then, one of the following assertion holds :*

(i) $l(sw) = l(w) + 1$ *and $ss_1 \cdots s_m$ is a reduced decomposition of sw*

(ii) $l(sw) = l(w) - 1$ *and there exists $j \in \{1, \dots, m\}$ such that $s_1 \cdots s_{j-1}s_{j+1} \cdots s_m$ is a reduced decomposition of sw and $ss_1 \cdots s_{j-1}s_{j+1} \cdots s_m$ is a reduced decomposition of w.*

This lemma actually characterizes the Coxeter systems amongst the pairs (W, S), where S is a set of generators of order 2 of a group W [Bki, Chap. IV, §1, Théorème 1].

3 Real reflection groups

Let V be a finite dimensional real vector space. A reflection of V is an automorphism of order 2 whose set of fixed points is an hyperplane. A finite reflection group W in V is a finite subgroup of $GL(V)$ generated by reflections.

The group W is *crystallographic* if there is a W-invariant $\mathbf{Z}$-lattice of V, *i.e.*, if there exists a free $\mathbf{Z}$-submodule L of V stable under W such that the canonical map $L \otimes_{\mathbf{Z}} \mathbf{R} \to V$ is an isomorphism.

Note that this amounts to the existence of a W-stable $\mathbf{Q}$-structure on V, *i.e.*, a $\mathbf{Q}$-subspace $V_{\mathbf{Q}}$ of V stable under W such that the canonical map $V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R} \to V$ is an isomorphism [Bki, Chap. VI, §2, Proposition 9].

Let $\mathcal{A}$ be the set of reflecting hyperplanes of W — *i.e.*, the set of $\ker(s-1)$, where s is a reflection of W.

Then, $V - \bigcup_{H \in \mathcal{A}} H$ is in general non-connected : its connected components are the *chambers* of W.

Theorem 3.1 ([Bki, Chap. V, §3, Théorèmes 1 et 2]) *The group W acts simply transitively on the set of chambers ; the closure of a chamber is a fundamental domain for the action of W on V.*

Let C_1 be a chamber and S the set of reflections with respect to the walls of C_1 (a wall of C_1 is an hyperplane in $\mathcal{A}$ whose intersection with the closure of C_1 has codimension 1 in V).

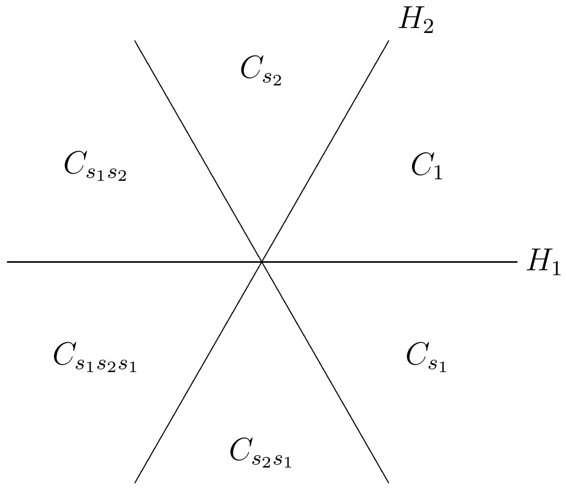
Theorem 3.2 ([Bki, Chap. V, §3, Théorème 1]) *The pair (W, S) is a Coxeter system.*

Taking into account the choice of the chamber C_1, the chambers are now parametrized by W. The chamber C_w corresponding to $w \in W$ is $w(C_1)$.

A *gallery* of length n is a sequence $D_0, \dots, D_n$ of adjacent chambers (*i.e.*, the intersection of the closures of D_i and D_{i+1} has codimension 1 in V). The following result can be deduced from [Bki, Chap. V, §3, Théorème 1] :

Proposition 3.3 *The minimal length of a gallery from C_0 to C_w is $l(w)$.*

Example : the chamber system for the group A_2.



Here, s_i is the orthogonal reflection with respect to H_i. The group W generated by s_1 and s_2 is a Coxeter group of type A_2.

4 Coxeter groups as reflection groups

Let S be a set and $M = (m_{s,s'})_{s,s' \in S}$ a Coxeter matrix. Let $V = \mathbf{R}^S$ and denote by $\{e_s\}_{s \in S}$ its canonical basis.

Define a bilinear form B_M on V by

$$B_M(e_s, e_{s'}) = -\cos \frac{\pi}{m_{s,s'}}.$$

(Note that $B_M(e_s, e_s) = 1$).

Let ρ_s be the reflection in V given by

$$\rho_s(x) = x - 2B_M(e_s, x)e_s.$$

One has $V = \mathbf{R}e_s \oplus H_s$, where H_s is the hyperplane orthogonal to e_s.

Let W be the group with set of generators S and relations

$$s^2 = 1$$

$$\underbrace{ss'ss'\cdots}_{m_{ss'}\ terms} = \underbrace{s'ss's\cdots}_{m_{ss'}\ terms} \text{ for those } s, s' \in S \text{ such that } m_{s,s'} \neq \infty.$$

Theorem 4.1 ([Bki, Chap. V, §4.3 et §4.4]) *The map $s \mapsto \rho_s$ extends to an injective group morphism $W \to GL(V)$, the reflection representation of W. Furthermore, (W, S) is a Coxeter system.*

