HOMOLOGICAL ALGEBRA AND REPRESENTATION THEORY

RAPHAËL ROUQUIER

Trinity Term 2008

Contents

1. Categories

1.1. Definitions.

1.1.1.

Definition 1.1. *A category $\mathcal{C}$ is the data of*

- *a set* $\mathrm{Ob}(\mathcal{C})$
- *a small set* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ *for every* $X, Y \in \mathrm{Ob}(\mathcal{C})$
- *a map* $\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}(X,Z)$, $(f,g) \mapsto g \circ f$ *for every* $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$

such that

- $\circ$ *is associative*
- *for every* $X \in \mathrm{Ob}(\mathcal{C})$*, there is an element* $\mathbf{1}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ *such that given* $Y \in \mathrm{Ob}(\mathcal{C})$ *and* $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$*, then* $f \circ \mathbf{1}_X = f$ *and given* $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$*, then* $\mathbf{1}_X \circ g = g$.

The elements of $\mathrm{Ob}(\mathcal{C})$ are the *objects* of $\mathcal{C}$ and the elements of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are called *morphisms* or *arrows* between X and Y. We denote by $\mathrm{Ar}(\mathcal{C})$ the set of arrows of $\mathcal{C}$. We sometimes write "$X \in \mathcal{C}$" to mean "$X \in \mathrm{Ob}(\mathcal{C})$".

Note that $\mathbf{1}_X$ is uniquely determined by the conditions of the definition.

A *small category* is a category $\mathcal{C}$ such that $\mathcal{C}$ is a small set.

Remark 1.2. The data of a small category with one object is the same as the data of a monoid with a unit. In general, one should think of a category as a monoid with "several objects": let $\mathcal{C}$ be a category. Recall that a magma is a set together with a partially defined associative multiplication law. Then, composition defines a structure of magma on $M = \coprod_{X,Y \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(X,Y)$: given $X \xrightarrow{f} Y$ and $Z \xrightarrow{g} T$, we say that $g \cdot f$ is defined when $Y = Z$ and we put then $g \cdot f = g \circ f$.

A map $X \xrightarrow{f} Y$ is a

- *monomorphism* (write $X \overset{f}{\hookrightarrow} Y$) if given $g, h : Y \to Z$ such that $gf = hf$, then $g = h$
- *epimorphism* (write $X \overset{f}{\twoheadrightarrow} Y$) if given $g, h : W \to X$ such that $fg = fh$, then $g = h$
- *isomorphism* (write $X \underset{\sim}{\overset{f}{\to}} Y$) if there exists $g : Y \to X$ such that $f \circ g = \mathbf{1}_Y$ and $g \circ f = \mathbf{1}_X$.

Example 1.3. The first and foremost example is the category Set. Its objects are the small sets and the morphisms are the maps between sets. The monomorphisms are the injections, the epimorphisms the surjections, and the isomorphisms the bijections.

We denote by $\mathcal{C}^{\mathrm{opp}}$ the *opposite category* to $\mathcal{C}$. We have $\mathrm{Ob}(\mathcal{C}^{\mathrm{opp}}) = \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and given $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(Y, Z)$, then the composition $g \circ^{\mathrm{opp}} f$ of f and g in $\mathcal{C}^{\mathrm{opp}}$ is $f \circ g$ (taken in $\mathcal{C}$).

A *subcategory* $\mathcal{C}'$ of $\mathcal{C}$ is a category $\mathcal{C}'$ with $\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$, with $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for $X, Y \in \mathrm{Ob}(\mathcal{C}')$, and where the composition in $\mathcal{C}'$ coincides with the one in $\mathcal{C}$. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is a *full subcategory* if $\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathrm{Ob}(\mathcal{C}')$. So, a full subcategory of $\mathcal{C}$ is determined by its set of objects and conversely, given any $I \subset \mathrm{Ob}(\mathcal{C})$, there is a unique full subcategory $\mathcal{C}'$ of $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C}') = I$. A *strictly full subcategory* $\mathcal{C}'$ of $\mathcal{C}$ is a full subcategory such that every object of $\mathcal{C}$ isomorphic to an object of $\mathcal{C}'$ is an object of $\mathcal{C}'$.

Remark 1.4. When describing graphically a category, we will usually omit the identity arrows.

1.1.2. *Functors.* Let $\mathcal{C}'$ be a category. A *functor* $F : \mathcal{C} \to \mathcal{C}'$ is the data of

- a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$
- a map $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$

such that $F(\mathbf{1}_X) = \mathbf{1}_{FX}$ and $F_{X,Z}(g \circ f) = F_{Y,Z}(g) \circ F_{X,Y}(f)$ for all $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$.

