Induced subgraphs of graphs with large chromatic number. VI. Banana trees

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Abstract

We investigate which graphs H have the property that in every graph with bounded clique number and sufficiently large chromatic number, some induced subgraph is isomorphic to a subdivision of H. In an earlier paper [6], one of us proved that every tree has this property; and in another earlier paper with M. Chudnovsky [2], we proved that every cycle has this property. Here we give a common generalization. Say a “banana” is the union of a set of paths all with the same ends but otherwise disjoint. We prove that if H is obtained from a tree by replacing each edge by a banana then H has the property mentioned. We also find some other multigraphs with the same property.

1 Introduction

All graphs in this paper are finite and simple. For some purposes it is convenient to use multigraphs instead of graphs; all multigraphs in this paper are finite and loopless. If G is a graph, $\chi(G)$ denotes its chromatic number, and $\omega(G)$ denotes its clique number, that is, the cardinality of the largest clique of G.

Let H be a multigraph, and let J be a graph obtained from H by replacing each edge uv by a path (of length at least one) joining u, v, such that these paths are vertex-disjoint except for their ends. Then J is a *subdivision* of H. We say a graph G is *H-subdivision-free* if no induced subgraph of G is a subdivision of H. We could ask:

- which multigraphs H have the property that for all κ there exists c such that every H-subdivision-free graph with clique number at most κ has chromatic number at most c?
- which multigraphs H have the property that for every subdivision J of H and for all κ there exists c such that every J-subdivision-free graph with clique number at most κ has chromatic number at most c?

The second question, while more complicated, is perhaps better. At least if we confine ourselves to "controlled" classes of graphs (defined later), we know the answer to the second question, while the first remains open. The second question could be rephrased as asking for which multigraphs H every graph with bounded clique number and large chromatic number contains a "long" subdivision of H, that is, one in which every edge is subdivided at least some prescribed number of times.

Let us say a multigraph H is *pervasive* in some class of graphs $\mathcal{C}$ if it has the second property above for graphs in the class; that is, for every subdivision J of H and for all $\kappa \geq 0$ there exists c such that every J-subdivision-free graph $G \in \mathcal{C}$ with $\omega(G) \leq \kappa$ satisfies $\chi(G) \leq c$. (The reader is referred to [3] for a more detailed introduction to the topic of pervasiveness.)

There are some earlier theorems that can be expressed in this language. First, Scott [6] proved that

1.1 *Every tree is pervasive in the class of all graphs.*

Second, we proved with Maria Chudnovsky [2] a conjecture of Gyárfás [4] that for all κ, ℓ, every graph with clique number at most κ and sufficiently large chromatic number has an induced cycle of length at least ℓ; and that can be reformulated as:

1.2 *The multigraph with two vertices and two parallel edges is pervasive in the class of all graphs.*

Actually, we proved in [3] that:

1.3 *For all $k \geq 0$, the multigraph with two vertices and k parallel edges is pervasive in the class of all graphs.*

One of the main theorems of this paper is the following common generalization:

1.4 *Let H be a multigraph obtained from a tree by adding parallel edges. Then H is pervasive in the class of all graphs.*

What is known in the converse direction? Chalopin, Esperet, Li and Ossona de Mendez [1] proved:

1.5 *Every graph that is pervasive in the class of all graphs is a forest of chandeliers,*

where

- a *chandelier* is a graph obtained from a tree by adding a new vertex called the *pivot* adjacent to its leaves;
- a *tree of chandeliers* is either a chandelier or obtained from a smaller tree of chandeliers by identifying some vertex with the pivot of a new chandelier; and
- a *forest of chandeliers* is a graph where every component is a tree of chandeliers.

A *string graph* is the intersection graph of a set of curves in the plane. In fact a result stronger than 1.5 was shown in [1], namely that:

1.6 *Every graph that is pervasive in the class of string graphs is a forest of chandeliers.*

With M. Chudnovsky [3], we proved a converse to this:

1.7 *Every forest of chandeliers is pervasive in the class of string graphs.*

The goal of this paper is to investigate pervasiveness in other classes of graphs. Before we go on, we need some definitions.

If $X \subseteq V(G)$, $G[X]$ denotes the subgraph induced on X, and we write $\chi(X)$ for $\chi(G[X])$. If $\rho \geq 0$ is an integer, then for $v \in V(G)$, $N_G^\rho[v]$ means the set of vertices of G with distance at most ρ from v; and $\chi^\rho(G)$ denotes the maximum over all vertices v of $\chi(N_G^\rho[v])$, or zero for the null graph. Sometimes we speak of "G-distance" (to mean distance in G) rather than just distance, in case there may be some ambiguity. Let us say an *ideal* is a class of graphs such that for all graphs G, H, if G is an induced subgraph of H and $H \in \mathcal{C}$ then $G \in \mathcal{C}$. An ideal $\mathcal{C}$ is

- *colourable* if there exists k such that all members of $\mathcal{C}$ have chromatic number at most k;
- *ρ-bounded* if there exists τ such that $\chi^\rho(G) \leq \tau$ for all $G \in \mathcal{C}$;
- *ρ-controlled* if every ρ-bounded subideal of $\mathcal{C}$ is colourable; and
- *controlled* if it is ρ-controlled for some $\rho \geq 0$.

Roughly, if a graph in a controlled ideal has large chromatic number, then some ball of bounded radius in the graph also has large chromatic number. We proved the following in [3], a significant extension of 1.7:

1.8 *Every forest of chandeliers is pervasive in every controlled ideal.*

We would like to know which multigraphs are pervasive in the ideal of all graphs. Subdividing edges in a graph or multigraph does not change whether the graph or multigraph is pervasive, so it is enough to decide which graphs are pervasive. (We could have written this paper just working with graphs, but sometimes multigraphs are more convenient.) Every such graph is a forest of chandeliers, so let H be a forest of chandeliers; in view of the results of [3], what do we still need to prove, to show that H is pervasive in the ideal of all graphs? Let J be a subdivision of H, and let $\mathcal{C}$ be the ideal of all J-subdivision-free graphs; we need to show that the members of $\mathcal{C}$ with bounded clique number also have bounded chromatic number. Suppose not; so for some $\kappa \geq 0$, there is a noncolourable subideal $\mathcal{D}$ of $\mathcal{C}$ such that all graphs $G \in \mathcal{D}$ satisfy $\omega(G) \leq \kappa$. In particular, since H is a forest of chandeliers, 1.8 implies that $\mathcal{D}$ is not ρ-controlled, for any ρ. Let $\rho \geq 0$. Since $\mathcal{D}$ is not ρ-controlled, there is a noncolourable ρ-bounded subideal of $\mathcal{D}$. Thus, a forest of chandeliers H is not pervasive in the ideal of all graphs if and only if for some subdivision J of H, some $\kappa \geq 0$ and all $\rho \geq 0$ there is a noncolourable ρ-bounded ideal of J-subdivision-free graphs all with clique number at most κ.

Let us say a multigraph H is *widespread* if for every subdivision J of H and all $\kappa \geq 0$ there exists $\rho \geq 0$ such that every ρ-bounded ideal of J-subdivision-free graphs G with $\omega(G) \leq \kappa$ is colourable. Then we have shown that a forest of chandeliers H is not pervasive in the ideal of all graphs if and only it is not widespread.

We know which graphs are pervasive in controlled ideals, and roughly speaking, the concept of "widespread" is the complementary property; a graph is both pervasive in controlled ideals and widespread if and only if it is pervasive in the ideal of all graphs, which is what we really want to determine.

Scott [6] conjectured that every multigraph is pervasive in the ideal of all graphs, but this was disproved in [5]. Now we have a different question: which graphs H are widespread? Originally we expected that the answer would be "if and only if H is a forest of chandeliers", but in this paper we give some counterexamples, that is, widespread graphs that are not forests of chandeliers. So now our best guess is a resuscitated version of Scott's conjecture, the following:

1.9 Conjecture: *Every multigraph is widespread.*

We are very far from proving this; for instance we still do not know whether every forest of chandeliers, or indeed every chandelier, is widespread. All the multigraphs that we have proved to be widespread are subdivisions of outerplanar graphs.

We proved in [3] that:

1.10 *For all $\rho \geq 2$ and every multigraph J, every ρ-bounded class of J-subdivision free graphs is 2-bounded.*

Thus 1.9 is equivalent to the following, which is nicer (although we do not use 1.11 in this paper):

1.11 Conjecture: *For all graphs J and for all integers $\tau \geq 0$, there exists c such that if G is a graph with chromatic number more than c, then either some induced subgraph of G is a subdivision of J or $\chi^2(G) > \tau$.*

If $e = uv$ is an edge of a multigraph G, *fattening* e means replacing e by some set of parallel edges all joining u, v. In this paper we will prove the following three results:

1.12 *Let T be a tree, and let H be a multigraph obtained by fattening the edges of T. Then H is widespread.*

1.13 *Let H be a multigraph obtained from a cycle by fattening all except one of its edges. Then H is widespread.*

1.14 *Let H be a multigraph obtained from a triangle K_3 by fattening two of its edges and replacing the third by two parallel edges. Then H is widespread.*

We remark that the multigraphs H of 1.13 and 1.14 are in general not forests of chandeliers. Note also that 1.4 follows immediately from 1.12 and 1.8 (as fattening the edges of a tree gives a multigraph whose subdivisions are banana trees, and banana trees are trees of chandeliers).

