EXACT BOUNDS FOR JUDICIOUS PARTITIONS OF GRAPHS

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Abstract. Edwards showed that every graph of size $m \geq 1$ has a bipartite subgraph of size at least $m/2 + \sqrt{m/8 + 1/64} - 1/8$. We show that every graph of size $m \geq 1$ has a bipartition in which the Edwards bound holds, and in addition each vertex class contains at most $m/4 + \sqrt{m/32 + 1/256} - 1/16$ edges. This is exact for complete graphs of odd order, which we show are the only extremal graphs without isolated vertices. We also give results for partitions into more than two classes.

1. Introduction

Many classical problems in graph theory demand that a certain quantity be maximized or minimized. For instance, given a graph G, the Max Cut problem asks for the largest bipartite subgraph of G. Our aim in this paper to consider problems in which *several* quantities must be minimized or maximized *simultaneously*. Problems of this type are in general more difficult, since the quantities are not usually independent. As in [2], by a *judicious partitioning problem* we mean a partitioning problem in which we require all vertex classes (or all pairs of vertex classes, or all triples, and so on) to satisfy inequalities simultaneously.[1]

For the Max Cut problem, it is easy to see by considering random partitions that every graph of size m contains a bipartite subgraph of size at least $m/2$. We can do a little better by considering partitions into two almost equal vertex classes: for instance, if $|G| = 2n$ then in a partition into two equal classes we expect to have $m/2 + m/(4n-2)$ edges between the classes, so G contains a bipartite graph with at least this many edges. Edwards [6], [7] proved the essentially best possible result that every graph of order n and size m contains a bipartite subgraph of size at least

$$\frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8}. \qquad (1)$$

[1]Work for this article was partly completed at the Institute for Advanced Study.

Recently, simpler proofs of the result of Edwards have been given by Erdős, Gyárfás and Kohayakawa [9] and Hofmeister and Lefmann [10]. (In fact, simple proofs can be read out of Lehel and Tuza [11] and Locke [12].) Alon [1] proved that there is some $c > 0$ such that if $m/2$ is a sufficiently large square then we can improve on (1) by $cm^{1/4}$, while it is never possible to improve on (1) by more that $O(m^{1/4})$. Further results on bipartite subgraphs have also been given by Erdős, Faudree, Pach and Spencer [8]. Max Cut and the more general Max k-Cut, which asks for the maximum size of a k-partite subgraph, are NP-hard, and have been the subject of vigorous investigation in both combinatorics and computer science.

Given a graph G, the Max Cut problem is equivalent to the problem of minimizing $e(V_1) + e(V_2)$ over bipartitions $V(G) = V_1 \cup V_2$. In this paper we shall study bipartitions in which we aim to control the values of $e(V_1)$ and $e(V_2)$. In particular, we are interested in the problem of minimizing $\max\{e(V_1), e(V_2)\}$, and more generally, for $k \geq 2$, of minimizing

$$\max\{e(V_1), \ldots, e(V_k)\}$$

over partitions $V(G) = \bigcup_{i=1}^{k} V_i$. Thus rather than bounding the l_1 norm of the sequence $(e(V_i))_{i=1}^{k}$, we are bounding the l_∞ norm. Random partitions give us less help here than for Max Cut: although in a random partition $V(G) = V_1 \cup V_2$ we expect each of $e(V_1)$ and $e(V_2)$ to have $m/4$ edges, bounding both quantities simultaneously is much harder. Even proving a bound such as $(1 + o(1))5m/16$, for instance, is not at all straightforward. Indeed, the possible presence of large degrees in the graph means that a simple random partitioning will not suffice (see [2]).

It was proved in [2] that every graph G has a partition into k sets, with at most

$$e(G)/\binom{k+1}{2}$$

edges contained in any set. Note that this is best possible, as seen by considering K_{k+1}. However, this is some way from the m/k^2 that random partitions would suggest. Indeed, for graphs with more edges it is possible to do much better. A deterministic partial partitioning combined with martingale methods was used in [2] to prove that for $k \geq 2$, every graph of size m has a partition into k sets, each of which contains at most

$$\frac{m}{k^2} + (3m)^{4/5}(\log k)^{2/5}$$

edges.

How much can these bounds be improved? We cannot expect to be able to do better than $m/k^2 + c_k\sqrt{m}$, since any partition of K_{kn+1} into k classes has at least

$$\begin{aligned}\binom{n+1}{2} &= \frac{1}{k^2}\binom{kn+1}{2} + \frac{k-1}{2k}n \\ &= \frac{m}{k^2} + \frac{k-1}{k^2}\sqrt{2m} + O(1)\end{aligned}$$

edges in some class.