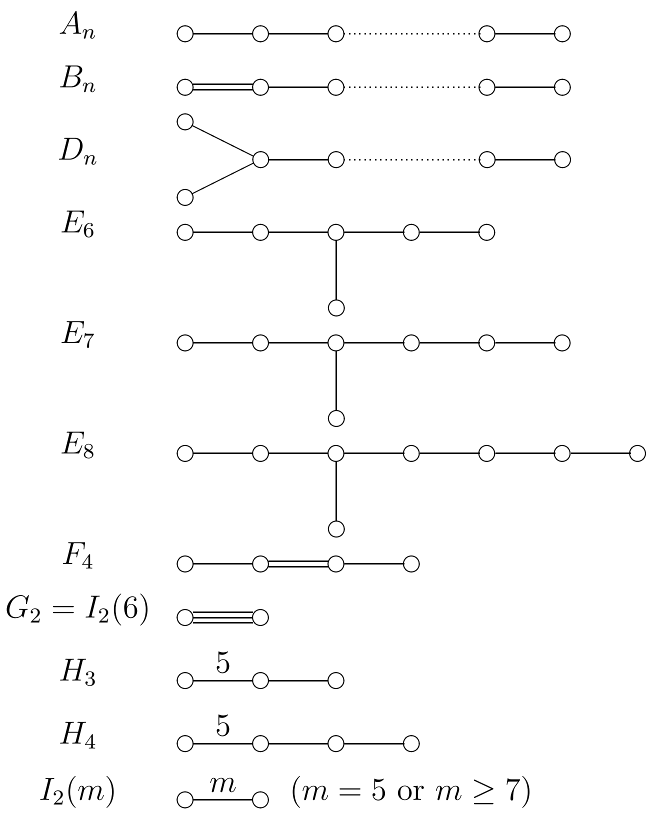
When S is finite, B_M is positive definite if and only if W is finite.

Summarizing Theorems 3.2 and 4.1, we deduce

Theorem 4.2 *The constructions of §3 and §4 give rise to inverse bijections between the set of conjugacy classes of finite subgroups of $GL_n(\mathbf{R})$ generated by reflections and the set of those rank n Coxeter matrices giving rise to a finite Coxeter group.*

5 Finite Coxeter groups

The classification of Coxeter graphs giving rise to irreducible finite Coxeter groups is the following [Bki, Chap. VI, §4, Théorème 1] (the number attached to the name of the diagram is the number of nodes of the diagram) :



In the list above, all the groups are crystallographic except H_3, H_4 and $I_2(m)$, $m = 5$ or $m \geq 7$.

6 Root systems and Weyl groups

Let V be a finite dimensional real vector space, Φ a finite subset of V and $\Phi^\vee$ a finite subset of V^* parametrized by $\Phi : \Phi \to \Phi^\vee$, $\alpha \mapsto \alpha^\vee$.

Assume

(1) the vector space V is generated by Φ

(2) for all $\alpha \in \Phi$, we have $\langle \alpha^\vee, \alpha \rangle = 2$ and the reflection $s_\alpha : V \to V$, $x \mapsto x - \langle \alpha^\vee, x \rangle \alpha$ stabilizes Φ

(3) we have $\alpha^\vee(\Phi) \subset \mathbf{Z}$ for all $\alpha \in \Phi$

(4) for $\alpha \in \Phi$, we have $2\alpha \notin \Phi$.

Then, Φ is a *root system* in V (sometimes called reduced, because of (4)). Note that given Φ, there is at most one set $\Phi^\vee$ parametrized by Φ with the required properties.

If $\Phi = \Phi_1 \cup \Phi_2$ and Φ_i (together with $\Phi_i^\vee = \{\alpha^\vee\}_{\alpha \in \Phi_i}$) is a root system in V_i, the subspace of V generated by Φ_i, for $i \in \{1, 2\}$, then we say that Φ is the direct sum of the root systems Φ_1 and Φ_2. The root system Φ is *irreducible* if it is non-empty and it is not the direct sum of two non-empty root systems.

The *Weyl group* of the root system Φ is the subgroup of $GL(V)$ generated by the reflections s_α for $\alpha \in \Phi$. Note that W is a crystallographic finite reflection group with $\mathbf{Z}$-lattice the $\mathbf{Z}$-submodule of V generated by Φ. A converse actually holds [Bki, Ch. VI, §2, Proposition 9] :

Proposition 6.1 *Let W be a crystallographic reflection group in a finite dimensional real vector space V. Then, there is a root system Φ in V with Weyl group W.*

Note that if W is irreducible, then the root system Φ is unique up to isomorphism if and only if W is not of type B_n, $n \geq 3$ (cf Remark 2).

Let C be a chamber of W with walls $L_1, \ldots, L_n$. Then, there is a unique root $\alpha_i \in \Phi$ orthogonal to L_i and lying in the same half-space delimited by L_i as C.

The set $\Delta = \{\alpha_i\}_{1 \leq i \leq n}$ is called a *basis* of Φ.

Let $\Phi^+ = \{\alpha \in \Phi | \alpha = \sum n_i \alpha_i, n_i \geq 0\}$ (the *positive* roots) and $\Phi^- = \{\alpha \in \Phi | \alpha = \sum n_i \alpha_i, n_i \leq 0\}$ (the *negative* roots).

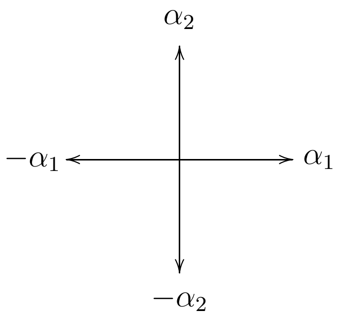
Proposition 6.2 ([Bki, Chap. VI, §1, Théorèmes 2 et 3]) *The set* Δ *is a basis of* V *and* $\Phi = \Phi^+ \cup \Phi^-$.