2 Distant subgraphs with large chromatic number

With 1.9 in mind, let us see what we need. We have a graph J (a subdivision of the initial multigraph H), and a number κ, and we need to show that if we choose ρ large enough, then every ρ-bounded ideal of J-subdivision-free graphs with clique number at most κ is colourable. At this stage we prefer not to specify H, J, and see how far we can progress in general. So H, J might as well both be K_ν^1, the graph obtained from a complete graph K_ν by subdividing every edge once; because if ν is large enough then there is an induced subgraph of K_ν^1 which is a subdivision of any fixed graph. So, we are given ν and κ, and let us choose ρ very large in terms of ν, κ. Now we need to show that every ρ-bounded ideal $\mathcal{C}$ of K_ν^1-subdivision-free graphs with clique number at most κ is colourable. Choose some such $\mathcal{C}$; then since it is ρ-bounded, there exists τ such that $\chi^\rho(G) \leq \tau$ for all $G \in \mathcal{C}$. Altogether then we have four numbers κ, ν, ρ, τ, where κ, ν are given, and ρ is some large function of the numbers κ, τ that we can choose, and then after selecting ρ, the number τ is given. We need to prove for such a quadruple of numbers, there is a number c, such that every graph G that is K_ν^1-subdivision-free and satisfies $\chi^\rho(G) \leq \tau$ and $\omega(G) \leq \kappa$ also satisfies $\chi(G) \leq c$. We begin by proving some lemmas about such graphs G. We need first:

2.1 *For all $\kappa \geq 0$ and $d, s \geq 0$ there exists $k \geq 0$ with the following property. Let G be a connected graph with $\omega(G) \leq \kappa$. Let $x_1, \ldots, x_k \in V(G)$ be distinct, and let $v \in V(G)$ such that the distance between v and x_i is at most d for $1 \leq i \leq k$. Then there exists $u \in V(G)$ and a subset $S \subseteq \{1, \ldots, k\}$ with $|S| = s$, and for each $i \in S$ a path Q_i between u and x_i of length at most d, such that each Q_i is a shortest path in G between u, x_i, and for all distinct i, j, Q_i, Q_j are disjoint except for u, and have the same length, and there is no edge between $V(P_i) \setminus \{u\}$ and $V(P_j) \setminus \{u\}$.*

Proof. For fixed κ, s we proceed by induction on d. The result is vacuous for $d = 0$ and true for $d = 1$, so we assume $d > 1$ and that setting $k = k'$ satisfies the result for $d - 1$. We may therefore assume that every vertex of G has distance less than d from at most $k' - 1$ of $x_1, \ldots, x_k$. Let k_1 be such that every graph with at least k_1 vertices either has a clique of cardinality κ or a stable set of cardinality s; and let $k = ((d+1)(k'-1)+1)k_1$. We claim that k satisfies the theorem. For given $G, x_1, \ldots, x_k, v$ as in the theorem, let P_i be a shortest path in G between v, x_i for $1 \leq i \leq k$; thus each P_i is an induced path. Let D be the digraph with vertex set $\{1, \ldots, k\}$ in which there is an edge from i to j if some vertex of P_i has distance less than d from v_j. This digraph has outdegree at most $(d+1)(k'-1)$, and so its underlying graph is $2(d+1)(k'-1)$-degenerate and therefore $(2(d+1)(k'-1)+1)$-colourable; and so there exists $I \subseteq \{1, \ldots, k\}$ with $|I| \geq k/(2(d+1)(k'-1)+1) = k_1$ such that for all distinct $i, j \in I$, every vertex of P_i has distance at least d from v_j. It follows that

- all the paths $P_i (i \in I)$ have length exactly d (since $v \in V(P_i)$ and so has distance d from x_j);
- all the paths $P_i (i \in I)$ are pairwise disjoint except for v; and
- for all distinct $i, j \in I$, every edge between $V(P_i) \setminus \{v\}$ and $V(P_j) \setminus \{v\}$ joins two neighbours of v.

From the choice of k_1, since $\omega(G) \leq \kappa$, there exists $J \subseteq I$ with $|J| = s$ such that the neighbours of v in $P_i(i \in J)$ are pairwise nonadjacent; and then setting $Q_i = P_i(i \in J)$ satisfies the theorem. This proves 2.1. ▮

2.2 *For all $\nu, d \geq 0$, there exist $k, \ell \geq 0$ with the following property. Let G be a graph, let $X_1, \ldots, X_k$ be nonempty connected subgraphs of G, and let $v_1, \ldots, v_\ell \in V(G)$, such that*

- *for all distinct $i, j \in \{1, \ldots, k\}$, every vertex in X_i has distance at least three from every vertex of X_j;*
- *for all distinct $i, j \in \{1, \ldots, \ell\}$, the distance between v_i, v_j is at least $2d + 2$; and*
- *for $1 \leq i \leq k$ and $1 \leq j \leq \ell$, the distance between X_i and v_j is at most d.*

Tnen G is not K^1_ν-subdivision-free.

Proof. Let $s = \nu(\nu - 1)/2$; choose k_1 such that setting $k = k_1$ satisfies 2.1; and let $\ell_1 = \nu\binom{k_1}{s}$. We claim that k, ℓ satisfy the theorem. For let $G, X_1, \ldots, X_k, v_1, \ldots, v_\ell$ be as in the theorem.

For $1 \leq i \leq k$ and $1 \leq j \leq \ell$, let the shortest path between X_i and v_j have length d_{ij}. For each value of j, there are only $(d + 1)^k$ possibilities for the sequence $d_{1j}, \ldots, d_{kj}$, and so there exists $J_1 \subseteq \{1, \ldots, \ell\}$ with $|J_1| \geq \ell(d+1)^{-k} = \ell_1$ such that for each $i \in \{1, \ldots, k\}$, the numbers $d_{ij}(j \in J_1)$ all have some common value, say d_i. Since there are only $d + 1$ possibilities for d_i, there exists $I_1 \subseteq \{1, \ldots, k\}$ with $|I_1|/(d+1) = k_1$ such that the numbers $d_i(i \in I_1)$ all have some common value, say D. Thus for each $i \in I_1$ and each $j \in J_1$, the distance between X_i and v_j is D. Let P_{ij} be some shortest path between X_i and v_j, and let its end in X_i be x_{ij}. For each j, let G_j be the subgraph induced on the union of the paths $P_{ij}(i \in I_1, j \in J_1)$. For distinct $j, j' \in J_1$, since the distance between $v_j, v_{j'}$ is at least $2d + 2$ and every vertex of G_j has distance at most d from v_j and the same for $G_{j'}$, it follows that $G_j, G_{j'}$ are disjoint and there is no edge joining them.

Suppose that for some distinct $i, i' \in I_1$ and $j \in J_1$, some vertex z of P_{ij} belongs to or has a neighbour in $X_{i'}$. Since every path between x_{ij} and $X_{i'}$ has length at least three, it follows that z is not x_{ij} or its neighbour in P_{ij}, and so there is a path between u_j and z of length at most $D - 2$, and hence a path between U_j and $X_{i'}$ of length at most $D - 1$, a contradiction. Thus no vertex of P_{ij} belongs to or has a neighbour in $X_{i'}$.

For each $j \in J_1$, by 2.1 applied to G_j there exist $I_j \subseteq I_1$ with $|I| = s$, and a vertex $u_j \in V(G_j)$, and induced paths Q_{ij} of G_j between u_j and x_{ij} for each $i \in I_j$, such that the paths $Q_{ij}(i \in I_j)$ are pairwise disjoint except for u_j, and there are no edges between them not incident with u_j. Since $\ell_1 = \nu\binom{k_1}{s}$, there exists $J \subseteq J_1$ with $|J| = \nu$ such that the sets $I_j(j \in J)$ are all equal, equal to some I say. Since $|I| = s = \nu(\nu-1)/2$, we can number the members of I as $i_{j,j'}$ where $j, j' \in J$ and $j < j'$. For all $j, j' \in J$ with $j < j'$, let $i = i_{j,j'}$; the subgraph $Q_{ij} \cup X_i \cup Q_{ij'}$ is connected, and so includes an induced path $R_{jj'}$ of G between $v_j, v_{j'}$. But then the vertices $v_j(j \in J)$ and the paths $R_{jj'}$ provide an induced subgraph isomorphic to a subdivision of K^1_ν. This proves 2.2. ▮

The main result of this section is the following.

