Consecutive holes

Alex Scott
Oxford University, Oxford, UK

Paul Seymour[1]
Princeton University, Princeton, NJ 08544, USA

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Abstract

A *hole* in a graph is an induced subgraph which is a cycle of length at least four. We prove that for all $\nu > 0$, every triangle-free graph with sufficiently large chromatic number contains holes of ν consecutive lengths. In particular, this implies two well-known conjectures of Gyárfás [3], namely that every triangle-free graph with sufficiently large chromatic number has a hole of length at least k, and every such graph has an odd hole of length at least k. It was not known until now that every such graph has a hole of length > 7.

1 Introduction

All graphs in this paper are finite and without loops or parallel edges. A *hole* in a graph is an induced subgraph which is a cycle of length at least four, and a hole is *odd* if its length is odd. A *triangle* in G is a three-vertex complete subgraph, and a graph is *triangle-free* if it has no triangle. In this paper we are concerned with the chromatic number of triangle-free graphs that have no holes of certain specified lengths.

What can we say about the hole lengths in triangle-free graphs with large chromatic number? There are two well-known conjectures of Gyárfás [3], the second stronger than the first, the following.

1.1 Conjecture: *For all k, there exists n such that if G is triangle-free with chromatic number at least n, then*

- *G has a hole of length at least k*
- *G has an odd hole of length at least k.*

(Both these conjectures were formulated by Gyárfás [3] in greater generality, as statements about graphs with bounded clique number instead of just triangle-free graphs, but we have nothing to say about graphs with clique number more than two.) Until now, even the weaker first conjecture was not known for $k = 8$; but in this paper we prove both these conjectures, for all k.

We proved recently [4] that every graph with bounded clique size and sufficiently large chromatic number contains an odd hole, but for triangle-free graphs this is trivial. Let G be a triangle-free graph with sufficiently large chromatic number. All that was already known seems to be:

- G contains an odd hole of length at least seven, proved by Chudnovsky and the authors [2].
- G contains a hole of length a multiple of three, proved by Bonamy, Charbit and Thomassé [1].

Let us say a set F of integers is *constricting* if there exists n such that every triangle-free graph with chromatic number at least n contains a hole with length in F. Which sets are constricting? Certainly every constricting set is infinite, because there are graphs with arbitrarily large chromatic number and arbitrarily large girth.

Here is basically the only source of counterexamples that we know. Let G_1 be the null graph; for each $i > 1$, let G_i be a triangle-free graph with girth at least $2^{|V(G_{i-1})|}$ and chromatic number at least i; and let F be the set of all cycle lengths that do not occur in any G_i. Then F is not constricting, and yet F has upper density 1. This shows that not every infinite set is constricting, not even sets with upper density one.

Lower density seems to be closer to the truth. As far as we know, a set is constricting if and only if it has strictly positive lower density, but we are far from proving the implication in either direction. A more approachable question is: suppose that F contains at least one out of every ν consecutive integers; are all such sets F constricting? We prove a strengthening, the following:

1.2 *For all integers $\nu > 0$ there exists n such that if G is triangle-free with chromatic number at least n, then for some t, G has a hole of length $t + i$ for $1 \le i \le \nu$.*

This implies the conjectures of 1.1, and also the result of [1].

2 Chromatic number and radius

The proof breaks into three cases, depending on the chromatic number of the subgraphs within a fixed distance of a vertex, so next let us describe that more exactly. If $X \subseteq V(G)$, the subgraph of G induced on X is denoted by $G[X]$, and we often write $\chi(X)$ for $\chi(G[X])$. The *distance* (denoted by $d_G(u,v)$ or $d(u,v)$) between two vertices u,v of G is the length of a shortest path between u,v, or ∞ if there is no such path. If $v \in V(G)$ and $\rho \geq 0$ is an integer, $N_G^\rho(v)$ or $N^\rho(v)$ denotes the set of all vertices u with distance exactly ρ from v, and $N_G^\rho[v]$ or $N^\rho[v]$ denotes the set of all v with distance at most ρ from v.

Since we are only concerned with triangle-free graphs, it follows that $\chi(N^1(v)) \leq 1$, but there may be vertices v such that $\chi(N_G^2[v])$ is large, and such vertices cause difficulties. If we can find an induced subgraph H with large chromatic number such that $\chi(N_H^2[v])$ is bounded for every vertex v, then we might as well replace G by H. If we cannot find such a subgraph, then we will prove that for all $l \geq 4$, G has a hole of length l (if its chromatic number is large enough in terms of l).

Next we assume $\chi(N_G^2[v])$ is bounded for every vertex v. If there is an induced subgraph H with large chromatic number in which $\chi(N_H^3[v])$ is bounded for every v, we might as well pass to that; and if not, we prove that G contains holes of any fixed length (except very short ones) if $\chi(G)$ is large enough. And the same for $\chi(N_G^\rho[v])$ for all bounded ρ.

Finally, we assume $\chi(N_G^\rho[v])$ is bounded for every vertex v, for some appropriately large ρ. (We need ρ to be exponentially large in terms of ν.) In that case we prove that G contains holes of ν consecutive lengths (but the smallest of them might be arbitrarily large).

Let us say this more precisely. Let $\nu \geq 0$; a *hole ν-sequence* in a graph G is a sequence $C_1, \ldots, C_\nu$ of holes in G, such that $|E(C_{i+1})| = |E(C_i)|+1$ for $1 \leq i < \nu$ (thus, ν holes with consecutive lengths). Let $\mathbb{N}$ denote the set of nonnegative integers, and let $\phi : \mathbb{N} \to \mathbb{N}$ be a nondecreasing function. For $\rho \geq 1$, let us say a graph G is *(ρ, ϕ)-controlled* if for every nonnull induced subgraph H of G, there is a vertex v of H such that $\chi(H) \leq \phi(\chi(N_H^\rho[v]))$.

We will show the following three statements:

2.1 *Let $\nu \geq 2$; then there exist $\rho > 0$ and a nondecreasing function ϕ with the following property. If G is a triangle-free graph then either G is (ρ, ϕ)-controlled or G admits a hole ν-sequence.*

2.2 *Let $\rho > 2$ and $l \geq \rho(2\rho + 5)$ be integers. For every nondecreasing function $\phi : \mathbb{N} \to \mathbb{N}$ there is a nondecreasing function ϕ' with the following property. Let G be a (ρ, ϕ)-controlled triangle-free graph. Then either G is $(2, \phi')$-controlled or G has a hole of length l.*

2.3 *Let $\phi : \mathbb{N} \to \mathbb{N}$ be a nondecreasing function; then for all $l \geq 4$ there exists n such that every $(2, \phi)$-controlled triangle-free graph with chromatic number more than n has a hole of length l.*

2.3 is easy for $l \leq 6$, and in another paper [2] (with Maria Chudnovsky) we proved it for $l = 7$, in the expectation that that would be the easiest of the open cases. By a happy coincidence, $l = 7$ turns out to be the one case that is not handled by the proof method of the present paper.

Let us see that these three together imply 1.2.

Proof of 1.2, assuming 2.1, 2.2, 2.3. Let $\nu \geq 2$, and let ρ and ϕ be as in 2.1. Let $l_i = \rho(2\rho + 5)$, and for $i = 1, \ldots, \nu - 1$ let $l_i = l_0 + i$. By 2.2, for each $i \in \{0, \ldots, \nu - 1\}$ there is a function ϕ' as in 2.2 (with l replaced by l_i); define $\phi_i = \phi'$. Thus $\phi_0, \ldots, \phi_{\nu-1}$ are all non-decreasing functions; define

$$\psi(\kappa) = \max(\phi_0(\kappa), \ldots, \phi_{\nu-1}(\kappa))$$

for $\kappa \geq 0$. Thus ψ is nondecreasing. Now by 2.3 (with ϕ replaced by ψ) for $l = 4, \ldots, \nu - 3$ there exists n as in 2.3; let $n_l = n$. Let $n = \max(n_4, \ldots, n_{\nu+3})$.

We claim that every triangle-free graph with chromatic number more than n admits a hole ν-sequence. For let G be such a graph, and suppose it admits no hole ν-sequence. From the choice of ρ and ϕ, it follows that G is (ρ, ϕ)-controlled. For some $i \in \{0, \ldots, \nu - 1\}$, G has no hole of length l_i; so from the choice of ϕ_i, G is $(2, \phi_i)$-controlled and hence $(2, \psi)$-controlled. For some $l \in \{4, \ldots, \nu + 3\}$, G has no hole of length l; and so from the choice of n_l, $\chi(G) \leq n_l \leq n$. This proves 1.2. ▮

The three statements 2.1, 2.2, 2.3 will be proved in separate parts of the paper, and in reverse order. What we are proving is a considerable strengthening of 1.1, and we expect it would be of interest if we show how to prove 1.1 alone. This is much easier, and does not need anything after section 3 of this paper. We sketch a proof of it at the end of section 3.