The *Cartan matrix* is $(\langle \alpha, \beta^\vee \rangle)_{\alpha,\beta \in \Delta}$.

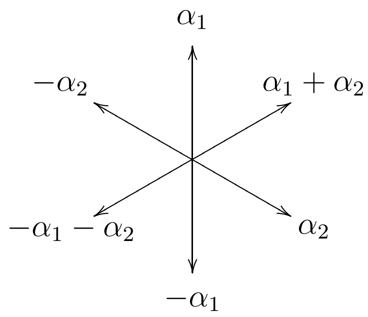
Define $S = \{s_\alpha\}_{\alpha \in \Delta}$. Then, (W, S) is a Coxeter system.

The rank 2 root systems

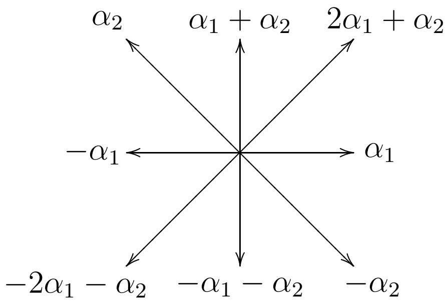
Type $A_1 \times A_1$



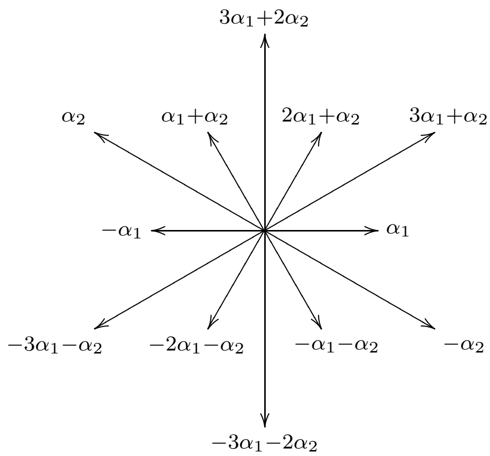
Type A_2



Type $B_2 = C_2$



Type G_2



The action of W on Φ gives another interpretation of the length function [Bki, Chap. VI, §1, Corollaire 2 de la Proposition 17] :

Proposition 6.3 *Let* $w \in W$. *The cardinality of* $\Phi^- \cap w(\Phi^+)$ *is the length* $l(w)$.

The set $\Phi^\vee$ defines a root system in V^* (the root system *inverse* or *dual* to Φ). There is an isomorphism of groups

$$W(\Phi) \to W(\Phi^\vee),$$
$$u \mapsto {}^t u^{-1}$$

sending s_α on $s_{\alpha^\vee}$. Through this isomorphism, $W(\Phi)$ operates on V^*.

Remark 2 *Note that the root systems (V, Φ) and (V^*, Φ^*) are not isomorphic in general : for example, the root system of type C_n is the inverse of the root system of type B_n ; when $n \geq 3$, these root systems are not isomorphic.*

7 Affine Weyl groups

Let Φ be a root system in a finite dimensional real vector space V. We construct a subgroup of the group $\mathrm{Aff}(V^*)$ of affine transformations of V^* as follows :

For $\alpha \in \Phi$ and $k \in \mathbf{Z}$, let $H_{\alpha,k}$ be the affine hyperplane of V^* defined by

$$H_{\alpha,k} = \{x \in V^* | \langle \alpha, x \rangle = k\}.$$

Let $s_{\alpha,k}$ be the orthogonal reflection with respect to $H_{\alpha,k}$:

$$s_{\alpha,k}(x) = x - (\langle \alpha, x \rangle - k)\alpha^\vee.$$

The affine Weyl group associated to Φ is the subgroup $\mathcal{W}$ of $\mathrm{Aff}(V^*)$ generated by the $s_{\alpha,k}$, $\alpha \in \Phi$, $k \in \mathbf{Z}$.

Let Q be the subgroup of $\mathrm{Aff}(V^*)$ generated by the translations by elements of $\Phi^\vee$.

Proposition 7.1 ([Bki, Chap. VI, §2, Proposition 1]) *We have $\mathcal{W} = Q \rtimes W$.*

An *alcove* is a connected component of $V^* - \bigcup_{\alpha\in\Phi,k\in\mathbf{Z}} H_{\alpha,k}$ (note that the set $\{H_{\alpha,k}\}_{\alpha\in\Phi,k\in\mathbf{Z}}$ is the set of reflecting hyperplanes of $\mathcal{W}$).

Theorem 7.2 ([Bki, Chap. VI, §2.1]) *The group $\mathcal{W}$ acts simply transitively on the set of alcoves. The closure of an alcove is a fundamental domain for the action of $\mathcal{W}$ on V^*.*