2.3 *For all $\nu, d, c, \tau \geq 0$ there exists $c' \geq 0$ with the following property. Let G be a K^1_ν-subdivision-free graph, such that $\chi^{2d+7}(G) \leq \tau$, and let $Z \subseteq V(G)$ with $\chi(Z) > c'$. Then there exist subsets $Z_1, Z_2 \subseteq Z$ such that $\chi(Z_i) > c$ for $i = 1, 2$ and the G-distance between Z_1, Z_2 is more than d.*

Proof. Let $k, \ell \geq 1$ satisfy 2.2 with d replaced by $d+3$. Let $c_k = \ell\tau$, and for $i = k-1, \ldots, 0$ define $c_i = 2c_{i+1} + 2c$. Let $c' = c_0$. We claim that c' satisfies the theorem. For let G, Z be as in the theorem. Choose $k' \leq k$ maximum such that there exist connected subgraphs $X_1, \ldots, X_{k'}$ of $G[Z]$ and a subset $A \subseteq Z$ with the following properties:

- for $1 \leq i < j \leq k'$, the G-distance between X_i, X_j is at least three;
- for $1 \leq i \leq k'$, there exists d_i with $3 \leq d_i \leq d+3$ such that every vertex in A has G-distance exactly d_i from X_i; and
- $\chi(A) > c_{k'}$.

(This is possible since setting $k' = 0$ and $A = Z$ satisfies the bulletted statements.) Since $c_{k'} > (\ell-1)\tau$ and $\chi^{2d+7}(G) \leq \tau$, there exist vertices $v_1, \ldots, v_\ell \in A$, pairwise with G-distance at least $2d+8$. Consequently, $k' < k$ by 2.2. Choose a connected component A_1 of A with maximum chromatic number, and let $z_0 \in A_1$. For $i \geq 0$ let L_i be the set of vertices in A_1 with $G[A_1]$-distance from z_0 equal to i. For $i \geq 0$ let $M_i = L_0 \cup \cdots \cup L_i$. Thus each M_i induces a connected graph. For $r \geq 0$ let M_i^r denote the set of vertices in A_1 with G-distance from M_i at most r. Thus for sufficiently large i, $M_i = A_1$; and so there exists i such that $M_i^{d+3} > 2c_{k'+1} + c$. Choose i minimum with this property.

Suppose that $\chi(M_i^2) \leq c$. Then $\chi(M_i^{d+3} \setminus M_i^2) > (2c_{k'+1} + c) - c$, and every vertex in $M_i^{d+3} \setminus M_i^2$ has G-distance from M_i at least three and at most $d+3$. For $3 \leq j \leq d+3$ let B_j be the set of vertices in $M_i^{d+3} \setminus M_i^2$ with G-distance exactly j from M_i. It follows that $\chi(B_j) > c_{k'+1}$ for some $j \in \{3, \ldots, d+3\}$. Let $X_{k'+1} = M_i$ and $d_{k'+1} = j$; then since $\chi(B_j) > c_{k'+1}$, this contradicts the maximality of k'. This proves that $\chi(M_i^2) > c$.

Now $\chi(M_i^{d+3}) > 2c_{k'+1} + c \geq \tau$, and so $i > 0$ since $\chi^{2d+7}(G) \leq \tau$. From the minimality of i it follows that $\chi(M_{i-1}^{d+3}) \leq 2c_{k'+1} + c$. Since $\chi(A_1) > c_{k'}$, it follows that $\chi(A_1 \setminus M_{i-1}^{d+3}) > c_{k'} - (2c_{k'+1} + c) = c$. But $M_i^2 \subseteq M_{i-1}^3$, so the G-distance between M_i^2 and $A_1 \setminus M_{i-1}^{d+3}$ is at least $d+1$. Since both the sets are subsets of A_1 and hence of Z, and both sets have chromatic number more than c, this proves 2.3. ∎

2.4 *For all $\nu, k, d, c, \tau \geq 0$ there exists $c' \geq 0$ with the following property. Let G be a K_ν^1-subdivision-free graph, such that $\chi^{2d+7}(G) \leq \tau$, and let $Z \subseteq V(G)$ with $\chi(Z) > c'$. Then there exist subsets $Z_1, \ldots, Z_k \subseteq Z$ such that $\chi(Z_i) > c'$ for $i = 1, \ldots, k$ and the G-distance between every two of $Z_1, \ldots, Z_k$ is more than d.*

Proof. We proceed by induction on k. Let c'' satisfy the theorem with k replaced by $k-1$; and let c' satisfy 2.3 with c replaced by c''. We claim that c' satisfies the theorem. For let G, Z be as in the theorem. By 2.3 exist subsets $Z_1, Z_2 \subseteq Z$ such that $\chi(Z_i) > c''$ for $i = 1, 2$ and the G-distance between Z_1, Z_2 is more than d. By the inductive hypothesis applied with Z replaced by Z_2, there are $k-1$ subsets $Y_1, \ldots, Y_{k-1}$ of Z_2, each with chromatic number at least c and pairwise with G-distance at least $d+1$. But then $Z_1, Y_1, \ldots, Y_{k-1}$ satisfy the theorem. ∎

3 Pineapple trees

If $X, Y \subseteq V(G)$, we say that Y *covers* X if $X \cap Y = \emptyset$ and every vertex in X has a neighbour in Y. If in addition $G[Y]$ is connected we call the pair (X, Y) a *pineapple* in G. It is a *levelled* pineapple if there exists $z_0 \in Y$ such that for some k, every vertex in Y is joined to z_0 by a path of $G[Y]$ of length less than k, and there is no path in $G[X \cup Y]$ of length less than k from z_0 to X.

Now let T be a tree, with a vertex r called its *root*. We call (T, r) a *rooted tree*. For $u, v \in V(T)$, we say v is an *ancestor* of u and u is a *descendant* of v if v belongs to the path of T between u, r. We define *parent* and *child* in the natural way. We say $u, v \in V(T)$ are *incomparable* if neither is a descendant of the other. Let $L(T)$ be the set of vertices of T with no children (thus, $L(T)$ is the set of leaves of T different from r, except when $V(T) = \{r\}$). Now let G be a graph. For each vertex $v \in V(T)$ let $C_v \subseteq V(G)$, and for each vertex $v \in V(T) \setminus L(T)$ let (X_v, Y_v) be a levelled pineapple in G with $X_v \cup Y_v = C_v$, and with the following properties:

- all the sets $C_v(v \in V(T))$ are nonempty and pairwise disjoint;
- for all incomparable $u, v \in V(T)$ there is no edge between C_u, C_v;
- if $u, v \in V(T)$ are distinct, and u is a descendant of v, then there are no edges between C_u and Y_v, and if also $u \in L(T)$ then X_v covers C_u.

(Note that we only demand that X_v covers C_u when v's descendant u is a leaf. We leave open whether there are edges between C_u and X_v when $u \in V(T) \setminus L(T)$ is a descendant of v; this will be resolved later.) We call the system

$$(T, r, ((X_v, Y_v) : v \in V(T) \setminus L(T)), (C_v : v \in L(T)))$$

a *pineapple tree* in G and (T, r) is its *shape*. Let us call the union of all the sets $C_v(v \in V(T))$ the *vertex set* of the pineapple tree. In this section we prove:

3.1 *For all $\nu, c, d, \tau \geq 0$, and every rooted tree (T, r), there exists c' with the following property. Let G be a K^1_ν-subdivision-free graph, such that $\chi^{2d+7}(G) \leq \tau$, and let $Z \subseteq V(G)$. with $\chi(Z) > c'$. Then there is a pineapple tree*

$$(T, r, ((X_v, Y_v) : v \in V(T) \setminus L(T)), (C_v : v \in L(T)))$$

in G, with vertex set a subset of Z, such that $\chi(C_v) > c$ for each $v \in L(T)$, and for all incomparable $u, v \in L(T)$, the G-distance between C_u, C_v is at least $d + 1$.

Proof. We may assume that $d \geq 1$; and we proceed by induction on $V(T)$. If $V(T) = \{r\}$, then $r \in L(T)$, and we define $C_r = V(G)$ and the theorem holds. Thus we may assume that $r \notin L(T)$, and the result holds for smaller trees. Let $r_1, \ldots, r_k$ be the children of r, and for $1 \leq i \leq k$ let T_i be the component of $T \setminus r$ containing r_i. Inductively for $i = 1, \ldots, k$, there exists c_i satisfying the theorem with T, r, c' replaced by T_i, r_i, c_i. Let c'' be the maximum of $c_1, \ldots, c_k$. Choose $c_0 \geq \tau$ such that 2.4 holds with c, c' replaced by c'', c_0. We claim that setting $c' = 2c_0$ satisfies the theorem. For let G, Z be as in the theorem. Choose a component A of $G[Z]$ with $\chi(A) = \chi(Z)$, and choose $z_0 \in A$. For $j \geq 0$ let L_j be the set of vertices in A with $G[Z]$-distance j from z_0, and choose j such that

$\chi(L_j) \geq \chi(A)/2 > c_0$. Then $j > 1$, since $\chi^{2d+7}(G) \leq \tau \leq c_0$; let $X_r = L_{j-1}$ and $Y_r = L_0 \cup \cdots \cup L_{r-2}$. Then (X_r, Y_r) is a levelled pineapple, and X_r covers L_j, and there are no edges between Y_r, L_j.