3 Radius 2

In this section we prove 2.3. We begin with the following:

3.1 *Let $\phi : \mathbb{N} \to \mathbb{N}$ be nondecreasing, and let G be triangle-free and $(2, \phi)$-controlled.*

- *If $\chi(G) > \phi(2)$ then G has a 5-hole.*
- *If $\chi(G) > \phi(3)$ then G has a 6-hole.*
- *If $\chi(G) > \phi(\phi(\phi(2\phi(2) + 2) + 1) + 1)$ then G has a 7-hole.*
- *If G has no 4-hole, then for all $l \geq 5$, if $\chi(G) > \phi(l - 3)$ then G has an l-hole.*

Proof. The first statement was proved in [2], but we repeat the proof because it is easy. Suppose that $\chi(G) > \phi(2)$, and let v be a vertex such that $\chi(G) \leq \phi(\chi(N^2[v]))$. It follows that $\chi(N^2[v]) > 2$, and so there are two adjacent vertices $x, y \in N^2(v)$. Since G is triangle-free, x, y, v, together with two vertices of $N^1(v)$ adjacent to x, y respectively, form a 5-hole.

For the second statement, let $\chi(G) > \phi(3)$, and choose a vertex v such that $\chi(G) \leq \phi(\chi(N^2[v]))$. It follows that $\chi(N^2[v]) > 3$, and so $\chi(N^2(v)) > 2$; and hence there is an odd hole P in $G[N^2(v)]$. Let P have vertices p_1-p_2-$\cdots$-p_n-p_1 in order, where $n \geq 5$. Choose $S \subseteq N^1(v)$ minimal such that every vertex in $V(P)$ has a neighbour in S. Let $s_i \in S$ be adjacent to p_i for $1 \leq i \leq n$. For each $s \in S$, some vertex in P is adjacent to s and to no other vertex in S, from the minimality of S. Consequently we may assume that p_3 is adjacent to $s_3 \in S$ and has no other neighbour in S. If p_1 is nonadjacent to s_3 then v-s_1-p_1-p_2-p_3-s_3 is a 6-hole as required, so we may assume that p_1 is adjacent to s_3, and similarly p_5 is adjacent to s_3. If s_2, s_4 are nonadjacent to p_4, p_2 respectively then v-s_2-p_2-p_3-p_4-s_4-v is a 6-hole, so we may assume that one of s_2, s_4 is adjacent both of p_2, p_4, say s_2. But then s_3-p_1-p_2-s_2-p_4-p_5-s_3 is a 6-hole.

The third statement is proved in [2].

For the fourth statement, assume that G has no 4-hole, and let $l \geq 5$ be an integer. Assume that $\chi(G) > \phi(l-3)$, and choose a vertex v such that $\chi(G) \leq \phi(\chi(N^2[v]))$. It follows that $\chi(N^2(v)) > l-4$. Let $v_1, \ldots, v_t$ be the neighbours of v, and for $1 \leq i \leq t$ let A_i be the set of vertices in $N^2(v)$ adjacent to v_i. Since G has girth at least five, it follows that $A_1, \ldots, A_t$ are disjoint and form a partition of

$N^2(v)$. Choose $X \subseteq N^2(v)$ minimal such that $\chi(X) > l-4$; and it follows that every vertex of $G[X]$ has degree at least $l-4$ in $G[X]$. Choose an integer $n \geq 1$ maximum such that $n \leq l-3$ and there exist distinct $x_1, \dots, x_n \in X$ with the following properties:

- $x_1, \dots, x_n$ are the vertices in order of an induced path of $G[X]$
- $x_1, \dots, x_n$ all belong to distinct sets $A_1, \dots, A_t$.

We may renumber $v_1, \dots, v_t$ and $A_1, \dots, A_t$ such that $x_i \in A_i$ for $1 \leq i \leq n$. Certainly $n \geq 1$; suppose that $n < l-3$. Now x_n has at most one neighbour in A_i for $1 \leq i \leq n-1$, since G has no 4-hole; and since A_n is stable, and x_n has degree at least $l-4$ in $G[X]$, it follows that x_n has a neighbour in $X \cap A_j$ for some $j > n$, contrary to the maximality of n. Thus $n = l-3$. Since $l \geq 5$, v-v_1-x_1-$\cdots$-x_n-v_n-v is an l-hole as required. This proves 3.1. ∎

Let $X \subseteq V(G)$. A *t-trellis on X in G* is a subgraph H of G with the following properties.

- $X \subseteq V(H)$, and $V(H) \setminus X$ consists of the disjoint union of four sets $\{a_1, , \dots, a_t\}$, $\{b_1, \dots, b_t\}$, $\{a_{x,j} : x \in X, 1 \leq j \leq t\}$ and $\{b_{x,j} : x \in X, 1 \leq j \leq t\}$.
- The edges of H are as follows:
  - $a_j b_j$ for $1 \leq j \leq t$;
  - $x a_{x,j}$ and $x b_{x,j}$ for $x \in X$ and $1 \leq j \leq t$; and
  - $a_{x,j} a_j$ and $b_{x,j} b_j$ for $x \in X$ and $1 \leq j \leq t$.

  (Thus, to construct H we start with $K_{s,2t}$, with bipartition X and Y say, where $|X| = s$; subdivide all its edges; and then add a matching pairing up the vertices in Y.)
- For all distinct $u, v \in V(H)$, if u, v are adjacent in G and nonadjacent in H then there exist $x, x' \in X$ and $j \in \{1, \dots, t\}$ such that $\{u, v\} = \{a_{x,j}, b_{x',j}\}$. (In particular, X is stable.)

We also need a modification of this. An *extended t-trellis on X in G* is a subgraph H of G with the following properties.

- $X \subseteq V(H)$, and $V(H) \setminus X$ consists of the disjoint union of four sets $\{a_0, a_1, , \dots, a_t\}$, $\{b_0, b_1, \dots, b_t\}$, $\{a_{x,j} : x \in X, 0 \leq j \leq t\}$ and $\{b_{x,j} : x \in X, 0 \leq j \leq t\}$, together with one more vertex c_0.
- The edges of H are as follows:
  - $a_0 c_0$ and $c_0 b_0$;
  - $a_j b_j$ for $1 \leq j \leq t$;
  - $x a_{x,j}$ and $x b_{x,j}$ for $x \in X$ and $0 \leq j \leq t$; and
  - $a_{x,j} a_j$ and $b_{x,j} b_j$ for $x \in X$ and $0 \leq j \leq t$.
- For all distinct $u, v \in V(H)$, if u, v are adjacent in G and nonadjacent in H then there exist $x, x' \in X$ and $j \in \{0, \dots, t\}$ such that $\{u, v\} = \{a_{x,j}, b_{x',j}\}$.

We need both these definitions; we will show that certain graphs contain extended trellises, and to do so we first show they contain trellises, and then find the extension.

3.2 *For every integer $l \geq 8$, there exists $t \geq 0$ with the following property. Let G be a graph, let $X \subseteq V(G)$ with $|X| = t$, and let H be an extended t-trellis on X. Then G has an l-hole.*

Proof. By Ramsey's theorem, there exists $t \geq 0$ such that if $\mathcal{A}$ is the set of all triples (i, i', j) with $1 \leq i < i' \leq t$ and $1 \leq j \leq t$, and we partition $\mathcal{A}$ into two subsets $\mathcal{A}_1$, $\mathcal{A}_2$, then there exist $R, S \subseteq \{1, \ldots, n\}$ with $|R|, |S| \geq l$, such that the triples (i, i', j) with $i < i' \in R$ and $j \in S$ either all belong to $\mathcal{A}_1$ or all belong to $\mathcal{A}_2$. We claim that n satisfies the theorem.

For let G, X, H be as in the theorem. Let $X = \{x_1, \ldots, x_t\}$, and let $\mathcal{A}_1$ be the set of all triples (i, i', j) with $1 \leq i < i' \leq t$ and $1 \leq j \leq t$ such that $a_{i,j}, b_{i',j}$ are nonadjacent, and let $\mathcal{A}_2$ be the set of all such triples such that $a_{i,j}, b_{i',j}$ are adjacent. From the choice of t, we may assume that for some $k \in \{1, 2\}$, $(i, i', j) \in \mathcal{A}_k$ for all i, i', j with $1 \leq i < i' \leq l$ and $1 \leq j \leq l$.

For $1 \leq i < l$ let P_i be the path x_i-$a_{i,i+1}$-a_{i+1}-$a_{i+1,i+1}$-x_{i+1}. If $k = 1$ let Q_i be the path x_i-$a_{i,i+1}$-a_{i+1}-b_{i+1}-$b_{i+1,i+1}$-x_{i+1}, and if $k = 2$ let Q_i be the path x_i-$a_{i,i+1}$-$b_{i+1,i+1}$-x_{i+1}. Thus P_i has length four, and Q_i has length five if $k = 1$, and three if $k = 2$.