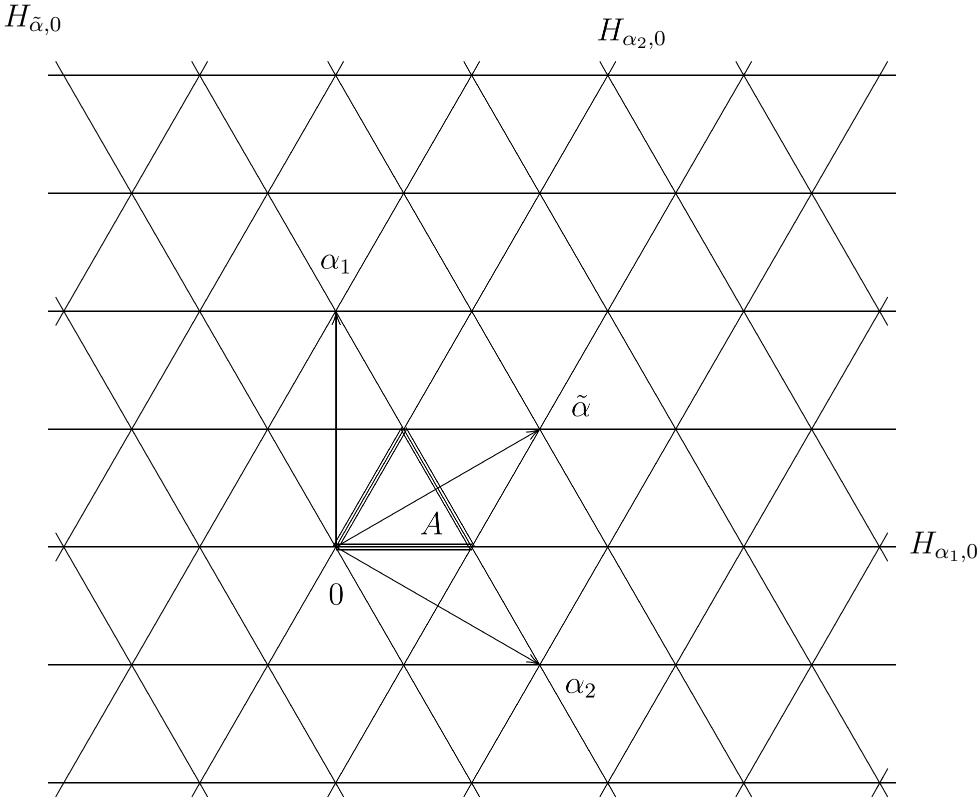
Let C be a chamber for W. Then, there is a unique alcove $A \subset C$ such that 0 is in the closure of A.

Let $\tilde{S}$ be the set of reflections with respect to the walls of A. The pair $(\mathcal{W}, \tilde{S})$ is Coxeter group.

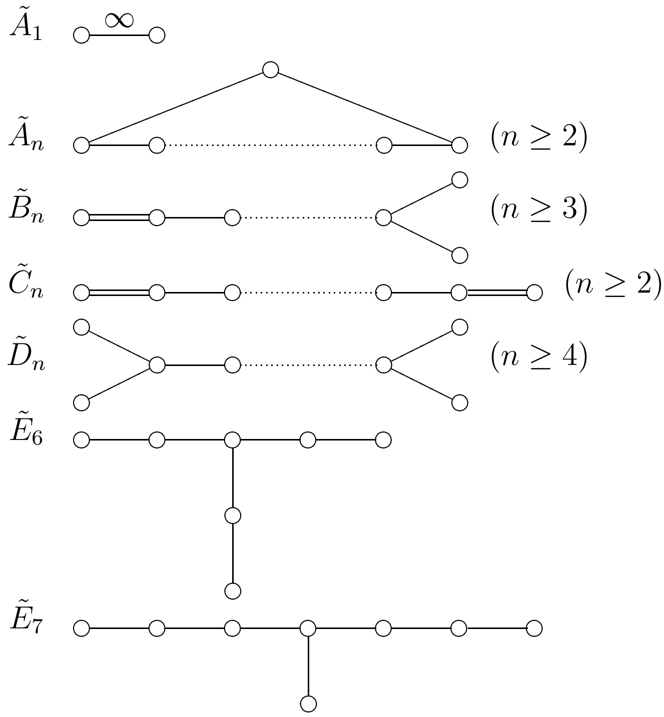
If Φ is irreducible, there is a root $\tilde{\alpha} = \sum n_i \alpha_i$ in Φ such that if $\beta \in \Phi$, $\beta = \sum m_i \alpha_i$, then $m_i \leq n_i$: $\tilde{\alpha}$ is the *highest root*. This root is orthogonal to the wall of A which doesn't contain 0 [Bki, Chap. VI, §1, Proposition 25].

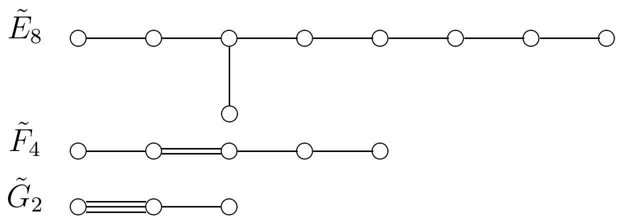
The length of an element $w \in \mathcal{W}$ (relatively to $\tilde{S}$) is the minimal length of a gallery of alcoves from A to $w(A)$.

Example : Type $\tilde{A}_2$.



The classification of the irreducible affine Weyl groups (or of their Coxeter graphs) is the following [Bki, Chap. VI, §4, Théorème 4] :





8 Braid groups and Iwahori-Hecke algebras

Let (W, S) be a Coxeter system.

Definition 8.1 *The* braid monoid B_W^+ *associated with* (W, S) *is the monoid with set of generators* $\{\sigma_s^+\}_{s\in S}$ *and relations :*

$$\underbrace{\sigma_s^+\sigma_{s'}^+\sigma_s^+\sigma_{s'}^+\cdots}_{m_{ss'}\ terms} = \underbrace{\sigma_{s'}^+\sigma_s^+\sigma_{s'}^+\sigma_s^+\cdots}_{m_{ss'}\ terms} \text{ for } s, s' \in S \text{ such that } m_{s,s'} \text{ is finite.}$$

The braid group B_W *associated with* (W, S) *is the group with set of generators* $\{\sigma_s\}_{s\in S}$ *and relations :*

$$\underbrace{\sigma_s\sigma_{s'}\sigma_s\sigma_{s'}\cdots}_{m_{ss'}\ terms} = \underbrace{\sigma_{s'}\sigma_s\sigma_{s'}\sigma_s\cdots}_{m_{ss'}\ terms} \text{ for } s, s' \in S \text{ such that } m_{s,s'} \text{ is finite.}$$

The following result is due to Deligne [De, Proposition 4.17] and Brieskorn-Saito [BrSa, Proposition 5.5] :

Theorem 8.2 *Assume* W *is finite. Then, the morphism* $B_W^+ \to B_W$ *given by* $\sigma_s^+ \mapsto \sigma_s$ *is injective.*

Thanks to this result, we can identify B_W^+ with the submonoid of B_W consisting of those elements which can be written as products of generators σ_s, when W is finite.