From 2.4 there exist $Z_1, \ldots, Z_k \subseteq L_j$, each with chromatic number more than c'', and pairwise at G-distance more than d. From the choice of c_i, for each i there is a pineapple tree

$$(T_i, r_i, ((X_v, Y_v) : v \in V(T_i) \setminus L(T_i)), (C_v : v \in L(T_i)))$$

in G, with vertex set a subset of Z_i, such that $\chi(C_v) > c'' \geq c_i$ for each $v \in L(T_i)$, and for all distinct $u, v \in L(T_i)$, the G-distance between C_u, C_v is at least $d+1$. But then

$$(T, r, ((X_v, Y_v) : v \in V(T) \setminus L(T)), (C_v : v \in L(T)))$$

is the required pineapple tree. ∎

4 Pruning a pineapple tree

Now we turn to the question whether there are edges between $X_u \cup Y_u$ and X_v when u is a descendant of v. Let

$$(T, r, ((X_v, Y_v) : v \in V(T) \setminus L(T)), (C_v : v \in L(T)))$$

be a pineapple tree in G, and let $d \geq 2$, such that for all incomparable $u, v \in L(T)$, the G-distance between C_u, C_v is at least $d+1$, where $C_v = X_v \cup Y_v$ for $v \in V(T) \setminus L(T)$. For each $u \in L(T)$ and each ancestor $v \in V(T) \setminus L(T)$ of u, let X_v^u be the set of vertices in X_v with a neighbour in C_u. Here are some observations about these subsets:

- For each $v \in V(T) \setminus L(T)$, if $u, u' \in L(T)$ are distinct descendants of v, then $X_v^u \cap X_v^{u'} = \emptyset$; for $d \geq 2$, so the distance between $C_u, C_{u'}$ is at least three, and so no vertex in X_v has neighbours in both sets.
- We may assume that for each $v \in V(T) \setminus L(T)$, every vertex in X_v belongs to X_v^u for some descendant $u \in L(T)$ of v; for any other vertices in X_v may be removed from X_v without violating the definition of a pineapple tree.
- For all distinct $u, v, v' \in V(T)$, if $u \in L(T)$, and v is an ancestor of u, and v' is incomparable with u, then there are no edges between X_v^u and $C_{v'}$; because the distance between $C_u, C_{v'}$ is at least three, and every vertex in X_v^u has a neighbour in C_u, and so has no neighbour in $C_{v'}$.

We say an *aligned triple* is a triple (u, v, w) such that $u \in L(T)$, w is an ancestor of u, v is an ancestor of w, and u, v, w are all different. We say that an aligned triple (u, v, w) is *pruned* if either every vertex in X_v^u has a neighbour in Y_w, or none does; and the pineapple tree is *pruned* if every aligned triple is pruned.

4.1 *For all $\nu, c, d, \tau \geq 0$, and rooted trees (T, r), there exists c' with the following property. Let G be a K_ν^1-subdivision-free graph, such that $\chi^{2d+7}(G) \leq \tau$, and let $Z \subseteq V(G)$. with $\chi(Z) > c'$. Then there is a pruned pineapple tree as in 3.1.*

Proof. Let $c'' = 2^{h^2}c$ where h is the length of the longest path in T with one end r. Let c' satisfy 3.1 with c replaced by c''. We claim that c' satisfies the theorem. For let G, Z be as in the theorem; then by 3.1 there is a pineapple tree as in 3.1, in the usual notation. Thus we may choose a pineapple tree

$$(T, r, ((X_v, Y_v) : v \in V(T) \setminus L(T)), (C_v : v \in L(T)))$$

in G, satisfying the following conditions:

- its vertex set is a subset of Z;
- for all incomparable $u, v \in L(T)$, the G-distance between C_u, C_v is at least $d + 1$, where $C_v = X_v \cup Y_v$ for $v \in V(T) \setminus L(T)$;
- for each $u \in L(T)$, let n_u be the number of pairs (v, w) such that (u, v, w) is a pruned aligned triple; then $\chi(C_u) > c'2^{-n_u}$.

Choose this tree such that in addition the sum of the numbers $n_u(u \in L(T))$ is maximum. We claim this tree is pruned. For if not, choose an aligned triple (u, v, w) that is not pruned. Let A be the set of vertices in X_u^v with a neighbour in Y_w, and $B = X_v^u \setminus A$. Every vertex in C_u has a neighbour in one of A, B, and so we can choose one of A, B, say W_v^u, such that the set of vertices in C_u with a neighbour in W_v^u, say C'_u, has chromatic number at least $\chi(C_u)/2$ and hence more than $c'2^{-n_u-1}$. But then replacing X_v^u by W_v^u and C_u by C'_u gives a new pineapple with the sum of the numbers $n_u(u \in L(T))$ larger, which is impossible. This proves that the pineapple tree is pruned.

For each $u \in L(T)$, $n(u) \leq h(h-1)/2 \leq h^2$ and so $\chi(C_u) > c'2^{-n_u} \geq c$, and so this pineapple tree satisfies the theorem. ∎

5 A Ramsey theorem for trees

Let $h \geq 0$ and $t \geq 1$, and let (T, r) be a rooted tree in which every path from r to a member of $L(T)$ has length h, and every vertex in $V(T) \setminus L(T)$ has t children (and hence degree $t + 1$, except for r). We call (T, r) a *uniform t-ary tree of height h*. We need the following.

5.1 *Let $q, h \geq 0$ and $t \geq 1$. Let (T', r) be a uniform (qt)-ary tree of height h, and let ϕ be a map from $L(T')$ to the set $\{1, \ldots, q\}$. Then there is a subtree T of T' containing r, such that (T, r) is a uniform t-ary tree of height h, and such that for some $x \in \{1, \ldots, q\}$, $\phi(u) = x$ for all $u \in L(T)$.*

Proof. We proceed by induction on h. For $h = 0$ the result is true, so we assume that $h > 0$ and the result holds for $h - 1$. Let $r_1, \ldots, r_{qt}$ be the children of r in T', and for $1 \leq i \leq qt$ let T_i be the component of $T \setminus r$ containing r_i. For $1 \leq i \leq qt$, from the inductive hypothesis there is a subtree T'_i of T_i containing r_i, such that (T'_i, r_i) is a uniform t-ary tree of height $h - 1$, and such that for some $x_i \in \{1, \ldots, q\}$, $\phi(u) = x_i$ for all $u \in L(T'_i)$. Choose $x \in \{1, \ldots, q\}$ such that $x_i = x$ for at least t values of i; then the union of t of the corresponding trees T_i, together with r, gives the desired tree T'. ∎

Let

$$(T, r, ((X_v, Y_v) : v \in V(T) \setminus L(T)), (C_v : v \in L(T)))$$

be a pineapple tree. It is *barren* if for every aligned triple (u, v, w), no member of X_v^u has a neighbour in Y_w; and it is *fruitful* if for every aligned triple (u, v, w), every member of X_v^u has a neighbour in Y_w. We need a further strengthening of 3.1, the following.

5.2 *For all $\nu, c, d, \tau \geq 0$, and every rooted tree (T, r), there exists c' with the following property. Let G be a K_ν^1-subdivision-free graph such that $\chi^{2d+7}(G) \leq \tau$, and let $Z \subseteq V(G)$. with $\chi(Z) > c'$. Then there is a pineapple tree as in 3.1 which is either barren or fruitful.*