Our aim in this paper is to show that a bound of form $m/k^2 + c_k\sqrt{m}$ can be guaranteed. Indeed, we give a bound that, surprisingly, for every value of k, is best possible for infinitely many values of m. We also show that, for any k, our bounds are exact for complete graphs of order $nk + 1$, for any positive integer n, and that these are the only extremal graphs without isolated vertices.

For bipartitions, we do more: we extend the result of Edwards by showing that there is a bipartition that satisfies both the optimal bound for $\max\{e(V_1), e(V_2)\}$ and the bound (1) of Edwards.

2. Bipartitions

We begin by proving a result for bipartitions. We shall determine the extremal graphs for this bound at the end of the section; a result for partitions into k sets is given in the next section.

Let us note that for a positive integer l, any partition of the graph K_{2l+1} must have at least $\binom{l+1}{2}$ edges in one class. Writing $m = e(K_{2l+1}) = \binom{2l+1}{2}$, we get $\binom{l+1}{2} = \frac{m}{4} + \frac{l}{4}$ and

$$\frac{l}{4} = \sqrt{\frac{m}{32} + \frac{1}{256}} - \frac{1}{16},$$

and so the bound (2) in Theorem 1 below is best possible whenever m is of form $\binom{2l+1}{2}$. It is surprising that this bound can be achieved, since we are minimizing more than one quantity simultaneously. As a bonus, we shall see that we can in addition demand that the bound (1) of Edwards is satisfied (it is equivalent to (3) below).

Theorem 1. *Let G be a graph with m edges. Then there is a partition $V(G) = V_1 \cup V_2$ with*

$$e(V_i) \le \frac{m}{4} + \sqrt{\frac{m}{32} + \frac{1}{256}} - \frac{1}{16} \qquad (2)$$

for $i = 1, 2$*, and*

$$e(V_1, V_2) \geq \frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8} \tag{3}$$

Proof. Let us consider partitions $V(G) = V_1 \cup V_2$ with $e(V_1) \geq e(V_2)$ and

$$|\Gamma(x) \cap V_2| \geq |\Gamma(x) \cap V_1| \tag{4}$$

for all $x \in V_1$, and

$$e(V_1, V_2) \geq \frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8}. \tag{5}$$

Such partitions exist: let $V(G) = U_1 \cup U_2$ be a partition of G with $e(U_1, U_2)$ maximal. We may assume $e(U_1) \geq e(U_2)$. Now $|\Gamma(x) \cap U_1| \leq |\Gamma(x) \cap U_2|$ for every $x \in U_1$, or else we could move x from U_1 to U_2 to get a partition with more edges going between the two sets, and hence (4) is satisfied. Furthermore, since $e(U_1, U_2)$ is maximal, we know from (1) that (5) is satisfied.

Let $V(G) = V_1 \cup V_2$ be a partition of $V(G)$ with $e(V_1) \geq e(V_2)$ that satisfies (4) and (5) with $e(V_1)$ minimal. If $e(V_1)$ satisfies (2) then we are done. Otherwise, suppose

$$e(V_1) = \frac{m}{4} + \alpha, \tag{6}$$

so

$$\alpha \geq \sqrt{\frac{m}{32} + \frac{1}{256}} - \frac{1}{16} \tag{7}$$

Summing (4) over all $x \in V_1$, it follows that

$$e(V_1, V_2) \geq \frac{m}{2} + 2\alpha$$

and so

$$e(V_2) = m - e(V_1) - e(V_1, V_2) \leq \frac{m}{4} - 3\alpha.$$

Now let $H = G[V_1]$ and pick $v \in V(H)$ with $d_H(v)$ minimal nonzero. Consider the partition $V(G) = W_1 \cup W_2$, where $W_1 = V_1 \setminus v$ and $W_2 = V_2 \cup \{v\}$. Clearly (W_1, W_2) satisfies (4) and $e(W_1) < e(V_1)$, so we must have either $e(W_2) > e(W_1)$ or else (W_1, W_2) does not satisfy (5). We claim that (W_1, W_2) satisfies (2) and (5).