Suppose that l is a multiple of four, say $l = 4p$. Then the union of $P_1, \ldots, P_{p-1}$ and the path x_1-$a_{1,1}$-a_1-$a_{p,1}$-x_p is a hole of length l as required. Thus we may assume that l is not a multiple of four.

If $k = 2$, choose integers $p, q \geq 0$ such that $l = 4p+3q$ and $q > 0$; then the union of Q_i $(1 \leq i < q)$, P_i $(q \leq i < p + q)$, and x_1-$a_{1,1}$-$b_{p+q,1}$-x_{p+q} is the desired hole.

Thus we may assume that $k = 1$. If $l \neq 11$, then l can be expressed as $4p + 5q$ where p, q are nonnegative integers and $q > 0$; and the union of Q_i $(1 \leq i < q)$, P_i $(q \leq i < p + q)$, and x_1-$a_{1,1}$-a_1-b_1-$b_{p+q,1}$-x_{p+q} is the desired hole.

Finally we may assume that $l = 11$. If $a_{1,0}, b_{2,0}$ are nonadjacent then the union of Q_1 and x_1-$a_{1,0}$-a_0-c_0-b_0-$b_{2,0}$-x_2 is the desired hole; while if $a_{1,0}, b_{2,0}$ are adjacent then the union of P_2, x_1-$a_{1,0}$-$b_{2,0}$-x_2, and x_1-$a_{1,3}$-a_3-$a_{3,3}$-x_3 is the desired hole. This proves 3.2. ∎

We also need another definition. Let $x \in V(G)$, let N be some set of neighbours of x, and let $C \subseteq V(G)$ be disjoint from $N \cup \{x\}$, such that every vertex in C is nonadjacent to x and has a neighbour in N. In this situation we call (x, N) a *cover* of C in G. For $C, X \subseteq V(G)$, a *multicover of* C in G is a family $(N_x : x \in X)$ such that

- for each $x \in X, (x, N_x)$ is a cover of C;
- for all distinct $x, x' \in X$, x' has no neighbour in $\{x\} \cup N_x$ (and in particular all the sets $\{x\} \cup N_x$ are pairwise disjoint).

If in addition we have

- for all distinct $x, x' \in X$, no vertex in $N_{x'}$ has a neighbour in N_x,

we call $(N_x : x \in X)$ a *stable multicover.*

3.3 *For all $t, \kappa \geq 0$, there exist $\tau, m \geq 0$ with the following property. Let G be a triangle-free graph such that every induced subgraph of G with chromatic number more than κ has a 5-hole. Let $C \subseteq V(G)$ with chromatic number more than τ; and let $(N_x : x \in X)$ be a multicover of C with $|X| \geq m$. Then there exists $Y \subseteq X$ with $|Y| = t$ and an extended t-trellis on Y in G.*

Proof. For $0 \le s \le t$ let $m'_s = 5t \cdot 5^{t-s}$, and let $m' = m'_0$. For $0 \le s \le t$ let $m_s = 5t(20m')^{t-s}$, and let $m = m_0$. Let $\tau_t = \tau'_t = \kappa + 1$, and for $s = t-1, \dots, 0$ let

$$\tau_s = 5(m_s + 1) + m_s^{m'+1} 5^{m_s} \tau' + 2^{m_s} 5^{m_s} \tau_{s+1},$$

and

$$\tau'_s = 5(m'_s + 1) + 5^{m'_s} \tau'_{s+1}.$$

Let $\tau = \tau_t$. We claim that τ, m satisfy the theorem. Let G be a triangle-free graph such that every induced subgraph of G with chromatic number more than κ has a 5-hole. We shall prove the following, which implies the theorem:

(1) *Let $C \subseteq V(G)$ and let $(N_x : x \in X)$ be a multicover of C, such that either*

- *$\chi(C) > \tau$ and $|X| = m$, or*
- *$\chi(C) > \tau'$ and $|X| = m'$ and $(N_x : x \in X)$ is stable.*

Then there exists $Y \subseteq X$ with $|Y| = t$ and an extended t-trellis on Y in G.

If $X' \subseteq X$, and $N'_x \subseteq N_x$ for each $x \in X'$, and $C' \subseteq C$, and every vertex in C' has a neighbour in N'_x for each $x \in X'$, then $(N'_x : x \in X')$ is a multicover of C', and we say it is *contained in* $(N_x : x \in X)$. Consequently, to prove (1), we may assume that:

(2) *Either*

(Case 1) *$\chi(C) > \tau$ and $|X| \ge m$ and there do not exist $C' \subseteq C$ with $\chi(C') > \tau'$ and $X' \subseteq X$ with $|X'| \ge m'$ and a stable multicover $(N'_x : x \in X')$ of C' contained in $(N_x : x \in X)$, or*

(Case 2) *$\chi(C) > \tau'$ and $|X| \ge m'$ and $(N_x : x \in X)$ is stable.*

Now we construct a t-trellis on a subset of X as follows. We begin with the 0-trellis on X, H_0 say, and let $C_0 = C$. Inductively, suppose that $s < t$, and we have constructed an s-trellis H_s on a subset $X_s \subseteq X$, with vertex set the disjoint union of X_s, $\{a_1, , \dots, a_s\}$, $\{b_1, \dots, b_s\}$, $\{a_{x,j} : x \in X_s, 1 \le j \le s\}$ and $\{b_{x,j} : x \in X_s, 1 \le j \le s\}$ in the usual notation, satisfying:

- $a_{x,j}, b_{x,j} \in N_x$ for each $x \in X_s$ and $1 \le j \le s$;
- $a_j, b_j \in C$ for $1 \le j \le s$; and
- in case 1, $|X_s| = m_s$, and in case 2, $|X_s| = m'_s$.

Moreover, assume that there is a subset $C_s \subseteq C$, satisfying

- no vertex in $V(H_s)$ has a neighbour in C_s;
- for each $v \in C_s$ and each $x \in X_s$, there is a neighbour of v in N_x that has no neighbour in $V(H_s)$ except x; and
- in case 1, $\chi(C_s) > \tau_s$, and in case 2, $\chi(C_s) > \tau'_s$.

For each $x \in X_s$, let N'_x be the set of vertices in N_x with no neighbour in $V(H_s)$ except x. Then $(N'_x : x \in X_s)$ is a multicover of C_s, and is stable in case 2.

Since $\chi(C_s) > \tau'_s \geq \kappa$, there is a 5-hole P in $G[C_s]$, with vertices p_1-p_2-$\cdots$-p_5-p_1 say, in order. For each $x \in X_s$, and $1 \leq i \leq 5$, let $D_i(x)$ be the set of vertices in N'_x adjacent to p_i, and select $d_i(x) \in D_i(x)$. Thus the union of $V(P)$ and $\{d_i(x) : 1 \leq i \leq 5, x \in X_s\}$ has cardinality at most $5(|X_s|+1)$, and since G is triangle-free, there exists $C_s^1 \subseteq C_s$ with $\chi(C_s^1) \geq \chi(C_s) - 5(|X_s|+1)$, such that no vertex in C_s^1 is adjacent to any of the vertices $d_i(x)$ or to any vertex in P (and in particular, $C_s^1 \cup V(P) = \emptyset$).

For each $x \in X_s$, no vertex is in more than two of $D_1(x), \ldots, D_5(x)$, because G is triangle-free. For each $v \in C_s^1 \setminus V(P)$ and $x \in X_s$, since v has a neighbour in N'_x, it follows that there exist adjacent vertices p_k, p_{k+1} of P such that some neighbour of v belongs to $N'_x \setminus (D_k(x) \cup D_{k+1}(x))$ (reading subscripts modulo 5); choose some such k and define $c_x(v) = k$. There are $5^{|X_s|}$ possibilities for the X_s-tuple $(c_x(v) : x \in X)$, and so there exists $C_s^2 \subseteq C_s^1$ with $\chi(C_s^2) \geq \chi(C_s^1)/5^{|X_s|}$, such that $c_x(v) = c_x(v')$ for all $x \in X_s$ and all $v, v' \in C_s^2$. Moreover, since there are only five possibilities for $c_x(v)$, there exists $k \in \{1, \ldots, 5\}$ and $Y_s \subseteq X_s$ with $|Y_s| = |X_s|/5$ such that $c_x(v) = k$ for all $x \in Y_s$ and $v \in C_s^2$. Thus $\chi(C_s^2) \geq (\chi(C_s) - 5(|X_s|+1))/5^{|X_s|}$, and so in case 1

$$\chi(C_s^2) > (\tau_s - 5(m_s+1))/5^{m_s} = m_s^{m'+1}\tau' + 2^{m_s}\tau_{s+1},$$

and in case 2

$$\chi(C_s^2) > (\tau'_s - 5(m'_s+1))/5^{m'_s} = \tau'_{s+1}.$$

Let $a_{s+1} = p_k$ and $b_{s+1} = p_{k+1}$, and for each $x \in X_{s+1}$ let $a_{x,s+1} = d_k(x)$ and $b_{x,s+1} = d_{k+1}(x)$.