Proof. Choose $t \geq 1$ and $h \geq 0$ such that every vertex of T has at most t children and every path of T with one end r has length at most h. Let $q = 2^{2^{2h}}$. Let (T', r') be a uniform uniform (qt)-ary tree of height 2^h, and choose c' such that 4.1 is satisfied with T replaced by T'. We claim that c' satisfies the theorem. For let G, Z be as in the theorem. By 4.1 there is a pruned pineapple tree

$$(T, r, ((X_v, Y_v) : v \in V(T) \setminus L(T)), (C_v : v \in L(T)))$$

as in 4.1. For each $u \in L(T')$, let q_u be the function with domain the set of all ordered pairs (i, j) with $0 \leq i < j < 2^h$, defined as follows. For each such pair (i, j), let $v, w \in V(T')$ be the ancestors of u with distance i and j from r respectively; let $q_u(i, j) = 0$ if no member of X_v^u has a neighbour in Y_w, and $q_u(i, j) = 1$ if every member of X_v^u has a neighbour in Y_w. (Since the pineapple tree is pruned and all the sets X_v^u are nonempty, this is well-defined.) Thus each q_u is a map into a domain with at most q elements, and so by 5.1 there is a subtree T'' of T' containing r', such that (T'', r') is a uniform t-ary tree of height h, and such that all the functions $q_u (u \in L(T''))$ are equal. Let the common value of all the $q_u (u \in L(T''))$ be a function f. Let H be the graph with vertex set $\{0, \ldots, 2^h - 1\}$ in which for $0 \leq i < j < 2^h$, i, j are adjacent if $f(i, j) = 1$. By Ramsey's theorem applied to H, there exists $I \subseteq \{0, \ldots, 2^h - 1\}$ with $|I| = h$ such that all the values $f(i, j) (i, j \in I, i < j)$ are equal. Let $I = \{i_0, \ldots, i_{h-1}\}$ where $0 \leq i_0 < \cdots < i_{h-1} < 2^h$. Choose $s \in V(T'')$ with distance i_0 from r'. Let N be the set of descendants of s in T'' whose distance from r' belongs the set $I \cup \{2^h\}$. Let S be the tree with vertex set N in which u, v are adjacent if one is a descendant in T'' of the other and no third vertex of N belongs to the path of T'' between them. Then (S, s) is a rooted tree in which every path from s to $L(S)$ has length h and every vertex in $V(S) \setminus L(S)$ has at least t children. Consequently

$$(S, s, ((X_v, Y_v) : v \in V(S) \setminus L(S)), (C_v : v \in L(S)))$$

is a pineapple tree, and it is either barren or fruitful, and since (S, s) has a rooted subtree isomorphic to (T, r), the result follows. This proves 5.2. ∎

We can eliminate barren pineapple trees, because of the following.

5.3 *For all $\nu, \kappa \geq 0$, there exists $n \geq 1$ with the following property. Let $\tau \geq 0$ and let (T, r) be a rooted tree where T is a path of length n^2 with ends r, u say. Let G be a K_ν^1-subdivision-free graph, such that $\chi^2(G) \leq \tau$ and $\omega(G) \leq \kappa$. Then there is no barren pineapple tree in G with shape (T, r), such that in the usual notation $\chi(C_u) > n\tau$.*

Proof. Say an *immersion* of K_n into G is a map ϕ, such that

- ϕ maps $V(K_n)$ injectively into $V(G)$, and
- ϕ maps each edge $e = uv$ of K_n to an induced path $\phi(e)$ of G between $\phi(u), \phi(v)$, of length at least two;
- for all distinct $e, f \in E(K_n)$, the paths $\phi(e), \phi(f)$ are vertex-disjoint except possibly for a common end;
- for all distinct $e, f \in E(K_n)$ with no common end, there is no edge of G between $\phi(e), \phi(f)$;
- for all distinct $e, f \in E(K_n)$ with a common end v say, there is at most one edge between $V(\phi(e) \setminus \phi(v))$ and $V(\phi(f) \setminus \phi(v))$ and such an edge joins the two neighbours of $\phi(v)$.

It is an easy application of Ramsey's theorem to prove that there exists n such that if G is K^1_ν-subdivision-free and $\omega(G) \le \kappa$, there is no immersion of K_n in G. (We omit the proof; see theorem 3.2 of [3] for the proof of a similar result.)

We claim that n satisfies the theorem. For let T, r, G be as in the theorem, and suppose that there is a barren pineapple tree with shape (T, r) as described in the theorem. Then $|L(T)| = 1$; let $u \in L(T)$. Let the vertices of T be $t_0\text{-}\cdots\text{-}t_{n^2}$ in order, where $t = t_0$. (Thus $u = t_{n^2}$.) For $0 \le k < n^2$ let us write X_k for X_{t_k} and Y_k for Y_{t_k} for convenience. Since $\chi(C_u) > n\tau$ and $\chi^2(G) \le \tau$, there exist n vertices $v_1, \dots, v_n$ in C_u, pairwise with G-distance at least three. For $1 \le i \le n$ and $0 \le k < n^2$, let x^i_k be a neighbour of v_i in X_k. Let H be a graph with vertex set $\{v_1, \dots, v_n\}$ in which all pairs of vertices are adjacent. Number the edges of H as $e_0, \dots, e_{m-1}$ where $m = n(n-1)/2$. Let $0 \le k < m$ and let e_k have ends v_i, v_j say where $i < j$. Let P_k be an induced path of G between v_i, v_j, consisting of the edges $v_i x^i_k$, $v_j x^j_k$ and an induced path joining x^i_k, x^j_k with interior in Y_k (this exists since Y_k covers X_k and $G[Y_k]$ is connected). Then the paths $P_0, \dots, P_{m-1}$ are pairwise vertex-disjoint except possibly for a common end. Suppose that there is an edge e of G joining $P_k, P_{k'}$ say, where $k \ne k'$, and e is not incident with a common end of $P_k, P_{k'}$. Suppose first that some end v of e is an end of one of $P_k, P_{k'}$, say of P_k; then $v = v_i$ for some i. Since $v_1, \dots, v_n$ pairwise have G-distance at least three, the other end of e is not an end of $P_{k'}$, and so belongs to $X_{k'} \cup Y_{k'}$. It cannot belong to $Y_{k'}$ since there are no edges between C_u and $Y_{k'}$, from the definition of a pineapple tree. Consequently it belongs to $X_{k'}$, and so is adjacent to an end of P'_k. This end of $P_{k'}$ must be v_i, since $v_1, \dots, v_n$ pairwise have G-distance at least three, and so v_i is a common end of $P_k, P_{k'}$, contrary to the definition of e. Thus neither end of e belongs to C_u. Hence one end is in $X_k \cup Y_k$, and the other in $X_{k'} \cup Y_{k'}$. From the symmetry we may assume that $k' > k$, and so there are no edges between Y_k and $X_{k'} \cup Y_{k'}$ from the definition of a pineapple tree. Hence one end of e is in X_k. The other end of e is not in $Y_{k'}$ since $(u, t_k, t_{k'})$ is an aligned triple, and by hypothesis no vertex in X^u_k has a neighbour in $Y_{k'}$. Hence the other end of e is in $X_{k'}$. It follows that both ends of e have a neighbour in $\{v_1, \dots, v_k\}$, and so they have a common neighbour since $v_1, \dots, v_n$ pairwise have G-distance at least three, and this common neighbour is a common end of both $P_k, P_{k'}$. Consequently the vertices $v_1, \dots, v_n$ and paths $P_0, \dots, P_{m-1}$ define an immersion of K_n in G, which is impossible. This proves 5.3. ∎

We deduce:

5.4 *For all $\nu, c, d, \tau, \kappa \ge 0$, and every rooted tree (T, r), there exists c' with the following property. Let G be a K^1_ν-subdivision-free graph, such that $\chi^{2d+7}(G) \le \tau$, and let $Z \subseteq V(G)$ with $\chi(Z) > c'$.*

Then there is a fruitful pineapple tree

$$(T, r, ((X_v, Y_v) : v \in V(T) \setminus L(T)), (C_v : v \in L(T)))$$

in G, with vertex set a subset of Z, such that

- *$\chi(C_v) > c$ for each $v \in L(T)$; and*
- *for all incomparable $u, v \in L(T)$, the G-distance between C_u, C_v is at least $d+1$, where $C_v = X_v \cup Y_v$ for $v \in V(T) \setminus L(T)$.*

Proof. By adding a path to T if necessary, we may assume that there is a rooted subtree of (T, r) which is a path of length n as in 5.3. But then the result follows from 5.2 and 5.3. ∎

6 Banana trees

Now we use the previous results to prove the first of our main theorems, 1.12 and hence 1.4. We need the following. Let (T, r) be a rooted tree. If $u, v \in V(G)$, the path of T with ends u, v is denoted by $T(u, v)$. A path in T of positive length joining some $u \in L(T)$ to some ancestor of u is called a *limb* of (T, r), and we call u its *leaf* and v its *start*. Let $\mathcal{T} = (T_q : q \in Q)$ be a family of limbs in (T, r) (not necessarily all different), and let $k \geq 1$ be an integer. We make a graph J with vertex set Q as follows. We say that distinct $q_1, q_2 \in Q$ are adjacent in J if there are at least k vertices x of T, such that x belongs to the interiors of T_{q_1} and T_{q_2}, and x is not a vertex of any $T_q (q \in Q \setminus \{q_1, q_2\})$. We call J the *k-overlap graph* of $\mathcal{T}$. It is easy to see that J must be a forest. More important for us is the converse; that

6.1 *For every forest J and every $k \geq 1$, there is a rooted tree (T, r) and a family of limbs $\mathcal{T}$ in T such that the k-overlap graph of $\mathcal{T}$ is isomorphic to J, and no two members of $\mathcal{T}$ share an end.*

We leave the (easy) proof to the reader.