We first prove that (W_1, W_2) satisfies (5). Since $e(W_1) = e(V_1) - \delta$, where $\delta = \delta(H) = d_H(v)$, we have

$$\begin{aligned} e(W_1, W_2) &= \sum_{x \in W_1} |\Gamma(x) \cap W_2| \\ &= \sum_{x \in W_1} |\Gamma(x) \cap V_2| + \delta \\ &\geq \sum_{x \in W_1} |\Gamma(x) \cap V_1| + \delta \\ &= (2e(V_1) - \delta) + \delta \\ &= \frac{m}{2} + 2\alpha. \qquad (8) \\ &\geq \frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{4}, \qquad (9) \end{aligned}$$

by (7). We now prove (2). It follows from (9) that

$$\begin{aligned} e(W_2) &= m - e(W_1) - e(W_1, W_2) \\ &\leq m - \left(\frac{m}{4} + \alpha - \delta\right) - \left(\frac{m}{2} + 2\alpha\right) \\ &= \frac{m}{4} - 3\alpha + \delta. \qquad (10) \end{aligned}$$

Now $e(H) \geq \binom{\delta+1}{2}$, so

$$\frac{m}{4} + \alpha \geq \binom{\delta + 1}{2}$$

and hence, rearranging, we get

$$(11) \qquad \delta \leq \sqrt{\frac{m}{2} + 2\alpha + \frac{1}{4}} - \frac{1}{2}.$$

If (W_1, W_2) does not satisfy (2) then it follows from (6) and (10) that

$$(12) \qquad \min\{\alpha, \delta - 3\alpha\} > \sqrt{\frac{m}{32} + \frac{1}{256}} - \frac{1}{16}.$$

It follows from (11) that

$$\min\{\alpha, \delta - 3\alpha\} \leq \min\left\{\alpha, \sqrt{\frac{m}{2} + 2\alpha + \frac{1}{4}} - \frac{1}{2} - 3\alpha\right\}.$$

The right hand side is maximized when

$$4\alpha + \frac{1}{2} = \sqrt{\frac{m}{2} + 2\alpha + \frac{1}{4}},$$

which gives

$$\alpha = \sqrt{\frac{m}{32} + \frac{1}{256}} - \frac{1}{16}.$$

However, this contradicts (12), and hence (2) must be satisfied. □

Let us note that we use the result of Edwards only to prove (3). If we are content with $e(V_1, V_2) \geq m/2$ then the result follows directly from the proof above without appeal to (1).

It is easy to determine the extremal graphs without isolated vertices for Theorem 1. Keeping the notation of the proof, if we can do no better than equality in (2) then we must have equality throughout the proof. In particular, (V_1, V_2) must satisfy (2) with equality and, defining (W_1, W_2) as in the proof, (10), (11) and (12) are also satisfied with equality. Thus H must be a complete graph, possibly together with some isolated vertices. Furthermore, (4) is satisfied with equality, so there are no isolated vertices in H, and every vertex in H must have exactly $|H|-1$ neighbours in V_2. Now consider (W_1, W_2): every vertex in W_1 has $|W_1| = |H| - 1$ neighbours in W_1 and $|H|$ neighbours in W_2. If any vertex v in W_2 has more neighbours in W_2 than in W_1, then moving v from W_2 to W_1 yields a partition (V_1', V_2') that satisfies the conditions of the theorem strictly, unless v is adjacent to every vertex of W_1, in which case moving any other vertex from W_1 to W_2 will do. Thus (W_2, W_1) also satisfies (4) and (5) with equality and $e(W_2) \geq e(W_1)$, so $G[W_2]$ is complete. Counting the number of edges between W_1 and W_2, we see that all edges between them must be present, so G is complete. It is easy to check that $|G|$ must be odd, so the only extremal graphs are complete graphs of odd order.

3. Partitions into k classes

In this section we shall prove results for partitions of a graph into $k \geq 2$ vertex classes. Our main aim is to prove that for integers $k, m \geq 2$, every graph of size m has a partition into k vertex classes, each of which contains at most

$$\frac{m}{k^2} + \frac{k-1}{2k^2}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right)$$

edges. Let us note that, for $n \geq 1$ any partition of K_{nk+1} into k vertex classes has at least $n+1$ vertices in some class, and

$$\binom{n+1}{2} - \frac{1}{k^2}\binom{kn+1}{2} = \frac{k-1}{2k}n.$$

Since

$$n = \frac{1}{k}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right),$$

it follows that the bound above is best possible for complete graphs of order $nk + 1$.

Our approach for $k > 2$ will be to choose one vertex class at a time. Thus we shall begin by proving a lemma about lopsided partitions.