Let $p : B_W^+ \to W$ be given by $\sigma_s \mapsto s$.

The exchange lemma (Lemma 2.4) has the following consequence [Bki, Chapitre IV, §1, Proposition 5] :

Proposition 8.3 *Let* $w = s_1 \cdots s_m = s'_1 \cdots s'_m$ *be two reduced expressions of* $w \in W$*. Then,* $\sigma_{s_1}^+ \cdots \sigma_{s_m}^+ = \sigma_{s'_1}^+ \cdots \sigma_{s'_m}^+$.

This allows the construction of a very nice section q of p (q is not multiplicative !) : given $w \in W$ and $w = s_1 \cdots s_m$ a reduced expression of w, we put $q(w) = \sigma_{s_1}^+ \cdots \sigma_{s_m}^+$. Thanks to the last proposition, this is independent of the choice of the reduced expression of w. We have $pq(w) = w$.

Let $\bar{S}$ be the set of equivalence classes for the relation defined by : $s, t \in S$ are equivalent if there is a sequence $t = t_0, t_1, \ldots, t_m = s$ such that $m(t_i, t_{i+1})$ is finite and odd. For $s \in S$, we denote by $\bar{s}$ its class in $\bar{S}$.

One can then read off conjugacy amongst generators of W and B_W from the Coxeter diagram. Two elements $s, t \in S$ are conjugate in W if and only if $\bar{s} = \bar{t}$. Similarly, σ_s and σ_t are conjugate in B_W if and only if $\bar{s} = \bar{t}$. So, we have :

Proposition 8.4 *The map $\sigma_s \mapsto 1_{\bar{s}}$ extends to a group morphism $B_W \to \mathbf{Z}^{\bar{S}}$. Its kernel is the derived subgroup of B_W,* i.e., *this morphism identifies the largest abelian quotient B_W^{ab} of B_W with $\mathbf{Z}^{\bar{S}}$. Similarly, W^{ab} is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^{\bar{S}}$.*

Examples (using the notations from the examples of §2).

- $\mathfrak{S}_n$: $\bar{S} = \{\bar{s}_1\}$.
- B_n : $\bar{S} = \{\bar{s}_0, \bar{s}_{n-1}\}$ $(n \geq 2)$.
- $I_2(m)$: $\bar{S} = \{\bar{t}_1\}$ for odd m, $\bar{S} = \{\bar{t}_1, \bar{t}_2\}$ for even m.

Let $\{q_{\bar{s}}\}_{\bar{s} \in \bar{S}}$ and $\{q'_{\bar{s}}\}_{\bar{s} \in \bar{S}}$ be two sets of indeterminates. Let $\tilde{\mathcal{O}} = \mathbf{Z}[q_{\bar{s}}, q'_{\bar{s}}]_{\bar{s} \in \bar{S}}$.

The Iwahori-Hecke algebra $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}(W)$ of (W, S) is the $\tilde{\mathcal{O}}$-algebra with generators T_s, $s \in S$, and relations

$$(T_s - q_{\bar{s}})(T_s - q'_{\bar{s}}) = 0 \text{ for } s \in S$$

$$\underbrace{T_s T_{s'} T_s T_{s'} \cdots}_{m_{ss'} \text{ terms}} = \underbrace{T_{s'} T_s T_{s'} T_s \cdots}_{m_{ss'} \text{ terms}} \text{ for } s, s' \in S \text{ such that } m_{s,s'} \text{ is finite.}$$

This is the quotient of the monoid algebra $\tilde{\mathcal{O}} B_W^+$ by the ideal generated by the elements $(T_s - q_{\bar{s}})(T_s - q'_{\bar{s}})$ for $s \in S$. Let T_w be the image of σ_w^+ for $w \in W$.

The next theorem shows that $\tilde{\mathcal{H}}$ is a deformation of $\mathbf{Z}W$ [Bki, Chap. IV, §2, Exercice 23] :

Theorem 8.5 *The algebra $\tilde{\mathcal{H}}$ is free over $\tilde{\mathcal{O}}$, with basis $\{T_w\}_{w \in W}$. The morphism $\tilde{\mathcal{H}} \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{O}}/(q_{\bar{s}} - 1, q'_{\bar{s}} + 1)_{\bar{s} \in \bar{S}} \to \mathbf{Z}W$, $T_w \otimes 1 \mapsto w$, is an isomorphism.*

We assume from now on that W is finite.

Let S' be a subset of S and W' be the subgroup of W generated by S'. Then, by [De],

- the submonoid of B_W^+ generated by $\{\sigma_s^+\}_{s \in S'}$ is isomorphic to $B_{W'}^+$,

- the subgroup of B_W generated by $\{\sigma_s\}_{s\in S'}$ is isomorphic to $B_{W'}$,
- the specialization of the subalgebra of $\tilde{\mathcal{H}}(W)$ generated by $\{T_s\}_{s\in S'}$ obtained by sending to 0 those parameters not associated to elements $\bar{s}, s \in S'$, is isomorphic to the specialization of $\tilde{\mathcal{H}}(W')$ given by identifying those parameters associated to elements of $\bar{S}'$ which become equal in $\bar{S}$.