Next we define C_{s+1}. In case 2 we define $C_{s+1} = C_s^2$; so henceforth we assume we are in case 1. Let Z be the union of the sets $\{a_{x,s+1}, b_{x,s+1}\}(x \in Y_s)$; then $|Z| = 2m_s/5 \leq m_s$. Let $z \in Z$, and let $Y \subseteq Y_s$ with $|Y| = m'$. Let $D_{z,Y}$ be the set of vertices in C_s^2 such that for each $x \in Y$ there exists a vertex in N'_x adjacent to both v, z. For each $x \in Y$, let N''_x denote the set of vertices in N'_x adjacent to z; then $(N''_x : x \in Y)$ is a multicover of $D_{z,Y}$; and it is stable, since G is triangle-free. Since we are in case 1, it follows from (2) that $\chi(D_{z,Y}) \leq \tau'$. Now let D_z denote the set of vertices $v \in C_s^2$ such that for at least m' values of $x \in Y_s$ there exists a vertex in N'_x adjacent to both v, z; that is, D_z is the union of the sets $D_{z,Y}$ over all choices of Y. Since there are only at most $m_s^{m'}$ choices of Y, it follows that $\chi(D_z) \leq m_s^{m'}\tau'$. Thus the union of the sets D_z over all $z \in Z$ has chromatic number at most $m_s^{m'+1}\tau'$, and so there exists $C_s^3 \subseteq C_s^2$ with

$$\chi(C_s^3) \geq \chi(C_2^2) - m_s^{m'+1}\tau' > 2^{m_s}\tau_{s+1},$$

such that for every $v \in C_s^3$, and every $z \in Z$, there are fewer than m' values of $x \in Y_s$ such that some vertex in N'_x is adjacent to both v, z.

Fix $v \in C_s^3$ for the moment, and make a digraph J_v with vertex set Y_s in which for distinct $x, y \in Y_s$, y is adjacent from x in J_v if some vertex in N'_y is adjacent to v and to one of $a_{x,s+1}, b_{x,s+1}$. We have just seen that for all v, every vertex of the digraph J_v has indegree in J at most $2m' - 2$. It follows that in J_v, some vertex has indegree plus outdegree at most $4m' - 4$, and the same holds for every nonnull subdigraph of J_v; and so the undirected graph underlying J_v can be $4m'$-coloured. Hence there is a subset U_v say of Y_s of cardinality $|Y_s|/(4m') = m_{s+1}$ such that no edge of J_v has both ends in U_v. There are only $2^{|Y_s|}$ possibilities for U_v, and so there exists $C_s^4 \subseteq C_s^3$ with

$$\chi(C_s^4) \geq \chi(C_s^3)/2^{|Y_s|} > \tau_{s+1}$$

such that the sets U_v are equal for all $v \in C_s^4$. Let X_{s+1} be this common value of U_v, and let $C_{s+1} = C_s^4$. This completes the definition of C_{s+1} in case 1.

In both cases, the pairs $a_j, b_j (1 \le j \le s+1)$ and the vertices $a_{x,j}, b_{x,j} (x \in X_{s+1}, 1 \le j \le s+1)$ define an $(s+1)$-trellis H_{s+1} on X_{s+1}, and no vertex in H_{s+1} has a neighbour in C_{s+1}, and for all $v \in C_{s+1}$ and $x \in X_{s+1}$, some neighbour of v in N_x has no neighbour in $V(H_{s+1})$ except x. This completes the inductive definition of H_s and C_s for $0 \le s \le t$.

Thus there is a t-trellis on the set X_t, where $|X_t| = 5t$; next we need to convert it to an extended t-trellis on a subset of X_t of cardinality t. With the same notation as before (with $s = t$), since $\chi(C_t) > \tau'_t > \kappa$, there is a 5-hole P in $G[C_t]$, with vertices p_1-p_2-$\cdots$-p_5-p_1 say, in order. Let $x \in X_t$; a *handle* for x means a 3-vertex path a-c-b of P such that some vertex in N'_x is adjacent to a, and not to b, c, and some vertex in N'_x is adjacent to b and not to a, c. We claim that there is a handle for x. Choose $S \subseteq N'_x$ minimal such that every vertex in $V(P)$ has a neighbour in S. For $1 \le i \le 5$, choose $s_i \in S$ adjacent to p_i. Suppose first that some $s_1 \in S$ has only one neighbour in $V(P)$, say p_1. Then no other vertex in S is adjacent to p_1, from the minimality of S, and since s_3 is nonadjacent to p_2 it follows that p_1-p_2-p_3 is a handle for x. We may assume therefore that each s_i has at least two (and hence exactly two) neighbours in $V(P)$. Let s_1 be adjacent to p_1, p_4 say. From the minimality of S, one of p_1, p_4 has no more neighbours in S, say p_1. But then again p_1-p_2-p_3 is a handle for x. This proves the claim that for each $x \in X_t$ there is a handle for x. Since there are only five possibilities for handles, there exists $X_0 \subseteq X_t$ with $|X_0| = |X_t|/5 = t$ such that every vertex in X_0 has the same handle, say a_0-c_0-b_0. For each $x \in X_0$ let $a_{x,0} \in N'_x$ be adjacent to a_0 and not to b_0, c_0, and let $b_{x,0}$ be adjacent to b_0 and not to a_0, c_0. Then the pairs $a_j, b_j (1 \le j \le t)$, the path a_0-c_0-b_0, and the vertices $a_{x,j}, b_{x,j} (x \in X_{s+1}, 0 \le j \le s+1)$ define an extended t-trellis on X_0. This proves 3.3. ∎

From 3.2 and 3.3 we deduce:

3.4 *For all $\kappa \ge 0$ and $l \ge 8$, there exist $\tau, m \ge 0$ with the following property. Let G be a triangle-free graph such that every induced subgraph of G with chromatic number more than κ has a 5-hole. Let $C \subseteq V(G)$ with chromatic number more than τ; and let $(N_x : x \in X)$ be a multicover of C with $|X| \ge m$. Then G has an l-hole.*

Let G be a graph and let $t \ge 0$ be an integer. A *t-cable* in G consists of:

- t distinct vertices $x_1, \dots, x_t$, pairwise nonadjacent;
- for $1 \le i \le t$, a subset N_i of the set of neighbours of x_i, such that the sets $N_1, \dots, N_t$ are pairwise disjoint;
- for $1 \le i \le t$, disjoint subsets $Z_{i,i+1}, \dots, Z_{i,t}, Y_i$ of N_i;
- a subset $C \subseteq V(G)$ disjoint from $\{x_1, \dots, x_t\} \cup N_1 \cup \cdots \cup N_t$

satisfying the following conditions:

- for $1 \le i \le t$, every vertex in C has a neighbour in Y_i, and has no neighbours in $Z_{i,j}$ for $i+1 \le j \le t$, and is nonadjacent to x_i;
- for $i < j \le t$, x_i has no neighbours in N_j;

- for $i < j < k \leq t$, there are no edges between $Z_{i,j}$ and N_k;
- for all $i < j \leq t$, either
  - $Z_{i,j} = \emptyset$ and x_j has no neighbours in Y_i, or
  - every vertex in N_j has a neighbour in $Z_{i,j}$ and has no neighbours in Y_i.

We call C the *base* of the t-cable, and say $\chi(C)$ is the *chromatic number* of the t-cable. Given a t-cable in this notation, let $I \subseteq \{1, \dots, t\}$; then the vertices $x_i (i \in I)$, the sets $N_i (i \in I)$, the sets $Z_{i,j} (i, j \in I)$, the sets $Y_i (i \in I)$ and C define an $|I|$-cable; we call this a *subcable*.

Thus there are two types of pair (i, j) with $i < j \leq t$, and we aim next to apply Ramsey's theorem on these pairs to get a large subcable where all the pairs have the same type. Two special kinds of t-cables are of interest: *type 1* t-cables, where for all $i < j \leq t$, $Z_{i,j} = \emptyset$ and x_j has no neighbours in Y_i, and *type 2* t-cables, where for all $i < j \leq t$, every vertex in N_j has a neighbour in $Z_{i,j}$ and has no neighbours in Y_i. A type 1 t-cable with base C is just a multicover of C in disguise, so from 3.4 we have:

3.5 *For all $\kappa \geq 0$ and $l \geq 8$, there exist $\tau, m \geq 0$ with the following property. Let G be a triangle-free graph such that every induced subgraph of G with chromatic number more than κ has a 5-hole. If G admits a type 1 m-cable with chromatic number more than τ, then G has an l-hole.*

We need a similar theorem for type 2 cables.