Theorem 2. *Let G be a graph with m edges and let $0 \le p \le 1$. Then there is a partition $V(G) = V_1 \cup V_2$ with*

$$e(V_1) \le p^2 m + c(p, m) \tag{13}$$

and

$$e(V_2) \le (1-p)^2 m + c(p, m), \tag{14}$$

where

$$c(p, m) = p(1-p)\left(\sqrt{\frac{m}{2} + \frac{1}{16}} - \frac{1}{4}\right). \tag{15}$$

Proof. We may assume $0 < p < 1$. Let $q = 1 - p$. Let us consider a partition $V(G) = U_1 \cup U_2$ with $qe(U_1) + pe(U_2)$ minimal. We may assume that $e(U_1) - p^2 m \ge e(U_2) - q^2 m$, or else exchange p and q (note that $c(p)$ is symmetric in p and q). Now every $v \in U_1$ satisfies

$$q|\Gamma(v) \cap U_1| \le p|\Gamma(v) \cap U_2|, \tag{16}$$

or else we could have moved v from U_1 to U_2.

Let us now choose a partition $V(G) = V_1 \cup V_2$ that satisfies (16), that has $e(V_1) - p^2 m \ge e(V_2) - q^2 m$ and, subject to this, has $e(V_1)$ minimal. Suppose that (V_1, V_2) does not satisfy (13). Then let

$$e(V_1) = p^2 m + \alpha.$$

It follows from (16) that

$$\begin{aligned} e(V_1, V_2) &\ge \frac{2q}{p}(p^2 m + \alpha) \\ &= 2pqm + \frac{2q}{p}\alpha, \end{aligned}$$

and so

$$\begin{aligned} e(V_2) &= m - e(V_1) - e(V_1, V_2) \\ &\leq m(1 - p^2 - 2pq) - \left(1 + \frac{2q}{p}\right)\alpha \\ &= q^2 m - \frac{1+q}{p}\alpha. \end{aligned}$$

As in the proof of Theorem 1, we let $H = G[V_1]$ and pick a vertex v of minimal degree δ in H. Letting $W_1 = V_1 \setminus v$ and $W_2 = V_2 \cup \{v\}$, we obtain a partition (W_1, W_2) which satisfies (16), so has with $e(W_1) - p^2 m < e(W_2) - q^2 m$. Now

$$\begin{aligned} e(W_1, W_2) &= \sum_{x \in W_1} |\Gamma(x) \cap W_2| \\ &= \sum_{x \in W_2} |\Gamma(x) \cap V_2| + \delta \\ &\geq \frac{q}{p} \sum_{x \in W_1} |\Gamma(x) \cap V_1| + \delta \\ &= \frac{q}{p}(2e(V_1) - \delta) + \delta \\ &= \frac{2q}{p} e(V_1) + \frac{2p-1}{p}\delta \\ &= 2pqm + \frac{2q}{p}\alpha + \frac{2p-1}{p}\delta. \end{aligned}$$

If $e(W_1) > p^2 m$ then summing (16) over W_1 gives $e(W_1, W_2) \geq 2pqm$ and so $e(W_2) = m - e(W_1) - e(W_1, W_2) < (1 - p^2 - 2pq)m = q^2 m$, and thus both (13) and (14) are satisfied. If this is not the case then $e(W_1) = p^2 M + \alpha - \delta < p^2 m$. Now

$$\begin{aligned} e(W_2) &= m - e(W_1) - e(W_1, W_2) \\ &\leq m - (p^2 m + \alpha - \delta) - \left(2pqm + \frac{2q}{p}\alpha + \frac{2p-1}{p}\delta\right) \\ &= q^2 m - \frac{2-p}{p}\alpha + \frac{1-p}{p}\delta. \end{aligned}$$

If neither (V_1, V_2) nor (W_1, W_2) satisfies (13) and (14) then

$$\min\left\{\alpha, \frac{1-p}{p}\delta - \frac{2-p}{p}\alpha\right\} > pq\left(\sqrt{\frac{m}{2} + \frac{1}{16}} - \frac{1}{4}\right). \tag{17}$$

As in the proof of Theorem 1, we have

$$\delta \leq \sqrt{2p^2 m + 2\alpha + \frac{1}{4}} - \frac{1}{2}, \tag{18}$$

and so

$$\min\left\{\alpha, \frac{1-p}{p}\delta - \frac{2-p}{p}\alpha\right\}$$

is at most

$$\left\{\alpha, \frac{1-p}{p}\sqrt{2p^2 m + 2\alpha + \frac{1}{4}} - \frac{1-p}{2p} - \frac{2-p}{p}\alpha\right\}.$$