In several applications, the Iwahori-Hecke algebra arises with invertible parameters. Then, without loss of generality, one may assume one of the two parameters $q_{\bar{s}}, q'_{\bar{s}}$ to be -1. So, let $\mathcal{O} = \tilde{\mathcal{O}}[q_{\bar{s}}^{-1}]_{\bar{s}\in\bar{S}}/(q'_{\bar{s}}+1)_{\bar{s}\in\bar{S}} \simeq \mathbf{Z}[q_{\bar{s}}, q_{\bar{s}}^{-1}]_{\bar{s}\in\bar{S}}$ and $\mathcal{H} = \tilde{\mathcal{H}} \otimes_{\tilde{\mathcal{O}}} \mathcal{O}$.

The Iwahori-Hecke algebra has a trace map $\tau : \mathcal{H} \to \mathcal{O}$ given by $\tau(T_w) = \delta_{1,w}$ (*i.e.*, we have $\tau(hh') = \tau(h'h)$ for $h, h' \in \mathcal{H}$).

Denote by "ind" the one-dimensional representation $\mathcal{H} \to \mathcal{O}$ given by $\mathrm{ind}(T_s) = q_{\bar{s}}$.

Proposition 8.6 *Given $w, w' \in W$, one has $\tau(T_w T_{w'}) = \delta_{w^{-1},w'}\mathrm{ind}(T_w)$.*

This means that the set $\{\mathrm{ind}(T_w)^{-1}T_{w^{-1}}\}_{w\in W}$ is the dual basis of $\{T_w\}_{w\in W}$ with respect to τ.

More conceptually, the trace τ gives a structure of symmetric algebra to $\mathcal{H}$, *i.e.*, the morphism :

$$\begin{aligned} \mathcal{H} &\to \mathrm{Hom}_{\mathcal{O}}(\mathcal{H}, \mathcal{O}) \\ h &\mapsto (h' \mapsto \tau(hh')) \end{aligned}$$

is an isomorphism.

Together with the fact that $\mathcal{H}$ is a deformation of $\mathbf{Z}W$, this explains the structure of $\mathcal{H}$ over an algebraic closure K of the field of fractions of $\mathcal{O}$ (Tits' deformation theorem) [Bki, Chap. IV, §2, Exercice 27] :

Theorem 8.7 *The algebra $\mathcal{H} \otimes_{\mathcal{O}} K$ is semi-simple and isomorphic to KW.*

Much more precise is the following rationality theorem (Benard, Springer, Benson-Curtis, Hoefsmit, Lusztig..., cf [Ge]) :

Theorem 8.8 *Assume W is a finite Weyl group. Then, the algebra $\mathbf{Q}W$ is isomorphic to a direct product of matrix algebras over $\mathbf{Q}$ and the algebra $\mathcal{H} \otimes_{\mathcal{O}} \mathbf{Q}(\sqrt{q_{\bar{s}}})_{\bar{s}\in\bar{S}}$ is isomorphic to a direct product of matrix algebras over $\mathbf{Q}(\sqrt{q_{\bar{s}}})_{\bar{s}\in\bar{S}}$.*

The theorem above generalizes to finite Coxeter groups : if W is a finite reflection group over $K \subset \mathbf{R}$, then KW is isomorphic to a product of matrix algebras over K and $\mathcal{H} \otimes_{\mathcal{O}} K(\sqrt{q_{\bar{s}}})_{\bar{s}\in\bar{S}}$ is isomorphic to a direct product of matrix algebras over $K(\sqrt{q_{\bar{s}}})_{\bar{s}\in\bar{S}}$.

9 Pseudo-reflection groups

Let V be a finite dimensional vector space over a characteristic zero field K. A *pseudo-reflection* of V is an automorphism of finite order whose set of fixed points is an hyperplane.

Let G be a finite subgroup of $GL(V)$.

We denote by $S(V)$ the symmetric algebra of V.

The following theorem is due to Shephard-Todd and Chevalley. It shows that the regularity of the ring of invariants $S(V)^G$ characterizes pseudo-reflection groups [Bens, Theorem 7.2.1] :

Theorem 9.1 *The following assertions are equivalent :*

(i) The algebra $S(V)^G$ is a polynomial algebra.

(ii) The group G is generated by pseudo-reflections.

(iii) The $S(V)^G[G]$-module $S(V)$ is free of rank one.

When $K \subset \mathbf{R}$, a pseudo-reflection is actually a reflection. When $K = \mathbf{C}$, a pseudo-reflection need not have order 2 ; a group generated by complex pseudo-reflections is then called a *complex reflection group*.

The rationality theorem 8.8 for representations of Weyl groups or finite Coxeter groups extends to pseudo-reflection groups : the group algebra of G over K is a direct product of matrix algebras over K [Bena, Bes].

The irreducible complex reflection groups have been classified by Shephard and Todd [ShTo]. There are two infinite series : the groups $A_n \simeq \mathfrak{S}_{n+1}$, the groups $G(p, q, n)$ and 34 exceptional groups (the dimension of an exceptional group is at most 8).

Let us describe the groups $G(p, q, n)$ ($p > 1$, $q \geq 1$, $n \geq 1$ and $q|p$). It turns out that these groups have nice presentations, generalizing in some sense the presentation of Coxeter groups and sharing some of their properties. In particular, these groups have a presentation given by a set S consisting of n or $n + 1$ pseudo-reflections and two kinds of relations :

- braid relations (homogeneous relations)
- finite order relations.