3.6 *Let G be a triangle-free graph. For all $l \geq 5$, if G admits a type 2 $(l-3)$-cable with nonnull base, then G has an l-hole.*

Proof. Let $t = l - 3$ (and so $t \geq 2$) and assume G contains a type 2 t-cable with nonnull base. In the usual notation, let $v \in C$. Since every vertex in C has a neighbour in Y_t, there exists $y_t \in Y_t$ adjacent to v. Since every vertex in N_t has a neighbour in $Z_{t-1,t}$, there exists $z_{t-1} \in Z_{t-1,t}$ adjacent to y_t. Similarly for $i = t-2, t-3, \dots, 1$ there exists $z_i \in Z_{i,i+1}$ such that z_{i+1} is adjacent to z_i. Thus z_1-z_2-$\cdots$-z_{t-1}-y_t is a path. It is induced; for if $i, j \leq t$ and $j \geq i + 2$ then z_i has no neighbour in N_j, since $z_i \in Z_{i,i+1}$. Since x_1 is adjacent to z_1 and to none of $z_2, \dots, z_{t-1}, y_t$ (because $t \geq 2$ and x_1 has no neighbours in N_j for $j > 1$), and v is adjacent to y_t and nonadjacent to $x_1, z_1, \dots, z_{t-1}$, it follows that

$$x_1\text{-}z_1\text{-}z_2\text{-}\cdots\text{-}z_{t-1}\text{-}y_t\text{-}v$$

is an induced path. Now v has a neighbour $y_1 \in Y_1$; and we claim that y_1 is nonadjacent to $z_1, \dots, z_{t-1}, y_t$. Certainly y_1, z_1 are nonadjacent, since they are both adjacent to x_1 and G is triangle-free. For $2 \leq j \leq t-1$, y_1 is nonadjacent to z_j since every vertex in N_j has no neighbours in Y_1. For the same reason, y_1 is nonadjacent to y_t, since $t > 1$. Consequently

$$x_1\text{-}z_1\text{-}z_2\text{-}\cdots\text{-}z_{t-1}\text{-}y_t\text{-}v\text{-}y_1\text{-}x_1$$

is a hole of length $t + 3 = l$. This proves 3.6. ∎

We deduce:

3.7 *For all $\kappa \geq 0$ and $l \geq 8$, there exist $t, \tau \geq 0$ with the following property. Let G be a triangle-free graph such that every induced subgraph of G with chromatic number more than κ has a 5-hole. If G admits a t-cable with chromatic number more than τ then G has an l-hole.*

Proof. Let m, τ be as in 3.5. Let $n = l - 3$. Let t equal the Ramsey number $R(m, n)$; that is, the smallest integer t such for for every partition of the edges of K_t into two sets, there is either a K_m subgraph with all edges in the first set, or a K_n with all edges in the second. We claim that t, τ satisfy the theorem.

For let G admit a t-cable with base C and chromatic number more than τ. By Ramsey's theorem either

- there exists $I \subseteq \{1, \ldots, t\}$ with $|I| = m$ such that for all $i, j \in I$ with $i < j$, every vertex in N_j has a neighbour in $Z_{i,j}$ and has no neighbours in Y_i, or
- there exists $I \subseteq \{1, \ldots, t\}$ with $|I| = n$ such that for all $i, j \in I$ with $i < j$, $Z_{i,j} = \emptyset$ and x_j has no neighbours in Y_i.

Thus either there is an m-subcable of type 1, or an n-subcable of type 2, with base C in each case. In the first case the result follows from 3.5, and in the second from 3.6. This proves 3.7. ∎

3.8 *Let $\phi : \mathbb{N} \to \mathbb{N}$ be non-decreasing, and let $t, \tau \geq 0$. Then there exists τ' with the following property. let G be a triangle-free graph such that G is $(2, \phi)$-controlled and $\chi(G) > \tau'$. Then G admits a t-cable with chromatic number more than τ.*

Proof. Let $\tau_t = \tau$, and for $s = t-1, \ldots, 0$ let $\tau_s = \phi(2^s \tau_{s+1} + 1)$; and let $\tau' = \tau_0$. We claim that τ' satisfies the theorem. For let G be a triangle-free graph such that G is $(2, \phi)$-controlled and $\chi(G) > \tau'$. Consequently G admits a 0-cable with chromatic number more than τ_0. We claim that for $s = 1, \ldots, t$, G admits an s-cable with chromatic number more than τ_s. For suppose the result holds for some $s < t$; we prove it also holds for $s+1$. In the usual notation, since $\chi(C) > \tau_s = \phi(2^s\tau_{s+1}+1)$, there exists $x_{t+1} \in C$ such that $\chi(N^2_{G[C]}[x_{t+1}]) > 2^s\tau_{s+1} + 1$, and hence $\chi_{G[C]}(N^2(x_{t+1})) > 2^s\tau_{s+1}$. Let $D = N^2_{G[C]}(x_{t+1})$. For each $v \in D$, and $1 \leq i \leq s$, if some neighbour of v in Y_i is nonadjacent to x_{s+1} define $c_i(v) = 1$, and otherwise define $c_i(v) = 2$. There are only 2^s possibilities for the s-tuple $(c_1(v), \ldots, c_s(v))$, and so there exists $C' \subseteq D$ with $\chi(C') \geq \chi(D)/2^s > \tau_{s+1}$ and an s-tuple $(c_1, \ldots, c_s)$ such that $c_i(v) = c_i$ for all $v \in C'$ and $1 \leq i \leq s$.

Let $N_{s+1} = Y'_{s+1}$ be the set of neighbours of x_{s+1} in C. For $1 \leq i \leq s$ define $Z_{i,s+1}, Y'_i \subseteq Y_i$ as follows:

- if $c_i = 1$, let Y'_i be the set of vertices in Y_i nonadjacent to x_{i+1}, and let $Z_{i,s+1} = \emptyset$
- if $c_i = 2$, let Y'_i be the set of vertices in Y_i adjacent to x_{s+1}, and let $Z_{i,s+1}$ be the set of vertices in Y_i nonadjacent to x_i.

Note that in the second case, no vertex in $Z_{i,s+1}$ has a neighbour in C', and no vertex in Y'_i has a neighbour in Y'_{s+1}. It follows that $x_1, \ldots, x_{s+1}$, the sets $N_1, \ldots, N_{s+1}$, the sets $Z_{i,j}$ for $1 \leq i < j \leq$

$s+1$, the sets Y'_i for $1 \le i \le s+1$, and C', define an $(s+1)$-cable with chromatic number more than τ_{s+1}.

This proves that G admits a t-cable with chromatic number more than $\tau_t = \tau$, and so proves 3.8. ∎

Let us put these pieces together to prove 2.3, which we restate:

3.9 *Let $\phi : \mathbb{N} \to \mathbb{N}$ be a nondecreasing function; then for all $l \ge 4$ there exists n such that every $(2, \phi)$-controlled triangle-free graph with chromatic number more than n has a hole of length l.*

Proof. If $l \le 7$ the result follows 3.1, so we may assume that $l \ge 8$. Let $\kappa = \phi(2)$. By 3.1, every induced subgraph of G with chromatic number more than κ has a 5-hole. Let t, τ be as in 3.7; and let τ' be as in 3.8. Let $n = \tau'$. We claim that n satisfies the theorem. For let G be a $(2, \phi)$-controlled triangle-free graph with chromatic number more than n. By 3.8, G admits a t-cable with chromatic number more than τ; and by 3.7, G has an l-hole. This proves 3.9. ∎

If we just want to prove the first conjecture of 1.1, rather than the full strength of 1.2, the remainder of the paper is not needed; let us explain why. The following is proved in [2] (the proof just takes a few lines):

3.10 *Let $l \ge 3$ and $\kappa \ge 1$ be integers, and let G be a graph with no hole of length more than l, such that $\chi(N(v)), \chi(N^2(v)) \le \kappa$ for every vertex v. Then $\chi(G) \le (2l-2)\kappa$.*

For each $\kappa \ge 0$, let $\phi(\kappa) = (2l-2)\kappa$. It follows from 3.10 that if G has no hole of length more than l, and H is an induced subgraph of G, if $\chi(H) > \phi(\kappa)$ then $\chi(N^2_H[v]) > \kappa$ for some vertex v of H; that is, G is $(2, \phi)$-controlled. Then from 3.9 it follows that $\chi(G)$ is bounded, which proves the first assertion of 1.1. Indeed, we don't even need all of 3.9; instead of an l-hole, we are content with a hole of length at least l, and with this modification 3.9 is easier to prove. For instance, we could get by with trellises instead of extended trellises, since holes of length 11 are of no significance, and indeed we could just use 1-subdivisions of a large $K_{n,n}$ instead of trellises, since we are not picky about the exact length of the hole.

