PAMM · Proc. Appl. Math. Mech. **8**, 10969 – 10970 (2008) / **DOI** 10.1002/pamm.200810969

Adaptive multiresolution schemes for reaction-diffusion systems

Mostafa Bendahmane [*1], **Raimund Bürger** [**1], **Ricardo Ruiz** [***1], and **Kai Schneider** [†2]

[1] Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile.
[2] Centre de Mathématiques et d'Informatique, Université de Provence, 39 rue Joliot-Curie, 13453 Marseille cedex 13, France.

This note outlines a fully adaptive multiresolution scheme with local time stepping for the efficient numerical solution of (possibly degenerate) reaction-diffusion systems. A new numerical example showing Turing-type pattern formation is presented.

1 Introduction

In [1] we present a fully adaptive multiresolution (MR) scheme for spatially 2D, possibly degenerate reaction-diffusion systems, focusing on models of combustion, pattern formation, and chemotaxis. Solutions of these equations in these applications exhibit steep gradients, and in the degenerate case, sharp fronts and discontinuities. This calls for a concentration of computational effort in zones of strong variation. The MR scheme is based on finite volume discretizations with explicit time stepping. The MR representation of the solution is stored in a graded tree ("quadtree"), whose leaves are the non-uniform finite volumes on the borders of which the numerical divergence is evaluated. By a thresholding procedure, namely the elimination of leaves that are smaller than a threshold value, substantial data compression and CPU time reduction is attained. Our version of the MR method includes a locally varying adaptive time stepping strategy similar to that by Müller and Stiriba [5]. Numerical experiments presented in [1] illustrate the effectiveness of the adaptive MR method. It turns out that local time stepping accelerates the adaptive MR method by a factor of two, while the error remains controlled.

2 Reaction-diffusion system, multiresolution scheme and numerical example

We consider the domain $Q_T := \Omega \times (0,T)$, $\Omega := (0,1)^2$, its boundary $\Sigma_T := \partial\Omega \times (0,T)$, and the reaction-diffusion system

$$u_t = \gamma f(u,v) + \Delta u, \quad v_t = \gamma g(u,v) + d\Delta v \quad \text{on } Q_T\,, \tag{1}$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \qquad v(\boldsymbol{x},0) = v_0(\boldsymbol{x}) \quad \text{on } \Omega, \quad \nabla u \cdot \boldsymbol{n} = \nabla v \cdot \boldsymbol{n} = 0 \quad \text{on } \Sigma_T. \tag{2}$$

In [1], this system is studied with more general degenerate diffusion terms. This system models several phenomena including combustion [1, 7], but is considered here as a well-known model of pattern formation in mathematical biology. Under a number of structural conditions relating the functions $f(u,v)$ and $g(u,v)$ and their derivatives to the parameters γ and d, the system produces stationary solutions with Turing-type spatial patterns (see [6] for details). To produce this effect, we select the kinetics $f(u,v) = a - u + u^2 v$ and $g(u,v) = b - u^2 v$, with the parameters $a = -0.5$, $b = 1.9$, $d = 4.8$, and $\gamma = 800$ [6].

We employ a standard finite volume scheme to discretize, which is described here for a uniform grid. The domain Ω is partitioned into control volumes $(\Omega_{ij})_{(i,j)\in\Lambda}$, where Λ is an index set, defining $\Omega_{ij} := [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$, $\Delta x := x_{i+1/2} - x_{i-1/2}$, $\Delta y := y_{j+1/2} - y_{j-1/2}$, for all $(i,j) \in \Lambda$, and $\widetilde{\Delta x} := \min\{\Delta x, \Delta y\}$. Let $\bar{q}_{ij}(t)$ denote the cell average over Ω_{ij} of a quantity q at time t. The FV scheme is described here for the first equation of (1); for the second equation, we replace u by v and $f(u,v)$ by $g(u,v)$. Integrating the respective equation and averaging over Ω_{ij} yields the folowing expression, where $\mathcal{D}$ denotes the RHS of the PDE except for the reaction term:

$$\frac{1}{|\Omega_{ij}|}\iint_{\Omega_{ij}} u_t(\boldsymbol{x},t)\,d\boldsymbol{x} = \frac{1}{|\Omega_{ij}|}\iint_{\Omega_{ij}} \mathcal{D}\big(u(\boldsymbol{x},t), \nabla u(\boldsymbol{x},t)\big)\,d\boldsymbol{x} + \frac{1}{|\Omega_{ij}|}\iint_{\Omega_{ij}} f\big(u(\boldsymbol{x},t), v(\boldsymbol{x},t)\big)\,d\boldsymbol{x}.$$

We discretize $\mathcal{D}$ via $\bar{\mathcal{D}}_{ij} := -\frac{1}{\Delta x}(\bar{F}_{i+1/2,j} - \bar{F}_{i-1/2,j}) - \frac{1}{\Delta y}(\bar{F}_{i,j+1/2} - \bar{F}_{i,j-1/2})$ defining $\bar{F}_{i+1/2,j} := -\frac{1}{\Delta x}(\bar{u}_{i+1,j} - \bar{u}_{ij})$ and $\bar{F}_{i,j+1/2} := -\frac{1}{\Delta y}(\bar{u}_{i,j+1} - \bar{u}_{ij})$. The reaction term is approximated by $\bar{f}_{ij} \approx f(\bar{u}_{ij}, \bar{v}_{ij})$. A first-order Euler time discretization yields the formula $\bar{u}_{ij}^{n+1} = \bar{u}_{ij}^n + \Delta t\gamma\bar{f}_{ij} + \Delta t\bar{\mathcal{D}}_{ij}(\mathcal{S}(\bar{u}_{ij}^n), \widetilde{\Delta x})$, $\bar{v}_{ij}^{n+1} = \bar{v}_{ij}^n + \Delta t\gamma\bar{g}_{ij} + d\Delta t\bar{\mathcal{D}}_{ij}(\mathcal{S}(\bar{v}_{ij}^n), \widetilde{\Delta x})$, where $\mathcal{S}(\cdot)$ denotes the stencil used for computing $\bar{\mathcal{D}}_{ij}$. This scheme is stable under the CFL condition

$$\lambda\gamma(\|f_u\|_\infty + \|f_v\|_\infty + \|g_u\|_\infty + \|g_v\|_\infty) + 8\mu d \le 1, \quad \lambda := \Delta t/\widetilde{\Delta x}, \quad \mu := \Delta t/\widetilde{\Delta x}^{\,2}. \tag{3}$$

The adaptive MR method is an efficient space-time adaptive implementation of the FV scheme, which relies on storing the numerical solution on a sequence of nested dyadic grids defined on Ω. Basically, we define a finest grid on Ω with the

* e-mail: mostafab@ing-mat.udec.cl. MB is supported by Fondecyt project 1070682.
** Corresponding author. e-mail: rburger@ing-mat.udec.cl. RB is supported by Fondecyt project 1050728 and Fondap project 15000001.
*** e-mail: rruiz@ing-mat.udec.cl
† e-mail: kschneid@cmi.univ-mrs.fr