The latter expression is maximized when

$$\alpha = \frac{1-p}{p}\sqrt{2p^2 m + 2\alpha + \frac{1}{4}} - \frac{1-p}{2p} - \frac{2-p}{p}\alpha$$

and hence

$$\alpha = pq\left(\sqrt{\frac{m}{2} + \frac{1}{16}} - \frac{1}{4}\right),$$

which contradicts (17). □

It follows immediately by repeated applications of Theorem 2 that every graph G has a partition $V(G) = \bigcup_{i=1}^{k} V_i$, such that

$$e(V_i) \leq \frac{m}{k^2} + c_k\sqrt{m},$$

where c_k depends only on k. However, by being a little more careful we can do much better.

Theorem 3. *Let G be a graph with m edges. Then G has a vertex partition into k sets such that each set spans at most*

$$\frac{m}{k^2} + \frac{k-1}{2k^2}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right) \tag{19}$$

edges.

Proof. Let $m = \binom{kn+1}{2}$, where n need not be an integer. Applying Theorem 2 with $p = 1/k$, we get a partition $V(G) = V_1 \cup V_2$ with

$$e(V_1) \leq \frac{m}{k^2} + c(m,p) \tag{20}$$

and

$$e(V_2) \leq \left(\frac{k-1}{k}\right)^2 m + c(m,p), \tag{21}$$

where

$$\begin{aligned} c(m,p) &= \frac{p(1-p)}{2}\left(\sqrt{2m+\frac{1}{4}}-\frac{1}{2}\right) \\ &= \frac{k-1}{2k^2}\left(\sqrt{2m+\frac{1}{4}}-\frac{1}{2}\right). \end{aligned}$$

Now since

$$\begin{aligned} \sqrt{2m+\frac{1}{4}}-\frac{1}{2} &= \sqrt{nk(nk+1)-\frac{1}{4}}-\frac{1}{2} \\ &= \left(nk+\frac{1}{2}\right)-\frac{1}{2} \\ &= nk, \end{aligned}$$

we have

$$c(m,p) = \frac{k-1}{2k}n.$$

Thus

$$\begin{aligned} e(V_1) &\le \frac{m}{k^2}+c(m,p) \\ &= \frac{n(nk+1)}{2k}+\frac{k-1}{2k}n \\ &= \binom{n+1}{2} \end{aligned}$$

and

$$\begin{aligned} e(V_2) &\le \left(\frac{k-1}{k}\right)^2 m + c(m,p) \\ &= \frac{(k-1)^2}{2k}n(nk+1)+\frac{k-1}{2k}n \\ &= \frac{(k-1)}{2}(nk-n+1) \\ &= \binom{n(k-1)+1}{2}. \end{aligned}$$

Repeating this argument, we may divide G into k sets, each containing at most $\binom{n+1}{2}$ edges. Now

$$\begin{aligned} \binom{n+1}{2}-\frac{m}{k^2} &= \frac{n(n+1)}{2}-\frac{n(n+\frac{1}{k})}{2} \\ &= \frac{k-1}{2k}n. \end{aligned}$$

However, $2m = k^2n^2 + kn$, so

$$n = \frac{1}{k}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right)$$

and hence

$$e(V_i) \le \binom{n+1}{2} \le \frac{m}{k^2} + \frac{k-1}{2k^2}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right).$$

□

To investigate the extremal graphs for Theorem 3, it is enough to know the extremal graphs for Theorem 2. For $p = a/b$, where $(a, b) = 1$, and a positive integer n, consider K_{bn+1}. Let $m = e(K_{bn+1}) = \binom{bn+1}{2}$, so

$$n = \frac{1}{b}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right).$$

In any bipartition $V(K_{bn+1}) = V_1 \cup V_2$, either

$$\begin{aligned} e(V_1) &\ge \binom{an+1}{2} \\ &= p^2\binom{bn+1}{2} + \frac{an}{2}\frac{b-a}{b} \\ &= p^2m + p(1-p)\left(\sqrt{\frac{m}{2} + \frac{1}{16}} - \frac{1}{4}\right) \end{aligned}$$

or

$$\begin{aligned} e(V_2) &\ge \binom{(b-a)n+1}{2} \\ &= (1-p)^2\binom{bn+1}{2} + \frac{(b-a)n}{2}\frac{a}{b} \\ &= (1-p)^2m + p(1-p)\left(\sqrt{\frac{m}{2} + \frac{1}{16}} - \frac{1}{4}\right) \end{aligned}$$

Thus Theorem 2 is exact for graphs of order $bn+1$, and an easy calculation shows that these are the only complete graphs for which Theorem 2 is exact.