4 Bounded radius

In this section we prove 2.2, which we restate, somewhat reformulated:

4.1 *Let $\phi : \mathbb{N} \to \mathbb{N}$ be non-decreasing, and let $\rho > 2$ and $l \ge \rho(2\rho + 5)$ be integers. There is a nondecreasing function $\phi' : \mathbb{N} \to \mathbb{N}$, with the following property. Let G be a triangle-free graph with no l-hole such that G is (ρ, ϕ)-controlled. Then G is $(2, \phi')$-controlled.*

4.1 follows immediately from the following.

4.2 *Let $\phi : \mathbb{N} \to \mathbb{N}$ be non-decreasing, and let $\rho > 2$ and $l \ge \rho(2\rho + 5)$ be integers. There is a nondecreasing function $\phi' : \mathbb{N} \to \mathbb{N}$, with the following property. Let G be a triangle-free graph with no l-hole such that G is (ρ, ϕ)-controlled. Then G is $(\rho - 1, \phi')$-controlled.*

Proof. Let $l = \alpha\rho + \beta$, where $\alpha \geq 0$ is an integer and $0 \leq \beta < \rho$. Since $l \geq \rho(2\rho+5)$, it follows that $\alpha \geq 2\rho + 5$. For $\kappa \in \mathbb{N}$, define $\mu_\alpha(\kappa) = \phi(0) + 1$, and for $h = \alpha, \ldots, 2$ define

$$\mu_{h-1}(\kappa) = (\rho+1)\kappa + \phi(\phi(\mu_h(\kappa) + \kappa) + (2\rho+2)\kappa),$$

and $\mu_0(\kappa) = \phi(\mu_1(\kappa) + \kappa)$. Define $\phi'(\kappa) = \mu_0(\kappa)$. We see that ϕ' is nondecreasing.

Let G be a triangle-free graph with no l-hole such that G is (ρ, ϕ)-controlled. We will show that G is $(\rho-1, \phi')$-controlled. Let $\kappa \in \mathbb{N}$, such that $\chi(N^{\rho-1}(v)) \leq \kappa$ for every vertex v; we must show that $\chi(G) \leq \mu_0(\kappa)$. (If so, then the same argument applied to every induced subgraph H of G and every κ shows that G is $(\rho-1, \phi')$-controlled.) Suppose not.

Let $v \in V(G)$. Let T be a path $v = t_0\text{-}t_1\text{-}\cdots\text{-}t_\rho$, such that $d_G(v, t_\rho) = \rho$. For the moment fix such a path T. Let us say a path P is a *(v,T)-extension* if it has the following properties, where P has vertices $p_0\text{-}p_1\text{-}\cdots\text{-}p_n$ in order:

- P is induced, and $p_0 = t_\rho$, and $n \geq \rho$;
- $d_G(v, p_i) = \rho$ for $0 \leq i \leq n$;
- $d_G(t_i, p_j) \geq \rho$ for $0 \leq i \leq \rho$ and $\rho \leq j \leq n$; and
- $d_G(p_i, p_n) \geq \rho$ for $0 \leq i \leq n - \rho$.

(1) *If P as above is a (v,T)-extension, then $P \cup T$ is an induced path of length $\rho + n$.*

Because T is induced since $d_G(v, t_\rho) = \rho$, and P is induced by hypothesis. Moreover $V(P) \cap V(T) = \{t_\rho\}$ since $d_G(v, t_i) < \rho$ for $0 \leq i < \rho$, and $d_G(v, p_i) = \rho$ for $0 \leq i \leq n$. Suppose that some t_i is adjacent to some p_j, where $i < \rho$ and $j > 0$. Since $d_G(v, p_j) = \rho$ and $d_G(v, t_i) = i < \rho$, it follows that $i = \rho - 1$. Now $j \neq 1$ since G is triangle-free, so $j \geq 2$. Since $d_G(t_{\rho-1}, p_k) \geq \rho$ for $\rho \leq k \leq n$, it follows that $j < \rho$. Then the path $t_{\rho-1}\text{-}p_j\text{-}p_{j+1}\text{-}\cdots\text{-}p_\rho$ has length $\rho - j + 1 < \rho$, a contradiction since $d_G(t_{\rho-1}, p_\rho) \geq \rho$. This proves (1).

Let P, P' both be (v,T)-extensions. We say they are *parallel* if the last three vertices of P are the same as the last three of P', and in particular the last vertices of P, P' are equal.

(2) *Let $P_1, \ldots, P_k$ be (v,T)-extensions, pairwise parallel. Then there exists $s \in \{2\rho, 2\rho-2, 2\rho-4\}$ such that G has holes of lengths $|E(P_1)| + s, \ldots, |E(P_k)| + s$.*

Let z be the common last vertex of $P_1, \ldots, P_k$, and choose a path Z between v, z of length ρ. Since $T \cup Z$ is connected, there is an induced path Q between t_ρ, z with $V(Q) \subseteq V(T \cup Z)$. Let us first examine the length of Q. Let Z have vertices $z_0\text{-}z_1\text{-}\cdots\text{-}z_\rho$, where $z_0 = v$ and $z_\rho = z$. If no vertex in $\{z_1, \ldots, z_\rho\}$ has a neighbour in $\{t_1, \ldots, t_\rho\}$, then the two sets are disjoint, and $Q = T \cup Z$ and hence has length 2ρ. We assume then that some $z_j \in \{z_1, \ldots, z_\rho\}$ is adjacent to some $t_i \in \{t_1, \ldots, t_\rho\}$. Since $d_G(t_i, z) \geq \rho$ from the definition of a (v,T)-extension, the path $t_i\text{-}z_j\text{-}z_{j+1}\text{-}\cdots\text{-}z_\rho$ has length at least ρ, and so $j = 1$. Since z_j is adjacent to $t_0 = v$, and G is triangle-free, it follows that $i \geq 2$. Since $d_G(v, t_\rho) = \rho$, it follows that $i = 2$. So there is only one such edge, and in particular the two sets $\{z_1, \ldots, z_\rho\}$, $\{t_1, \ldots, t_{\rho-1}, t_\rho\}$ are disjoint, and Q has length $2\rho - 2$. We have proved then that Q has length 2ρ or $2\rho - 2$.

Now let P be one of $P_1, \dots, P_k$, and let P have vertices p_0-p_1-$\cdots$-p_n in order. Thus $p_0 = t_\rho$ and $p_n = z_\rho = z$. Both P, Q are induced, and their interiors are disjoint, since every vertex x of the interior of Q belongs to one of $V(Z) \setminus \{z\}, V(T) \setminus \{t_\rho\}$ and hence satisfies $d_G(v, x) < \rho$, while $d_G(v, x) = \rho$ for every vertex x of the interior of P. Suppose then that some vertex x in the interior of Q has a neighbour $p_j \in \{p_1, \dots, p_{n-1}\}$. From (1) it follows that $x \notin V(T)$, and so $x \in \{z_1, \dots, z_{\rho-1}\}$. Since $d_G(v, p_j) = \rho$, it follows that $d_G(v, x) = \rho - 1$, and so $x = z_{\rho-1}$. Consequently $d_G(p_j, p_n) \leq 2$, and so $j > n - \rho$ from the final condition in the definition of a (v, T)-extension. Since $d_G(p_{n-\rho}, p_n) \geq \rho$ from the same condition, it follows that the path $p_{n-\rho}$-$p_{n-\rho+1}$-$\cdots$-p_j-$z_{\rho-1}$-p_n has length at least ρ, and so $j \geq n - 2$. Now $j \neq n - 1$ since G is triangle-free, and $j \neq n$ by its definition, so $j = n - 2$.

Consequently there is at most one edge joining the interiors of P, Q, and any such edge is between $z_{\rho-1}$ and p_{n-2}. Let $s = |E(Q)|$ if there is no such edge, and $|E(Q)| - 2$ if there is such an edge. In either case G has a hole of length $|E(P)| + s$. Moreover, since the final three vertices of $P_1, \dots, P_k$ are the same, it follows that G has a hole of length $|E(P_i)| + s$ for $1 \leq i \leq k$. This proves (2).

Since $\chi(G) > \mu_0(\kappa)$, there exists z_0 such that $\chi(N^\rho_G[z_0]) > \mu_1(\kappa) + \kappa$, and hence $\chi(N^\rho_G(z_0)) > \mu_1(\kappa)$. Let $H_0 = G$ and let T_0 be the one-vertex subgraph with vertex z_0. For $1 \leq h \leq \alpha$, we define $y_h, y'_h, S_h, z_h, T_h, M_h, H_h$ as follows. Assume we have defined H_{h-1}, T_{h-1} and z_{h-1} such that $\chi(N^\rho_{H_{h-1}}(z_{h-1})) > \mu_{h-1}(\kappa)$ and T_{h-1} is an induced path of G with at most $\rho + 1$ vertices and with one end z_{h-1}. Let M_h be the subgraph induced on the set of all vertices v of H_{h-1} that satisfy

- $d_{H_{h-1}}(z_{h-1}, v) = \rho$; and
- $d_G(x, v) \geq \rho$ for every vertex x of T_{h-1}.