Let us say a *banana* is a graph formed by the union of a nonempty set of paths each of positive length, all with the same ends (s, t say) and otherwise disjoint, and its *thickness* is the number of these paths. We call s, t the *ends* of the banana. (A banana of thickness two is just a cycle, and so its ends are not determined from the graph; so we will therefore specify its ends separately whenever we use a banana of thickness two.) By a *banana in G* we mean an induced subgraph of G that is a banana. Two bananas B_1, B_2 in G are *orthogonal* if every vertex in $V(B_1 \cap B_2)$ is an end of both bananas, and there is at most one such vertex, and every edge of G between $V(B_1)$ and $V(B_2)$ is incident with a common end of B_1, B_2. A *banana tree* is a graph obtained from a tree T by replacing each edge uv by a banana with ends u, v, such that these bananas are orthogonal. To prove 1.12, we need to prove that every multigraph obtained by fattening the edges of a tree is widespread; and part of "widespread" involves proving that for every subdivision J of such a multigraph, there is a subdivision of J which is present as an induced subgraph. But such a graph J is just a banana tree, so 1.12 can be reformulated as follows.

6.2 *For every banana tree J and $\kappa \geq 0$ there exists $\rho \geq 0$ such that every ρ-bounded ideal of J-subdivision-free graphs G with $\omega(G) \leq \kappa$ is colourable.*

Let us simplify this further, eliminating the ideal. It is equivalent to the following.

6.3 *For every banana tree J and $\kappa \geq 0$ there exists $\rho \geq 0$ such that for all $\tau \geq 0$ there exists $c \geq 0$ such that $\chi(G) \leq c$ for every J-subdivision-free graph G with $\chi^{\rho}(G) \leq \tau$ and $\omega(G) \leq \kappa$.*

Thus, ρ is permitted to depend on J, κ but not on τ or G. In fact we will prove something stronger, that setting $\rho = 2|V(J)| + 7$ works. In other words, we will prove:

6.4 *For every banana tree J with $|V(J)| = n$ and $\kappa, \tau \geq 0$ there exists $c \geq 0$ such that $\chi(G) \leq c$ for every J-subdivision-free graph G with $\chi^{2n+7}(G) \leq \tau$ and $\omega(G) \leq \kappa$.*

Proof. We may assume that $|V(J)| \geq 3$, since otherwise the result is trivial. Since J is a banana tree, it is obtained from some tree S by substituting bananas for its edges. Let (T, r) be a rooted tree such that there is a family $\mathcal{T}$ of limbs in (T, r) with n-overlap graph isomorphic to S such that no two members of $\mathcal{T}$ share an end. Choose $\nu > 0$ such that there is an induced subgraph of K^1_{ν} which is a subdivision of J; and let c' satisfy 5.4 with c, d replaced by $0, n$. We claim that setting $c = c'$ satisfies the theorem. For let G be a J-subdivision-free graph G with $\chi^{2n+7}(G) \leq \tau$ and $\omega(G) \leq \kappa$. We must show that $\chi(G) \leq c'$; for suppose not. Now G is K^1_{ν}-subdivision-free, and so by 5.4 (setting $Z = V(G)$) there is a fruitful pineapple tree

$$(T, r, ((X_v, Y_v) : v \in V(T) \setminus L(T)), (C_v : v \in L(T)))$$

in G, such that for all incomparable $u, v \in L(T)$, the G-distance between C_u, C_v is at least $n+1$, where $C_v = X_v \cup Y_v$ for $v \in V(T) \setminus L(T)$. For each $u \in L(T)$, C_u is nonempty from the definition of a pineapple tree, and we may assume that $|C_u| = 1$, $C_u = \{c_u\}$ say. Let $\mathcal{T} = (T_q : q \in Q)$, and for each $q \in Q$ let T_q have leaf u_q and start v_q; and choose $x_q \in X_{v_q}$. There is an isomorphism between $V(S)$ and the n-overlap graph of $\mathcal{T}$; let the corresponding bijection from Q onto $V(S)$ map q to s_q for each $q \in Q$. Now let $s_q s_{q'}$ be an edge of S. Then from the definition of the n-overlap graph, there are at least n vertices of T that belong to the interiors of the limbs $T_q, T_{q'}$ and do not belong to any of the other members of $\mathcal{T}$. Let W_e be a set of n such vertices, and let $w \in W_e$. Then (u_q, v_q, w) is an aligned triple, and so x_q has a neighbour in Y_w, since the pineapple tree is fruitful. Similarly $x_{q'}$ has a neighbour in Y_w, and since $G[Y_w]$ is connected, there is an induced path P^w_e between $x_q, x_{q'}$ with interior in Y_w. Since the G-distance between c_{u_q} and $c_{u_{q'}}$ is at least $n+1$, because $u_q, u_{q'}$ are incomparable, it follows that the G-distance between $x_q, x_{q'}$ is at least $n-1$, and so P^w_e has length at least $n-1$. The union of the paths P^w_e over all $w \in W_e$ is a banana with ends $x_q, x_{q'}$, say B_e, and B_e is an induced subgraph of G.

Since the vertices $w \in W_e$ do not belong to any other member of $\mathcal{T}$, it follows that for all distinct edges e, f of S, the bananas B_e, B_f are disjoint except for their ends. Since each banana has thickness n, and each of its constituent paths is of length at least $n-1$, it follows that the vertices $x_1, \ldots, x_k$ and all the bananas B_e make a subgraph H of G which has an induced subgraph that is a subdivision of J. We claim that H is itself induced. For suppose not, and let $a, b \in V(H)$ be distinct and adjacent in G and not adjacent in H. Since the vertices $x_q (q \in Q)$ pairwise have G-distance at least $n-1$ and hence are nonadjacent (since $n \geq 3$), we may assume that a belongs to the interior of some banana B_e say, and hence $a \in Y_w$ for some w.

From the definition of a pineapple tree, the only vertices of the pineapple tree with neighbours in Y_w belong to Y_w, to X_w, or to X_v for some ancestor v of w; and if a vertex in X^u_v has a neighbour in Y_w where v is an ancestor of w and $u \in L(T)$, then u is a descendant of w. Consequently the only vertices of H with neighbours in Y_w belong to $Y_w \cup \{x_q, x_{q'}\}$ where $e = s_q s_{q'}$; and so b belongs to this set, and hence to P_e^w. Since P_e^w is induced, it follows that a, b are adjacent in H, a contradiction. This proves that H is induced. Consequently there is an induced subgraph of G isomorphic to a subdivision of J, a contradiction. This proves 6.4. ∎

7 Fattening a cycle

In this section we prove 1.13. In the proof of 6.4 we made use of the overlap graph, which exploited vertices of the tree that only belonged to two of the selected limbs. For 1.13 we will again apply 3.1, but now we need to use vertices of the tree that belong to more than two limbs. This will still give us bananas, but we have less control over which pairs the bananas join. We have the following.

7.1 *Let $(T, r, ((X_v, Y_v) : v \in V(T) \setminus L(T)), (C_v : v \in L(T)))$ be a pineapple tree in G, such that*

- *$\chi(C_v) > 0$ for each $v \in L(T)$;*
- *for all incomparable $u, v \in L(T)$, the G-distance between C_u, C_v is at least 3, where $C_v = X_v \cup Y_v$ for $v \in V(T) \setminus L(T)$; and*
- *for every aligned triple (u, v, w), every vertex in X_v^u has a neighbour in Y_w.*

Let $(T_q : q \in Q)$ be a family of limbs of (T, r), let v_q be the start of $T_q (q \in Q)$, and for all distinct $q, q' \in Q$ let the G-distance between $v_q, v_{q'}$ be at least three. Let $W \subseteq V(T)$ be a subset of the interior of each $T_q (q \in Q)$, where $|W| > (n-1)k(k-1)/2$. For every partition of Q into two nonempty sets I, J, there exist $q \in I$, $q' \in J$, and a banana B in G, with ends $v_q, v_{q'}$, thickness n and interior a subset of $\bigcup_{w \in W} Y_w$, such that there is no edge between $V(B)$ and $\{v_{q''} : q'' \in Q \setminus \{q, q'\}\}$.