The group given by the same presentation, but without the finite order relations can be seen as an analog of the braid group defined in §8 for real reflection groups. We will come back to this in §10.

9.1 $G(p,1,n)$

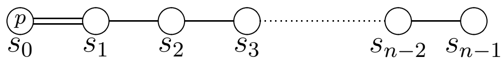
First, $G(p,1,n)$ is the group of n by n monomial complex matrices whose non-zero entries are p-th roots of unity. This group has a semi-direct product decomposition $G(p,1,n) = (\mathbf{Z}/p\mathbf{Z})^n \rtimes \mathfrak{S}_n \simeq (\mathbf{Z}/p\mathbf{Z}) \wr \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the subgroup of permutation matrices and $(\mathbf{Z}/p\mathbf{Z})^n$ is the subgroup of diagonal matrices.

Let $s_0 = \mathrm{diag}(\zeta, 1, \ldots, 1)$, where ζ is a primitive p-th root of unity. Keeping the notations of §2, Example (i), one sees that $G(p,1,n)$ is generated by the set of pseudo-reflections $\{s_0, s_1, \ldots, s_{n-1}\}$. They satisfy the following relations :

$$\text{braid relations} \begin{cases} s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 & \\ s_i s_j = s_j s_i & \text{if } |i-j|>1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } i \geq 1 \end{cases}$$

$$\text{finite order relations} \begin{cases} s_0^p = 1 \\ s_i^2 = 1 \text{ for } i \geq 1. \end{cases}$$

Actually, this gives a presentation for $G(p,1,n)$ by generators and relations. A convenient way to encode the relations is to use a generalization of the Coxeter diagrams :



Note that $G(2,1,n) = B_n$ and the presentation above is the Coxeter presentation.

Now, for $q|p$, we define $G(p,q,n)$ as the subgroup of $G(p,1,n)$ consisting of matrices where the product of the non-zero entries is a (p/q)-th root of unity.

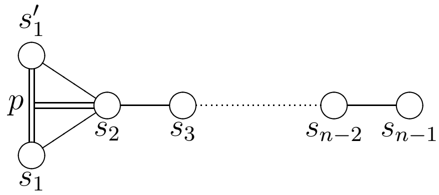
9.2 $G(p,p,n)$

Let us now look at $G(p,p,n)$. It is generated by the set of pseudo-reflections $\{s_1', s_1, \ldots, s_{n-1}\}$ where $s_1' = s_0 s_1 s_0^{-1}$. They satisfy the following relations :

$$\text{braid relations} \begin{cases} s_i s_j = s_j s_i & \text{if } |i-j|>1 \\ s_1' s_i = s_i s_1' & \text{for } i \geq 3 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } i \geq 1 \\ s_1' s_2 s_1' = s_2 s_1' s_2 & \\ s_2 s_1' s_1 s_2 s_1' s_1 = s_1' s_1 s_2 s_1' s_1 s_2 & \\ \underbrace{s_1 s_1' s_1 s_1' \cdots}_{p \text{ terms}} = \underbrace{s_1' s_1 s_1' s_1 \cdots}_{p \text{ terms}} & \end{cases}$$

$$\text{finite order relations} \begin{cases} {s'_1}^2 = 1 \\ s_i^2 = 1 \text{ for } i \geq 1. \end{cases}$$

This gives a presentation of $G(p,p,n)$ by generators and relations. The relations may be encoded in the following diagram :



Note that $G(p,p,2) = I_2(p)$ and the presentation above is a Coxeter presentation. Also, $G(2,2,n) = D_n$ and the presentation above is a Coxeter presentation.

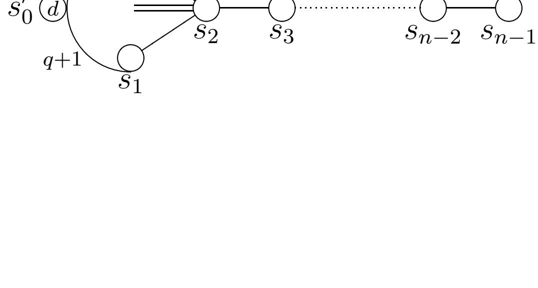
9.3 $G(p,q,n)$

Finally, let us consider $G(p,q,n)$ for $q|p$, $q \neq p$ and $q \neq 1$. We put $d = p/q$. This group is generated by the set of pseudo-reflections $\{s'_0, s'_1, s_1, \ldots, s_{n-1}\}$ where $s'_0 = s_0^q$. They satisfy the following relations :

$$\text{braid relations} \begin{cases} s_i s_j = s_j s_i & \text{if } |i-j| > 1 \\ s'_0 s_i = s_i s'_0 & \text{for } i \geq 2 \\ s'_1 s_i = s_i s'_1 & \text{for } i \geq 3 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } i \geq 1 \\ s'_1 s_2 s'_1 = s_2 s'_1 s_2 & \\ s'_0 s'_1 s_1 = s'_1 s_1 s'_0 & \\ s_2 s'_1 s_1 s_2 s'_1 s_1 = s'_1 s_1 s_2 s'_1 s_1 s_2 & \\ \underbrace{s_1 s'_0 s'_1 s_1 s'_1 s_1 \cdots}_{q+1 \textit{ terms}} = \underbrace{s'_0 s'_1 s_1 s'_1 s_1 s'_1 \cdots}_{q+1 \textit{ terms}} & \end{cases}$$

$$\text{finite order relations} \begin{cases} {s'_0}^d = 1 \\ {s'_1}^2 = 1 \\ s_i^2 = 1 \text{ for } i \geq 1. \end{cases}$$

We have obtained a presentation of $G(p,q,n)$ by generators and relations which we encode in the following diagram :



10 Topological construction of braid groups and Iwahori-Hecke algebras

Let V be a finite dimensional complex vector space and G a finite subgroup of $GL(V)$ generated by pseudo-reflections. Let $\mathcal{A}$ be the set of reflecting hyperplanes of G and $X = V - \bigcup_{H \in \mathcal{A}} H$. Let $\rho : X \to X/G$ be the projection map.