Since $\chi(N^{\rho-1}(x)) \leq \kappa$ for each vertex x of T_{h-1}, and $\chi(N^\rho_{H_{h-1}}(z_{h-1})) > \mu_{h-1}(\kappa)$, it follows that

$$\chi(M_h) > \mu_{h-1}(\kappa) - (\rho + 1)\kappa = \phi(\phi(\mu_h(\kappa) + \kappa) + (2\rho + 2)\kappa).$$

Since G is (ρ, ϕ)-controlled, there is a vertex $y_h \in M_h$ such that

$$\chi(N^\rho_{M_h}[y_h]) > \phi(\mu_h(\kappa) + \kappa) + (2\rho + 2)\kappa,$$

and hence with

$$\chi(N^\rho_{M_h}(y_h)) > \phi(\mu_h(\kappa) + \kappa) + (2\rho + 1)\kappa.$$

Let S_h be a path of H_{h-1} of length ρ between z_{h-1} and y_h. Let y'_h be adjacent to y_h in M_h. Let S'_h be a path of H_{h-1} of length ρ between z_{h-1} and y'_h. Let H_h be the subgraph induced on the set of all vertices v of M_h with the following properties:

- $d_{M_h}(y_h, v) = \rho$; and
- $d_G(x, v) \geq \rho$ for every $x \in V(S_h) \cup V(S'_h)$.

Since $\chi(N^\rho_{M_h}(y_h))) > \phi(\mu_h(\kappa) + \kappa) + (2\rho + 1)\kappa$, and $\chi(N^{\rho-1}(x)) \leq \kappa$ for each vertex x of $V(S_h \cup S'_h)$, and there are at most $2\rho + 1$ such vertices x, it follows that $\chi(H_h) > \phi(\mu_h(\kappa) + \kappa)$. Consequently there exists $z_h \in H_h$ such that $\chi(N^\rho_{H_h}[z_h]) > \mu_h(\kappa) + \kappa$, and hence with $\chi(N^\rho_{H_h}(z_h)) > \mu_h(\kappa)$. Let T_h be a path of M_h of length ρ between y_h, z_h. This completes the inductive definition of $y_h, y'_h, S_h, z_h, T_h, M_h, H_h$ for $1 \leq h \leq \alpha$.

(3) *For* $1 \le h \le \alpha$, $S_h \cup T_h$ *is an induced path* L_h *between* z_{h-1}, z_h *of length* 2ρ. *Also there is an induced path* L'_h *between* z_{h-1}, z_h *of length* $2\rho - 1$ *or* $2\rho + 1$ *with vertex set a subset of* $V(S'_h \cup T_h)$.

The first claim follows from (1). For the second, the graph formed by the union of S'_h, T_h and the edge $y_h y'_h$ is a path, but it might not be induced. If it is induced, it has length $2\rho+1$ as required; and since S'_h and T_h are both induced paths, we may assume that some vertex a of S'_h is adjacent to some vertex b of T_h, where $(a, b) \neq (y'_h, y_h)$. Since every vertex of S'_h has distance at most $\rho - 2$ from z_{h-1} except the last two, and every vertex of T_h has distance at least ρ from z_{h-1}, it follows that a is either y'_h or its neighbour in S'_h. Now $d_G(y'_h, z_h) = \rho$, so y'_h has no neighbour in T_h for y_h (because y'_h is not adjacent to the second vertex of T_h since G is triangle-free). Thus a is the penultimate vertex of S'_h. Consequently $b \neq y_h$ since G is triangle-free, and since $d_G(a, z_h) \ge \rho$, a has no neighbour in T_h different from the second vertex of T_h. We deduce that b is indeed the second vertex of T_h; and so there is an induced path between z_{h-1}, z_h of length $2\rho - 1$ with vertex set a subset of $V(S'_h \cup T_h)$. This proves (3).

Let there be q values of $h \in \{3, \dots, \alpha\}$ such that L'_h has length $\rho - 1$. For $3 \le h \le \alpha$, choose $L''_h \in \{L_h, L'_h\}$; then $L''_3 \cup L''_4 \cup \dots \cup L''_\alpha$ is an induced path between z_2 and z_α, and it is a (y_2, T_2)-extension, for every choice of $L''_3, L''_4, \dots, L''_\alpha$. Moreover, all these (y_2, T_2)-extensions are parallel (since the last ρ vertices of L_α, L'_α are the same). These paths have lengths every integer between $\rho(\alpha - 2) - q$ and $(\rho + 1)(\alpha - 2) - q$. From (2), G has holes of every length between $\rho\alpha - q$ and $(\rho + 1)\alpha - q - 6$, that is, all integers between $l - q - \beta$ and $l - q - \beta + \alpha - 6$. Since G has no l-hole, it follows that $l - q - \beta + \alpha - 6 < l$, that is, $\alpha < q + \beta + 6$. But by concatenating each of the paths $L''_3 \cup L''_4 \cup \dots \cup L''_\alpha$ with L_2, we obtain a (y_1, T_1)-extension of length exactly 2ρ more; and so there are (y_1, T_1)-extensions of all lengths between $\rho\alpha - q$ and $(\rho + 1)\alpha - q - 2$. Hence by (2) there are holes in G of all lengths between $\rho\alpha - q + 2\rho$ and $(\rho + 1)\alpha - q - 2 + 2\rho - 4$, that is, all lengths between $l - \beta - q + 2\rho$ and $l + \alpha - \beta - q + 2\rho - 6$. Since $\alpha < q + \beta + 6$ and $\alpha \ge 2\rho + 5$, it follows that $2\rho \le \alpha - 5 \le q + \beta$, and so $l - \beta - q + 2\rho \le l$. But

$$\alpha + l - \beta - q + 2\rho - 6 = l + (\alpha - q - 2) + (\rho - 1 - \beta) + (\rho - 3)$$

and hence the left side is at least l, because $\alpha - q - 2$, $\rho - 1 - \beta$ and $\rho - 3$ are all nonnegative. Consequently some hole has length l, a contradiction. This proves 4.2 and hence 4.1. ∎

5 Showers

Now we come to the third and most complicated part of the proof: proving 2.1. This will occupy the remainder of the paper.

What can we prove about hole lengths if for some large fixed ρ, $\chi(N^\rho[v])$ is bounded for every vertex v? In 4.1 we were able to guarantee the presence of a hole of any desired length (almost), but in these new circumstances that becomes impossible; for any fixed $\rho \ge 0$ and $l \ge 2$, there are graphs with arbitrarily large χ, and girth more than $l, \rho/2$; which implies that $\chi(N^\rho[v])$ is at most 2, and yet they have no l-hole. We will show the following, a reformulation of 2.1.

5.1 *Let $\nu \ge 2$ and $\kappa \ge 0$ be integers, and let G be a triangle-free graph such that $\chi(N^{\rho}_G[v]) \le \kappa$ for every vertex v, where $\rho = 3^{\nu+2}+4$. If G admits no hole ν-sequence then $\chi(G)$ is bounded.*

The proof will need a number of steps and preliminary lemmas. We begin with some definitions. A *levelling* in G is a sequence of pairwise disjoint subsets $(L_0, L_1, \dots, L_k)$ of $V(G)$ such that

- $|L_0| = 1$;
- for $1 \le i \le k$ every vertex in L_i has a neighbour in L_{i-1};
- for $0 \le i < j \le k$, if $j > i+1$ then no vertex in L_j has a neighbour in L_i.

We call L_k the *base* of the levelling. The *chromatic number* of a levelling is the chromatic number of its base. We observe first:

5.2 *For any integer $\tau \ge 0$, if $\chi(G) > 2\tau$ then G admits a levelling with chromatic number more than τ.*

Proof. Choose a component C of G with chromatic number equal to that of G, and let z be a vertex in that component. For each $i \ge 0$, let L_i be the set of vertices v of C such that $d_C(z, v) = i$, and choose j such that $L_0 \cup \dots \cup L_j = V(C)$. If $\chi(L_k) \le \tau$ for all k with $0 \le k \le j$, then $\chi(C) \le 2\tau$ (take two disjoint sets of colours both of size τ, and use them for the even and odd levels alternately), which is impossible; so there exists k such that $\chi(L_k) > \tau$. Then $(L_0, \dots, L_k)$ is the desired levelling. This proves 5.2. ∎

If $(L_0, \dots, L_k)$ is a levelling in G, we call the unique vertex in L_0 the *head* of the levelling, and we call $L_0 \cup \dots \cup L_k$ the *vertex set* of the levelling. A path P of $G[V]$ (where V is the vertex set of the levelling) with ends u, v is *monotone* (with respect to the given levelling) if there exist h, j with $0 \le h, j \le k$, such that $u \in L_h, v \in L_j$, and P has length $|j - h|$; and therefore P has exactly one vertex in L_i for each i between h, j, and has no other vertices.