Proof. We see first that since for all distinct $q, q' \in Q$, the G-distance between $v_q, v_{q'}$ is at least three, it follows that the vertices $v_q (q \in Q)$ are pairwise nonadjacent and no two have a common neighbour. Let I, J be two complementary nonempty subsets of Q, and let $w \in W$. For each $i \in I$ and $j \in J$, since v_i, v_j both have neighbours in Y_w and $G[Y_w]$ is connected, there is a path between v_i, v_j with interior in Y_w. Choose i, j and the path such that this path (P_w say) is as short as possible. It follows that no other member of $\{v_1, \ldots, v_k\}$ has a neighbour in P_w, since $v_1, \ldots, v_k$ are pairwise nonadjacent and no two have a common neighbour. Let $f_w = (i, j)$ where $i < j$. Since there are only $k(k-1)/2$ possibilities for the pair (i, j) (in fact fewer, since we insists that $i \in I$ and $j \in J$), there exist at least n values of w where the pairs f_w are all the same, equal to (q, q') say. But then the union of the corresponding paths P_w makes the desired banana B. ∎

The goal of the section is to prove 1.13, which is equivalent to the following:

7.2 *Let H be a multigraph obtained from a cycle by fattening all except one of its edges. Then for every subdivision J of H and all $\kappa \geq 0$ there exists $\rho \geq 0$ such that for all $\tau \geq 0$ there exists $c \geq 0$ such that if G is J-subdivision-free and $\omega(G) \leq \kappa$ and $\chi^\rho \leq \tau$ then $\chi(G) \leq c$.*

We will prove a strengthening of 7.2, that setting $\rho = 2|V(J)| + 7$ works, that is:

7.3 *Let J be obtained from a cycle of length m by substituting bananas for all except one of its edges. Then for all $\kappa, \tau \geq 0$ there exists $c \geq 0$ such that if G is J-subdivision-free, $\omega(G) \leq \kappa$ and $\chi^{2|V(J)|+7} \leq \tau$, then $\chi(G) \leq c$.*

Proof. Let $n = \max(|V(J)|, 5)$. Let (S, r) be a uniform 2-ary tree of height m, and let (T, r) be obtained from (S, r) by replacing each edge by a path P_e of length $2n^3$. (Thus $V(S) \subseteq V(T)$.) For each vertex $z \in V(S)$, we say its *height* is the S-distance from z to a vertex in $L(T)$. Choose ν such that K_ν^1 contains J; and choose c' such that 5.4 holds, taking $d = n + 1$ and $c = 0$. We claim that setting $c = c'$ satisfies the theorem. For let G be a J-subdivision-free graph with $\omega(G) \leq \kappa$ and $\chi^{2|V(J)|+7} \leq \tau$, and suppose that $\chi(G) > c$. Since G is K_{nu}^1-subdivision-free, by 5.4 there is a fruitful pineapple tree

$$(T, r, ((X_v, Y_v) : v \in V(T) \setminus L(T)), (C_v : v \in L(T)))$$

in G, such that for all incomparable $u, v \in L(T)$, the G-distance between C_u, C_v is at least n, where $C_v = X_v \cup Y_v$ for $v \in V(T) \setminus L(T)$. For each $u \in L(T)$ choose $c_u \in C_u$, and let $x_v \in X_v^u$ be adjacent to v. Since the vertices $c_u (u \in L(T))$ pairwise have G-distance at least $n + 2$, it follows that the vertices $x_v (v \in L(T))$ pairwise have G-distance at least n. For each edge $e = z_1 z_2 \in E(S)$, where z_1 is the parent of z_2 in S, let T_e be the union of $V(P_e) \setminus \{z_1\}$ and the set of descendants of z_2 in T. Let $L_e = T_e \cap L(T)$, and let $X_e = \{x_u : u \in L_e)\}$.

Let $x_1, \ldots, x_h$ be distinct vertices of G, pairwise with G-distance at least n, and let $B_1, \ldots, B_{h-1}$ be pairwise orthogonal bananas in G, each of thickness $n - 1$, such that B_i has ends x_i, x_{i+1} for $1 \leq i \leq h - 1$ We call this sequence of bananas a *banana path* on $(x_1, \ldots, x_h)$ with *interior* the union of the interiors of $B_1, \ldots, B_{h-1}$.

(1) *Let $e = z_1 z_2 \in E(S)$, where z_1 is the parent of z_2 in S, and let h be the height of z_1. Then there exist h vertices $x_1, \ldots, x_h \in X_e$ such that there is a banana path on $(x_1, \ldots, x_h)$ with interior a subset of $\bigcup_{v \in T_e \setminus L_e} Y_v$.*

To prove this we proceed by induction on h. If $h = 1$, then the result is true since then $z_2 \in L(T)$ and x_{z_2} satisfies the requirement. Thus we may assume that $h > 1$ and the result holds for $h - 1$. Let z_3, z_4 be the children of z_2 in S, and let f, g be the edges $z_2 z_3$ and $z_2 z_4$ respectively. From the inductive hypothesis, there exist $a_1, \ldots, a_{h-1} \in X_{z_3}$ such that there is a banana path on $(a_1, \ldots, a_{h-1})$ with interior in $\bigcup_{v \in T_f \setminus L_f} Y_v$, and $b_1, \ldots, b_{h-1} \in L_{z_4}$ similarly. It follows that there are no edges between any banana of the first banana path and any banana of the second, from the definition of a pineapple tree. Moreover, each of $a_1, \ldots, a_{h-1}$ has distance at least n from each of $b_1, \ldots, b_{h-1}$. Let P_e^* denote the set of vertices of T in the interior of P_e. Now each a_i is adjacent to a member of $\bigcup_{u \in L(T)} C_u$, and the corresponding limb includes P_e^*, and the same for each b_j. Since there are $2n^3 - 1 > (n - 1)2h(2h - 1)/2$ vertices of T in the interior of P_e, 7.1 implies that there exist $a_i \in \{a_1, \ldots, a_{h-1}\}$ and $b_j \in \{b_1, \ldots, b_{h-1}\}$, and a banana B with ends a_i, b_j and thickness n and interior a subset of $\bigcup_{w \in P_e^*} Y_w$, such that no other member of $\{a_1, \ldots, a_{h-1}, b_1, \ldots, b_{h-1}\}$ has a neighbour in $V(B)$. By reversing the sequence $(a_1, \ldots, a_{h-1})$ if necessary, we may assume that $i \geq h/2$, and similarly $j \leq h/2$. Now no vertex of the interior of B belongs to or has a neighbour in $\bigcup_{v \in T_f \setminus L_f} Y_v$, and the same for $\bigcup_{v \in T_g \setminus L_g} Y_v$; and so there is a banana path on $(a_1, \ldots, a_i, b_j, \ldots, b_{h-1})$ with interior in

$$\bigcup_{v \in T_f \setminus L_f} Y_v \cup \bigcup_{v \in T_g \setminus L_g} Y_v \cup \bigcup_{v \in P_e^*} Y_v \subseteq \bigcup_{v \in T_e \setminus L_e} Y_v.$$

This proves (1).

In particular, since r has height m, from (1) applied to some edge $e = rs$ of S incident with r, there exist m vertices $x_1, \dots, x_m \in X_e$ such that there is a banana path on $(x_1, \dots, x_m)$ with interior a subset of $\bigcup_{v \in T_e \setminus L_e} Y_v$. Now choose a path between x_1, x_m with interior in Y_r such that $x_2, \dots, x_{m-1}$ have no neighbours in it (this is possible since the pineapple (X_r, Y_r) is levelled). Adding this to the banana path gives a subdivision of J. This proves 7.3. ∎

8 The fat triangle

Now we turn to the third of our theorems, 1.14. We need a lemma, as follows.

8.1 *Let $\rho \geq 4$, $\tau \geq 0$, and $n \geq 0$, let G be a graph with $\chi^\rho(G) \leq \tau$, and let $X, Z \subseteq V(G)$ be disjoint, such that X covers Z and $\chi(Z) > n\tau$. Then there exist $x_1, \dots, x_n \in X$ with the following properties:*

- *$x_1, \dots, x_n$ pairwise have distance at least ρ in G; and*
- *for all distinct $i, j \in \{1, \dots, n\}$, there is a path between x_i, x_j with interior in Z, such that no other vertex in $\{x_1, \dots, x_n\}$ has a neighbour in this path.*

Proof. By a $\{1, \dots, k\}$-*colouring* of a graph we mean a colouring using $\{1, \dots, k\}$ as the set of colours. If $X_1, X_2 \subseteq V(G)$ with $X_1 \cap X_2 = \emptyset$, and κ_i is a colouring of $G[X_i]$ for $i = 1, 2$, we say they are *compatible* if their union in the natural sense is a colouring of $G[X_1 \cup X_2]$. For each $x \in X$, the subgraph G_x induced on the set of neighbours of x in Z has chromatic number at most τ; choose some $\{1, \dots, \tau\}$-colouring κ_x of G_x for each such v. Now choose $C \subseteq Z$ and $x_1, \dots, x_k \in X$ where $0 \leq k \leq n$, with the following properties:

- $G[C]$ is connected, and $x_1, \dots, x_k$ have no neighbours in C;
- $x_1, \dots, x_k$ pairwise have G-distance at least ρ;
- no $\{1, \dots, n\tau\}$-colouring of $G[C]$ is compatible with each of the colourings $\kappa_{x_i} (1 \leq i \leq k)$; and
- subject to these conditions, C is minimal.