We will often write "F" for "$F_{X,Y}$" and "FX" for "$F(X)$".

Let $G : \mathcal{C} \to \mathcal{C}'$ be a functor. A *morphism of functors* $\alpha : F \to G$ is the data of morphisms $\alpha(X) \in \mathrm{Hom}_{\mathcal{C}'}(FX, GX)$ for $X \in \mathcal{C}$ making the following diagram commutative, for all $X \xrightarrow{f} Y$ in $\mathcal{C}$:

$$\begin{array}{ccc} FX & \xrightarrow{F(f)} & FY \\ {\scriptstyle \alpha(X)}\downarrow & & \downarrow{\scriptstyle \alpha(Y)} \\ GX & \xrightarrow[G(f)]{} & GY \end{array}$$

We denote by $\mathrm{Id}_{\mathcal{C}}$ the *identity functor* of $\mathcal{C}$ and by $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ the *category of functors* from $\mathcal{C}$ to $\mathcal{C}'$.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor.

The *image* of F is the full subcategory of $\mathcal{C}'$ with set of objects $\{F(X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$. The *essential image* of F is the full subcategory $\mathrm{im}\, F$ of $\mathcal{C}'$ with objects those objects of $\mathcal{C}$ isomorphic to objects of the image of F.

We say that F is

- *faithful* if $\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{C}'}(FX, FY)$ is injective for all $X, Y \in \mathcal{C}$
- *full* if $F(X, Y) : \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{C}'}(FX, FY)$ is surjective for all $X, Y \in \mathcal{C}$
- *fully faithful* if it is full and faithful

- *essentially surjective* if the essential image of F is $\mathcal{C}'$
- an *equivalence of categories* if there is a functor $G : \mathcal{C}' \to \mathcal{C}$ such that $F \circ G \simeq \mathrm{Id}_{\mathcal{C}'}$ and $G \circ F \simeq \mathrm{Id}_{\mathcal{C}}$.
- an *isomorphism of categories* if there is a functor $G : \mathcal{C}' \to \mathcal{C}$ such that $F \circ G = \mathrm{Id}_{\mathcal{C}'}$ and $G \circ F = \mathrm{Id}_{\mathcal{C}}$.

We leave the following Proposition as an exercice:

Proposition 1.5. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor. Then, F is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

A functor $\mathcal{C} \to \mathcal{C}'$ is the same as a functor $\mathcal{C}^{\mathrm{opp}} \to \mathcal{C}'^{\mathrm{opp}}$ and this gives a canonical isomorphism of categories $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')^{\mathrm{opp}} \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}^{\mathrm{opp}}, \mathcal{C}'^{\mathrm{opp}})$.

A *contravariant functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a functor $\mathcal{C}^{\mathrm{opp}} \to \mathcal{C}$.

Remark 1.6. In defining a morphism of functors, it is common to specify only the effect on objects when the effect on morphisms is obvious in the context.

1.2. **Representability.**

1.2.1. Let $\mathcal{C}^{\wedge} = \mathrm{Fun}(\mathcal{C}^{\mathrm{opp}} \to \mathrm{Sets})$.

There is a canonical functor

$$\mathrm{can} : \mathcal{C} \to \mathcal{C}^{\wedge},\ M \mapsto \mathrm{Hom}_{\mathcal{C}}(-, M).$$

Let $M \in \mathcal{C}$ and $F \in \mathcal{C}^{\wedge}$.

We have canonical maps

$$\mathrm{Hom}_{\mathcal{C}^{\wedge}}(\mathrm{Hom}_{\mathcal{C}}(-, M), F) \to F(M), \quad f \mapsto f(M)(\mathrm{id}_M)$$

$$\text{and } F(M) \to \mathrm{Hom}_{\mathcal{C}^{\wedge}}(\mathrm{Hom}_{\mathcal{C}}(-, M), F), \quad x \mapsto (N \mapsto (\phi \mapsto F(\phi)(x)))\,.$$

The following Theorem, whose proof is trivial, is a fundamental result in the theory of categories.

Theorem 1.7 (Yoneda's Lemma)**.** *The maps above define inverse bijections, functorial in $M \in \mathcal{C}$ and $F \in \mathcal{C}^{\wedge}$*

$$\mathrm{Hom}_{\mathcal{C}^{\wedge}}(\mathrm{Hom}_{\mathcal{C}}(-, M), F) \underset{\sim}{\overset{\sim}{\rightleftarrows}} F(M)$$

In particular, the canonical functor $\mathcal{C} \to \mathcal{C}^{\wedge}$ is fully faithful.