The following result is due to Steinberg [St] :

Theorem 10.1 *The group G acts freely on X,* i.e., *ρ is an unramified Galois covering.*

Let $x_0 \in X$. The *braid group* associated to G is $\mathcal{B}_G = \Pi_1(X/G, \rho(x_0))$ and the *pure braid group* associated to G is $\mathcal{P}_G = \Pi_1(X, x_0)$. Then, thanks to Steinberg's theorem, we have an exact sequence :

$$0 \to \mathcal{P}_G \xrightarrow{\rho_*} \mathcal{B}_G \to G \to 1.$$

10.1 The real case

Assume G is the complexification of a real reflection group, *i.e.*, there is a real vector space V' with $V = V' \otimes_{\mathbf{C}} \mathbf{R}$ and such that G is a subgroup of $GL(V')$. Let C_1 be a chamber of G (a connected component of $V' - \bigcup_{H \in \mathcal{A}} H \cap V'$) and take $x_0 \in C_1$. Let S be the set of reflections of G with respect to the walls of C_1. For $s \in S$, let γ_s be the path $[0,1] \to X$ defined by

$$\gamma_s(t) = \frac{x_0 + s(x_0)}{2} + \frac{x_0 - s(x_0)}{2} e^{i\pi t}.$$

Let τ_s be the class in $\mathcal{B}_G$ of $\rho(\gamma_s)$.

Brieskorn [Br] and Deligne [De] have proven the following theorem :

Theorem 10.2 *The map $\sigma_s \mapsto \tau_s$ induces an isomorphism $B_G \to \mathcal{B}_G$.*

10.2 The complex case

Let $H \in \mathcal{A}$. Let e_H be the order of the pointwise stabilizer of H in G. This is a cyclic group, generated by a pseudo-reflection s with non-trivial eigenvalue $\exp(2i\pi/e_H)$. Let $x_H \in X$. Let y_H be the intersection of H with the affine line containing x_H and $s(x_H)$. We assume x_H is close enough to H so that the closed ball with center y_H and radius $||x_H - y_H||$ does not intersect any H', $H' \in \mathcal{A}$, $H' \neq H$. Let α be a path from x_0 to x_H in X. Let λ be the path in X from x_H to $s(x_H)$ defined by

$$\lambda(t) = y_H + (x_H - y_H) e^{2i\pi t/e_H}.$$

We define the path γ from x_0 to $s(x_0)$ by

$$\gamma(t) = \begin{cases} \alpha(3t) & \text{for } 0 \leq t \leq 1/3 \\ \lambda(3t-1) & \text{for } 1/3 < t \leq 2/3 \\ s(\alpha(3-3t)) & \text{for } 2/3 < t \leq 1 \end{cases}$$

Let τ be the class of γ in $\mathcal{B}_G$: τ is called a *generator of the monodromy* associated to s (or to H). The image of τ in G is s.

Theorem 10.2 has a counterpart for complex reflection groups, based on a case by case study [BrMaRo] (six of the irreducible exceptional complex reflection groups are not covered by this approach).

Let us explain this for the group $G = G(p, q, n)$.

Theorem 10.3 *Assume $G = G(p, q, n)$. Then, for every $s \in S$, there is a generator of the monodromy τ_s associated to s, such that the group $\mathcal{B}_G$ has a presentation with set of generators $\{\tau_s\}_{s\in S}$ and relations the braid relations.*

10.3 Iwahori-Hecke algebras

An analogue of Proposition 8.4 for complex reflection groups is : $\mathcal{B}_G^{ab}$ is isomorphic to $\mathbf{Z}^{\mathcal{A}/G}$.

For $\mathcal{C} \in \mathcal{A}/G$ and $H \in \mathcal{C}$, we put $e_\mathcal{C} = e_H$. Let $\tilde{\mathcal{O}}$ be the polynomial ring over the integers on the set of variables $\{q_{\mathcal{C},j}\}_{\mathcal{C}\in\mathcal{A}/G, 0\leq j\leq e_\mathcal{C}-1}$. We put $q_{H,j} = q_{\mathcal{C},j}$ for $H \in \mathcal{C}$ and $\mathcal{C} \in \mathcal{A}/G$.

For $H \in \mathcal{A}$, let τ_H be a generator of the monodromy associated to H. Let $\tilde{\mathcal{H}}$ be the quotient of the group algebra $\tilde{\mathcal{O}}\mathcal{B}_G$ by the ideal generated by the $(\tau_H - q_{H,0})(\tau_H - q_{H,1}) \cdots (\tau_H - q_{H,e_H-1})$, for $H \in \mathcal{A}$. Then, $\tilde{\mathcal{H}}$ is the *Iwahori-Hecke algebra* associated to G.

When G is the complexification of a real reflection group, then Theorem 10.2 induces an isomorphism with the Iwahori-Hecke algebra associated with the Coxeter system (G, S).

When $G = G(p, q, n)$, the algebra $\tilde{\mathcal{H}}$ is a deformation of the group algebra $\tilde{\mathcal{O}}G$ (it is free over $\tilde{\mathcal{O}}$, with rank $|G|$) [Ar].

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