(This is possible, since taking C to be a component of $G[Z]$ with maximum chromatic number and $k = 0$ satisfies all bullets except the last.) If some G_{x_i} contains no vertex with a neighbour in C then we may remove x_i from the list $x_1, \dots, x_k$; so we may assume that $x_1, \dots, x_k$ each have a neighbour in Z which has a neighbour in C. Now $k \leq n$, and if $k = n$ then the theorem holds, so we may suppose for a contradiction that $k < n$. Since only colours $1, \dots, \tau$ are used by the colourings $\kappa_1, \dots, \kappa_k$, it follows that $\chi(C) > (n-1)\tau \geq k\tau$; and so there exists $v \in C$ with G-distance more than ρ from each of $x_1, \dots, x_k$. Choose $x_{k+1} \in X$ adjacent to v. Thus x_{k+1} has G-distance at least ρ from each of $x_1, \dots, x_k$. Let C' be the set of vertices in C nonadjacent to x_{k+1}. Since $\kappa_{x_{k+1}}$ is compatible with each of the colourings $\kappa_{x_i} (1 \leq i \leq k)$ (because x_{k+1} has G-distance at least four from each of $x_1, \dots, x_k$), it follows that no $\{1, \dots, n\tau\}$-colouring of $G[C']$ is compatible with each of the colourings $\kappa_{x_i} (1 \leq i \leq k+1)$. Consequently there is a component C'' of $G[C']$ such that no $\{1, \dots, n\tau\}$-colouring of $G[C'']$ is compatible with each of the colourings $\kappa_{x_i} (1 \leq i \leq k+1)$. But this contradicts the minimality of C. Hence $k = n$. This proves 8.1. ∎

We deduce 1.14, reformulating it in the same way that 6.4 is a reformulation of 1.12:

8.2 *Let H be the multigraph obtained from K_3 by fattening two of its edges and replacing the third by two parallel edges, and let J be a subdivision of H. Let $n = |V(J)|$. For all $\kappa, \tau \geq 0$ there exists $c \geq 0$ such that $\chi(G) \leq c$ for every J-subdivision-free graph G with $\chi^{2n+7}(G) \leq \tau$ and $\omega(G) \leq \kappa$.*

Proof. We may assume that $n \geq 3$. Let (T, r) be the rooted tree where T is a path of length $3n$ and r is one end of T. Let the vertices of T be $t_0, \dots, t_{3n}$ where $r = t_0$; thus $L(T) = \{t_{3n}\}$. We write $u = t_{3n}$. Choose ν such that K^1_ν contains J. Choose c' to satisfy 5.4 with $c = 3\tau$ and $d = n$; we claim that setting $c = c'$ satisfies the theorem. For let G be J-subdivision-free and hence K^1_ν-subdivision-free, with $\chi^{2n+7}(G) \leq \tau$ and $\omega(G) \leq \kappa$, and suppose that $\chi(G) > c'$, By 5.4, there is a pineapple tree

$$(T, r, ((X_v, Y_v) : v \in V(T) \setminus L(T)), (C_v : v \in L(T)))$$

in G, such that

- $\chi(C_u) > c\tau$; and
- for every aligned triple (u, v, w) every vertex in X^u_v has a neighbour in Y_w.

For $0 \leq i < 3n$ let $X_i = X_{t_i}$ and $Y_i = Y_{t_i}$. We may assume that every vertex in X_i has a neighbour in C_u, because any other vertices in X_i may be removed from X_i (thus $X_{t_i} = X^{t_{3n}}_{t_i}$ in the earlier notation). Consequently for all i, j with $0 \leq i < j < 3n$, every vertex in X_i has a neighbour in Y_j.

By 8.1, there exist $x, x', x'' \in X_0$, pairwise at G-distance at least n, such that every two of them are joined by a path with interior in C_u in which the third has no neighbours. Now for $1 \leq i < 3n$, let $R(xx')$ be the set of $i \in \{1, \dots, 3n-1\}$ such that there is a path between x, x' with interior in Y_i containing no neighbour of x''; and define (xx'') and $R(x'x'')$ similarly. It follows that since x, x', x'' all have neighbours in Y_i and $G[Y_i]$ is connected, each value of $i \in \{1, \dots, 3n-1\}$ belongs to at least two of $R(xx'), R(xx''), R(x'x'')$. Consequently there exists $I \subseteq \{1, \dots, 3n-1\}$ with $|I| = n$ such that I is a subset of one of $R(xx'), R(xx''), R(x'x'')$, say $R(xx'')$; and since $|\{1, \dots, 3n-1\} \setminus I| = 2n-1$, there exists $J \subseteq \{1, \dots, 3n-1\} \setminus I$ with $|J| = n$ such that J is a subset of one of $R(xx'), R(x'x'')$, say $R(x'x'')$. As in the proof of 6.4, there is a banana consisting of n paths with ends x, x'', one path with interior in each of the sets $Y_i (i \in I)$; and similarly there is a banana with ends x', x'' with the interiors of its paths in the sets $Y_i (i \in J)$. The union of these two bananas is induced.

To obtain a subdivision of J, we need to add to this union two paths joining x, x'; and we will obtain these, one with interior in C_u via 8.1, and one with interior in Y_0. The first is immediate from the definition of x, x', x''. For the second we use the fact that (X_0, Y_0) is a levelled pineapple. Let $z_0 \in Y_0$ such that for some k, every vertex in Y_0 is joined to z_0 by a path of $G[Y_0]$ of length less than k, and there is no path in $G[X_0 \cup Y_0]$ from z_0 to X_0 of length less than k. For $0 \leq i \leq k$, let L_i be the set of vertices in $X_0 \cup Y_0$ with $G[X_0 \cup Y_0]$-distance i from z_0. Thus $Y_0 = L_0 \cup \dots \cup L_{k-1}$ and $X_0 = Y_k$. Now x, x' both have neighbours in L_{k-1}, say y, y' respectively, and since the G-distance between x, x' is at least n, it follows that $2k \geq n$, and in particular $k > 1$. Since $G[L_0 \cup \dots \cup L_{k-2}]$ is connected and y, y' both have neighbours in it, there is an induced path between y, y' with interior in $L_0 \cup \dots \cup L_{k-2}$, and which consequently contains no neighbours of x''. Adding the edges xy and $x'y'$ to this path gives the required path from x to x'. The subgraph consisting of the two bananas and these two paths is induced, and isomorphic to a subdivision of H. Since x, x', x'' pairwise have distance at least n, all these paths between pairs of x, x', x'' have length at least n; and so this same subgraph is also isomorphic to a subdivision of J, a contradiction. This proves 8.2. ∎

9 Bigger widespread graphs

The same methods can be combined to prove that more complicated graphs are widespread. For instance, in the proof of 1.13, all the limbs we used started from the root r, and all the limbs eventually become disjoint. We are free to make the tree T bigger by adding more vertices to its leaves, and extend the old limbs further to make new limbs, and add more limbs meeting the old paths just in their new sections. By this process we can make not just one cycle as in 1.13, but any multigraph each of whose blocks is such a cycle. We omit the details.

Can we make more 2-connected widespread graphs? Here is one construction. Take a path with vertices $v_1\text{-}\cdots\text{-}v_k$ in order where $k \geq 4$, fatten each edge, and add two more vertices a, b and edges $av_1, av_2, bv_{k-1}, bv_k$ and ab, making a multigraph H. We claim:

9.1 *H is widespread.*

Proof. We merely sketch the proof, since the result is such an oddity. Let J be a subdivision of H. Let us proceed as in the proof of 7.2, with a subdivided 2-ary tree T; but apply 5.4 to this tree with c larger than zero, large enough that 8.1 can be applied. For each $u \in L(T)$, choose three vertices $x_u, y_u, z_u \in X_r^u$, such that for every two of them there is a path between them with interior in C_u containing no neighbour of the third; and choose x_u, y_u, z_u with G-distance at least $n + 2$. Now because the limb of T from r to u has a final section W consisting of many vertices w each with only one child, these vertices are incomparable with the other leaves of T, and so there are two orthogonal bananas B_u, B'_u in G, both with interior in the union of $Y_w (w \in W)$, and each with both ends in $\{x_u, y_u, z_u\}$ (and joining distinct pairs from this set). This defines a banana path of length two. Now we apply the method of 7.2; we generate longer and longer banana paths, starting from the ones we just made of length two. The procedure of 7.2 has the convenient feature that the first and last banana of every banana path it generates is a banana of one of the initial banana paths of length two. So we may assume that we generate a $k-1$-term banana path where the first banana is B_u and the last is $B_{u'}$ for some $u, u' \in L(T)$. Let B_u have ends x_u, y_u say. By choosing one path from the banana B'_u (not to be confused with $B_{u'}$), joining z_u with one of x_u, y_u, and choosing one path via 8.1 joining z_u with the other of x_u, y_u, we obtain an induced path from x_u to y_u in which z_u is an internal vertex. Now do the same thing for u', and then add a path joining $z_u, z_{u'}$ with interior in Y_r. This provides the induced subgraph which is a subdivision of J. ∎

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