We say that the object M of $\mathcal{C}$ *represents* the functor $F : \mathcal{C}^{\mathrm{opp}} \to \mathrm{Sets}$ if $\mathrm{Hom}_{\mathcal{C}}(-, M) \xrightarrow{\sim} F$. Such an object M, endowed with the isomorphism $\mathrm{Hom}_{\mathcal{C}}(-, M) \xrightarrow{\sim} F$, is unique up to unique isomorphism.

1.2.2. Consider now $\mathcal{C}^{\vee} = \mathrm{Fun}(\mathcal{C}^{\mathrm{opp}}, \mathrm{Sets}^{\mathrm{opp}})$. There is a canonical isomorphism $\mathcal{C}^{\vee} \xrightarrow{\sim} \left((\mathcal{C}^{\mathrm{opp}})^{\wedge}\right)^{\mathrm{opp}}$. We deduce from §1.2.1 that the canonical functor $\mathcal{C} \to \mathcal{C}^{\vee},\ M \mapsto \mathrm{Hom}_{\mathcal{C}}(M, -)$ is fully faithful and that $\mathrm{Hom}_{\mathcal{C}^{\vee}}(G, \mathrm{Hom}_{\mathcal{C}}(M, -)) \xrightarrow{\sim} G(M)$.

Let $G : \mathcal{C} \to \mathrm{Sets}$ be a functor. We say that M *corepresents* G if $\mathrm{Hom}_{\mathcal{C}}(M, -) \xrightarrow{\sim} G$, where G is viewed as a functor $\mathcal{C}^{\mathrm{opp}} \to \mathrm{Sets}^{\mathrm{opp}}$. Such an object M (endowed with the isomorphism $\mathrm{Hom}_{\mathcal{C}}(M, -) \xrightarrow{\sim} G$) is unique up to unique isomorphism.

1.3. **Limits.** The limit (resp. the colimit) of a functor with value in $\mathcal{C}$ is, when it exists, an object of $\mathcal{C}$ going to a well-defined object of $C^\wedge$ (resp. $\mathcal{C}^\vee$) which is constructed using limits in the category of sets. This object will then be unique up to unique isomorphism.

In §1.3, I is a small category.

1.3.1. Let $F : I^{\mathrm{opp}} \to \mathrm{Sets}$ be a functor. We define the limit of F, denoted by $\lim F$, as the subset of $\prod_{i\in I} F(i)$ given by those families $(x_i)_{i\in I}$ such that for every $j \in I$ and $\phi : i \to j$, then $F(\phi)(x_j) = x_i$.

1.3.2. Let $F : I^{\mathrm{opp}} \to \mathcal{C}$ be a functor. A *limit* of F is, if it exists, an object of $\mathcal{C}$ denoted by $\lim F$ that represents the functor

$$\mathcal{C}^{\mathrm{opp}} \to \mathrm{Sets},\ M \mapsto \lim \mathrm{Hom}_{\mathcal{C}}(M, F)$$

where $\mathrm{Hom}_{\mathcal{C}}(M, F) : I^{\mathrm{opp}} \to \mathrm{Sets}$ is the functor $i \mapsto \mathrm{Hom}_{\mathcal{C}}(M, F(i))$. So, there is an isomorphism $\mathrm{Hom}_{\mathcal{C}}(-, \lim F) \xrightarrow{\sim} \lim \mathrm{Hom}_{\mathcal{C}}(-, F)$.

Similarly, given $G : I \to \mathcal{C}$ a functor, a *colimit* of G is an object of $\mathcal{C}$, if it exists, denoted by $\mathrm{colim}\, G$ that corepresents the functor

$$\mathcal{C} \to \mathrm{Sets},\ M \mapsto \lim \mathrm{Hom}_{\mathcal{C}}(G, M)$$

where $\mathrm{Hom}_{\mathcal{C}}(G, M) : I^{\mathrm{opp}} \to \mathrm{Sets}$ is the functor $i \mapsto \mathrm{Hom}_{\mathcal{C}}(G(i), M)$. So, there is an isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}\, G, -) \xrightarrow{\sim} \lim \mathrm{Hom}_{\mathcal{C}}(G, -)$.

We say that the limits and colimits above are *indexed* by the category I. We denote them also by $\lim_{i\in I} F(i)$ and $\mathrm{colim}_{i\in I} G(i)$.