Now suppose that Theorem 2 is exact for G. Using the notation of the proof of Theorem 2, we see that we must have equality throughout. In particular, it follows from (18) that $H = G[V_1]$ must be a complete graph. Furthermore, (16) must be satisfied with equality. Hence p must be rational, say $p = a/b$ with $(a, b) = 1$. Now (W_1, W_2) must satisfy (14) with equality. If any vertex v in W_2 is not adjacent to every vertex in W_1 then moving v to W_1 yields a bipartition satisfying both (13) and (14) strictly. Thus the bipartite graph between W_1 and W_2 is complete. Since $G[W_1]$ is also complete, a simple calculation shows that $G[W_2]$ must also be complete and so G must be complete.

We have shown that the extremal graphs for Theorem 2 are exactly complete graphs of order $nb + 1$ with $p = a/b$, where a, b and n are positive integers. For Theorem 3, our proof yields equality in (19) exactly when we have equality at each stage of the partitioning process. It follows that we have equality iff $G = K_{kn+1}$ for some positive integer n.

4. Conclusion

We have given bounds that are best possible for graphs with $m = \binom{rk+1}{2}$ edges, where k is the number of sets in our partitions and r is an arbitrary positive integer. However, it should be possible to do a bit better when m is not of this form. The situation here is similar to that of Max Cut: the bound (1) of Edwards is exact for infinitely many m, but Alon [1] has shown that for certain values it can be improved by $cm^{1/4}$. It would be very interesting to know the best possible improvement of (2) for every value of m. Natural lower bounds can be obtained by considering unions of complete graphs.

The approach we have used here might also be useful for similar problems on hypergraphs. It was proved in [3] that every 3-uniform hypergraph with m edges has a partition into k sets such that no set contains more than

$$\frac{m}{k^3} + 5m^{6/7}(\log k)^{1/2}$$

edges. It seems likely that a stronger result should hold. However, we have not yet been able to gain much improvement using our methods. A lower bound $m/k^3 + O(m^{1/3})$ is given by considering complete 3-uniform hypergraphs.

In general, judicious partitioning problems are much harder for hypergraphs than for graphs, and for $k > 3$ very little is known. Thus even partial results are of interest: for instance it would be of great interest to determine whether every r-uniform hypergraph with m edges has a partition into k sets, each of which contains

$$\frac{m}{k^r} + o(m)$$

edges. The best bound known so far is proved in [5], where it is shown that every r-uniform hypergraph with m edges has a partition into k

sets, each of which contains at most

$$\frac{r^2}{8\log r}\frac{m}{k^r}+c_r m^{2r/(2r+1)}$$

edges.

5. Addendum

Some time after submitting this paper, we discovered that some previous work had been done on this problem. T. D. Porter [13] showed that every graph with $m \geq 1$ edges has a bipartition in which each class contains at most $m/4+\sqrt{m/8}$ edges. More recently, Porter [14] showed that if k is a power of 2 then every graph G with $m \geq 1$ edges has a partition into k sets $V_1, \ldots, V_k$, each containing at most $m/k^2+\sqrt{m/k}$ edges, such that $\sum_{i=1}^{k} e(V_i) \leq m/k$. Porter [15] has also shown that, for $k \geq 2$, every graph with $m \geq 1$ edges has a partition into k sets with at most $m/k^2+k\sqrt{m}$ edges contained in each set. Porter and Bin Yang [16] show that every graph with $m \geq 1$ edges has a bipartition in which each class contains at most $m/4+\sqrt{m/18}$ edges, and for k a power of 2 a partition into k sets, each of which contains at most $m/k^2+\sqrt{2m}/k$ edges.

Stronger results follow immediately from our Theorems 1 and 2. In particular, note that for k a power of 2, we can find a partition of a graph G into k sets, each of which satisfies (19), by repeated application of Theorem 1: given a partition into 2^s sets, we bipartition each set using Theorem 1. A straightforward calculation shows each set in a partition obtained in this way satisfies (19); furthermore, it follows from (3) that there are at most m/k edges with both ends in the same vertex class.

Finally, we note that Shahrokhi and Szekely [17] showed that the problem of finding a judicious bipartition is NP-hard.

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