Remark 1.8. A limit (resp. a colimit) is sometimes also called a projective limit (resp. an inductive or a direct limit).

Remark 1.9. Let $F : I^{\mathrm{opp}} \to \mathcal{C}$ be a functor. The colimit of F (indexed by the category I^{opp}) should not be confused with the limit of F (indexed by the category I) — cf Example 1.10. The difference comes from the fact that Sets is not equivalent to $\mathrm{Sets}^{\mathrm{opp}}$: an equivalence between Sets and $\mathrm{Sets}^{\mathrm{opp}}$ sends $\emptyset$ (the initial object, characterized by the property that $\mathrm{Hom}_{\mathrm{Sets}}(\emptyset, M)$ has cardinality 1 for every set M) to a set with one element (a final object, characterized by the property that $\mathrm{Hom}_{\mathrm{Sets}}(M, \{x\})$ has cardinality 1 for every set M). But $\mathrm{Hom}_{\mathrm{Sets}}(M, \emptyset) = \emptyset$ for all $M \neq \emptyset$ while $\mathrm{Hom}_{\mathrm{Sets}}(\{x\}, M) \neq \emptyset$ for $M \neq \emptyset$, hence there is no such equivalence.

Example 1.10. Take for I a discrete category (*i.e.*, the only arrows are the identities). Then, lim is called a *product*, denoted by $\prod_I$, and colim a *coproduct*, denote by $\coprod_I$. When F is in addition constant with value M, we denote by M^I the limit of F and by $M^{(I)}$ its colimit.

Example 1.11. Consider $I = \begin{array}{ccc} 3 & \longrightarrow & 1 \\ \downarrow & & \\ 2 & & \end{array}$. Given $G : I \to \mathcal{C}$ a functor, we call $\mathrm{colim}\, G$ the *fibered coproduct* of $G(1)$ by $G(2)$ above $G(3)$ and we denote it by $G(1) \sqcup_{G(3)} G(1)$.

Given a functor $F : I^{\mathrm{opp}} \to \mathcal{C}$, we call $\lim F$ the *fibered product* of $F(1)$ by $F(2)$ above $F(3)$ and we denote it by $F(1) \times_{F(3)} F(2)$.

Example 1.12. Consider the category $I = \bullet \rightrightarrows \bullet$. The data of a functor $F : I^{\rm opp} \to \mathcal{C}$ or of a functor $G : I \to \mathcal{C}$ corresponds to the data of two arrows $M \overset{f}{\underset{g}{\rightrightarrows}} N$. A limit of F (resp. a colimit of G) is called a *kernel* (resp. a *cokernel*) of $M \overset{f}{\underset{g}{\rightrightarrows}} N$.

When $\mathcal{C}$ is an additive category and $g = 0$, this is called a *kernel* (resp. a *cokernel*) of f.

1.3.3. We will now explain how to compute limits (resp. colimits) from kernels (resp. cokernels) and products (resp. coproducts). We leave the proofs of the following two Propositions to the reader.

Let $F : I^{\rm opp} \to \mathcal{C}$ be a functor. Given $f : i \to j$ in I, the morphisms $\mathrm{id}_{F(i)} : F(i) \to F(i)$ and $F(f) : F(j) \to F(i)$ induce two morphisms $F(i) \times F(j) \overset{\mathrm{id}_{F(i)}}{\underset{F(f)}{\rightrightarrows}} F(i)$. We deduce a morphism

$$\prod_{i\in \mathrm{Ob}(I)} F(i) \rightrightarrows \prod_{f\in \mathrm{Ar}(I)} F(\mathrm{source}(f)) \ .$$

Proposition 1.13. *The limit of F is isomorphic to the kernel of*

$$\prod_{i\in \mathrm{Ob}(I)} F(i) \rightrightarrows \prod_{f\in \mathrm{Ar}(I)} F(\mathrm{source}(f)) \ .$$

Let $G : I \to \mathcal{C}$ be a functor. Given $f : i \to j$ in I, the morphisms $\mathrm{id}_{G(i)} : G(i) \to G(i)$ and $G(f) : G(i) \to G(j)$ induce two morphisms $G(i) \overset{\mathrm{id}_{G(i)}}{\underset{G(f)}{\rightrightarrows}} G(i) \coprod G(j)$.

We deduce a morphism

$$\coprod_{f\in \mathrm{Ar}(I)} G(\mathrm{source}(f)) \rightrightarrows \coprod_{i\in \mathrm{Ob}(I)} G(i) \ .$$

Proposition 1.14. *The colimit of G is isomorphic to the cokernel of*

$$\coprod_{f\in \mathrm{Ar}(I)} G(\mathrm{source}(f)) \rightrightarrows \coprod_{i\in \mathrm{Ob}(I)} G(i) \ .$$

1.3.4. We have explained in §1.3.1 how to compute the limit of a functor $F : I^{\rm opp} \to \mathrm{Sets}$.

Let $G : I \to \mathrm{Sets}$ be a functor. We define an equivalence relation $\sim$ on $\coprod_{i\in I} G(i)$ as the relation generated by $x \sim G(f)(x)$ for $i \in I$, $x \in G(i)$ and $f : i \to j$.

Let M be a set. Consider the composition of canonical morphisms

$$\mathrm{Hom}((\coprod_i G(i))/\sim, M) \hookrightarrow \mathrm{Hom}(\coprod_i G(i), M) \xrightarrow{\sim} \prod_i \mathrm{Hom}(G(i), M)).$$

Its image is contained in $\lim_i \mathrm{Hom}(G(i), M)$ and we obtain a bijection

$$\mathrm{Hom}((\coprod_i G(i))/\sim, M) \xrightarrow{\sim} \lim_i \mathrm{Hom}(G(i), M).$$

We deduce the following Proposition:

Proposition 1.15. *Let $G : I \to \mathrm{Sets}$ be a functor. Then*

$$\mathrm{colim}\, G \xrightarrow{\sim} \left(\coprod_{i\in I} G(i)\right) / \sim$$

where $\sim$ *is the equivalence relation generated by* $x \sim G(f)(x)$ *for* $i \in I$, $x \in G(i)$ *and* $f : i \to j$.

We will see that under certain assumptions, the equivalence relation above has a more direct description.

We say that the category I is *filtrant* if

- for any i, j objects of I, there exists $i \to k$ and $j \to k$ two arrows in I : $\begin{smallmatrix} i \searrow & \\ & k \\ j \nearrow & \end{smallmatrix}$
- for any arrows $f, g : i \to j$ in I, there exists $h : j \to k$ such that $hf = hg : i \rightrightarrows j \longrightarrow k$.

Assume I is filtrant. Then the equivalence relation above has the following description. Let $x_i \in G(i)$ and $x_j \in G(j)$. Then $x_i \sim x_j$ if and only if there exists $f : i \to k$ and $g : j \to k$ such that $G(f)(x_i) = G(g)(x_j)$.

Example 1.16. Let $\mathcal{C} = \text{Sets}$. The kernel of $M \underset{g}{\overset{f}{\rightrightarrows}} N$ is $\{m \in M | f(m) = g(m)\}$. Its cokernel is the quotient of N by the equivalence relation generated by $f(m) \sim g(m)$ for $m \in M$.

The product in Sets is the product of sets. The coproduct in Sets is the disjoint union.

1.4. **Quotients of categories.** Let $\mathcal{C}$ be a category. An *equivalence relation* on $\mathcal{C}$ is

- the data for every X, Y objects of $\mathcal{C}$, of an equivalence relation $\sim$ on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$
- such that if $f \sim f'$, then $gf \sim gf'$ (resp. $fg \sim f'g$), for every arrow g with source Y (resp. with target X).

Given $\sim$ an equivalence relation on $\mathcal{C}$, we define the *quotient category* $\mathcal{C}/\sim$ of $\mathcal{C}$ by the equivalence relation $\sim$ as follows: its objects are those of $\mathcal{C}$. We have $\mathrm{Hom}_{\mathcal{C}/\sim}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\sim$.

There is canonical *quotient functor* $\mathrm{can} : \mathcal{C} \to \mathcal{C}/\sim$.

The quotient category is characterised by the following universal property: we have $\mathrm{can}(f) = \mathrm{can}(f')$ when $f \sim f'$ and given any category $\mathcal{C}'$ and any functor $F : \mathcal{C} \to \mathcal{C}'$ such that $F(f) = F(f')$ whenever $f \sim f'$, then there exists a unique functor $G : \mathcal{C}/\sim \to \mathcal{C}'$ such that $F = G \circ \mathrm{can}$:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ & {\scriptstyle \mathrm{can}} \searrow \quad \nearrow {\scriptstyle G} & \\ & \mathcal{C}/\sim & \end{array}$$

More generally, given binary relations $f \sim f'$ on Hom-spaces of $\mathcal{C}$, there exists a quotient category, uniquely defined up to unique equivalence by the universal property above. It is equivalent to the quotient of $\mathcal{C}$ by the equivalence relation generated by $\sim$.