There is a notational problem with levellings; that while it seems most natural to number levels starting with the head as level zero, most of the action will be at or close to the base L_k, and we constantly have to refer to the parameter k. To obviate this, let us say a vertex v of the vertex set has *height* $k - i$ if $v \in L_i$ where $0 \le i \le k$. Thus vertices in L_k have height zero.

A *shower* in G is a sequence $(L_0, L_1, \dots, L_k, s)$ where $L_0, L_1, \dots, L_k$ are pairwise disjoint subsets of $V(G)$ and $s \in L_k$, such that

- $|L_0| = 1$;
- for $1 \le i < k$ every vertex in L_i has a neighbour in L_{i-1};
- for $0 \le i < j \le k$, if $j > i+1$ then no vertex in L_j has a neighbour in L_i; and
- $G[L_k]$ is connected.

The differences between a shower and a levelling are that, first, not every vertex in L_k needs to have a neighbour in L_{k-1}; second, that $G[L_k]$ is connected; and third, the distinguished vertex s. We call $L_0, \dots, L_k$ the *levels* of the shower, and s the *drain* of the shower. We define "head", "base", "vertex set", "monotone", "height" for showers just as for levellings. The set of vertices in L_k with a neighbour in L_{k-1} is called the *floor* of the shower.

If $\mathcal{S} = (L_0, \ldots, L_k, s)$ is a shower, with head z_0 and vertex set V, a *recirculator* for $\mathcal{S}$ is an induced path R with ends s, z_0 such that no internal vertex of R belongs to V and no internal vertex of R has any neighbours in $V \setminus \{s, z_0\}$. The *distance* $d_G(P_1, P_2)$ between two nonnull subgraphs P_1, P_2 of G is the minimum of $d_G(v_1, v_2)$ over all $v_1 \in V(P_1)$ and $v_2 \in V(P_2)$.

5.3 *Let $\tau, \kappa \geq 0$ be integers. Let G be a graph such that $\chi(N_G^8[v]) \leq \kappa$ for every vertex v. Let $(L_0, \ldots, L_k)$ be a levelling in G, where $\chi(L_k) > 22\tau + 2\kappa$. Then there is a shower $(V_0, \ldots, V_n, s)$ in G, with floor of chromatic number more than τ, and with a recirculator, such that*

- *$V_n \subseteq L_k$, and $V_{n-1} \subseteq L_{k-1}$; and*
- *$V_0, \ldots, V_{n-2} \subseteq L_0 \cup \cdots \cup L_{k-2}$.*

Proof. By replacing L_k by the vertex set of a component of $G[L_k]$ with maximum chromatic number, we may assume that $G[L_k]$ is connected. A *post* is a monotone path with an end in L_k. Since $\chi(L_k) > \kappa$, there exist two vertices of L_k with distance more than 8. It follows that there are two posts both of length three with distance at least three. Consequently we can choose two posts P, Q with the following properties:

- P, Q have the same length $k - h \geq 3$;
- $d_G(P, Q) \geq 3$;
- subject to these two conditions, h is minimum.

Let P have vertices p_k-p_{k-1}-$\cdots$-p_h and Q have vertices q_k-q_{k-1}-$\cdots$-q_h, where $p_i, q_i \in L_i$ for $h \leq i \leq k$. Let p_{h-1}, q_{h-1} be parents of p_h, q_h respectively. From the minimality of h, either

- p_{h-1}, q_{h-1} are adjacent, or
- some vertex is adjacent to p_{h-1} and to at least one of q_{h-1}, q_h, q_{h+1}, or
- some vertex is adjacent to q_{h-1} and to at least one of p_{h-1}, p_h, p_{h+1}.

In each case there is a connected induced subgraph M with $V(M) \subseteq L_0 \cup \cdots \cup L_h \cup \{p_{h+1}, q_{h+1}\}$, with at most seven vertices, and with $p_{h+1}, p_h, p_{h-1}, q_{h+1}, q_h, q_{h-1} \in V(M)$; and if there is a vertex in $V(M) \setminus V(P \cup Q)$, then it belongs to $L_{h-2} \cup L_{h-1} \cup L_h$, and has a neighbour in $\{p_{h+1}, p_h, p_{h-1}\}$ and one in $\{q_{h+1}, q_h, q_{h-1}\}$.

Let X be the set of vertices $x \in L_{k-1}$ such that there is a path R from x to p_{h+1} satisfying:

- R has length at most $k - h + 8$;
- every internal vertex of R belongs to $L_0 \cup \cdots \cup L_{k-2}$; and
- no vertex of $R \setminus p_{h+1}$ equals or is adjacent to any vertex in $\{p_{h+2}, \ldots, p_k\}$.

Define $Y \subseteq L_{k-1}$ similarly with P, Q exchanged.

(1) *Every vertex $v \in L_k$ with $d_G(v, p_k), d_G(v, q_k) \geq 7$ has a neighbour in $X \cup Y$.*

Let $v \in L_k$ with $d_G(v, p_k), d_G(v, q_k) \geq 7$, and let r_0-r_1-$\cdots$-$r_k = v$ be a path between $r_0 \in L_0$ and $v = r_k$. We claim that $r_{k-1} \in X \cup Y$. From the minimality of h, one of $r_{h-1}, \ldots, r_k$ has distance at most two from one of $p_{h-1}, \ldots, p_k$. Choose j maximum such that r_j has distance at most two from some vertex u say of $P \cup Q \cup M$. Thus $j \geq h-1$. If $j = k$, then $u \notin V(M) \setminus V(P \cup Q)$ because $k-h \geq 3$, and so u is one of $p_k, p_{k-1}, p_{k-2}, q_k, q_{k-1}, q_{k-2}$; which is impossible since $d_G(v, p_k), d_G(v, q_k) \geq 7$. Thus $j < k$. From the maximality of j, it follows that $d_G(r_j, u) = 2$, and none of $r_j, \ldots, r_k$ equals or is adjacent to any vertex in $P \cup Q \cup M$. From the symmetry we may assume that $u \in V(Q) \cup (V(M) \setminus V(P \cup Q))$. Let w be a vertex adjacent to both u, r_j. If $u \in L_k \cup L_{k-1}$ then $k - j \leq 3$, and so $d_G(v, q_k) \leq 6$, a contradiction; and if $u \notin L_k \cup L_{k-1}$ and $w \in L_k \cup L_{k-1}$ then $u = q_{k-2}$ and $k - j \leq 2$, and again $d_G(v, q_k) \leq 6$, a contradiction. So $u, w \notin L_k \cup L_{k-1}$. Now there is a path of $M \cup Q$ between u and p_{h-1}. If $u \notin V(Q)$ then this path has length at most three, and its union with the path r_{k-1}-r_{k-2}-$\cdots$-r_j-w-u is of length at most $k - 1 - j + 5 \leq k - h + 5$, since $j \geq h - 1$, and so $r_{k-1} \in X$ as required. If $u \in V(Q)$, then u is one of $q_{j-2}, q_{j-1}, q_j, q_{j+1}, q_{j+2}$, and so some path of $M \cup Q$ between u and p_{h-1} has length at most $(j + 2) - (h + 1) + 6$, and its union with the path r_{k-1}-r_{k-2}-$\cdots$-r_j-w-u has length at most

$$(j + 2) - (h + 1) + 6 + (k - 1 - j) + 2 = k - h + 8,$$

and again $r_{k-1} \in X$. This proves (1).

Now the set of vertices $v \in L_k$ such that $d_G(v, p_k) \leq 6$ or $d_G(v, q_k) \leq 6$ has chromatic number at most 2κ; and since $\chi(L_k) > 22\tau + 2\kappa$, there exists a subset $Z_0 \subseteq L_k$ with $\chi(Z_0) > 22\tau$ such that $d_G(v, p_k), d_G(v, q_k) \geq 7$ for each $v \in Z_0$. Every vertex in Z_0 has a neighbour in $X \cup Y$, by (1); so we may assume that there exists $Z_1 \subseteq Z_0$ with $\chi(Z_1) > 11\tau$, such that every vertex in Z_1 is adjacent to a vertex in X. For each vertex $x \in X$, there is a path R as in the definition of X; let R_x be a shortest such path. Then R_x has length at most $k - h + 8$, and at least $(k - 1) - (h + 1)$; so there are eleven possibilities for its length, the numbers between $k - h - 2$ and $k - h + 8$. For each c with $k - h - 2 \leq c \leq k - h + 8$, let X_c be the set of vertices $x \in X$ such that R_x has length c. Then there exist c and $Z_2 \subseteq Z_1$ with $\chi(Z_2) \geq \chi(Z_1)/11 > \tau$, such that every vertex in Z_2 has a neighbour in X_c. Moreover we may choose Z_2 such that $G[Z_2]$ is connected. Let V be the union of the vertex sets of all the paths R_x ($x \in X_c$). Note that $V \subseteq L_0 \cup \cdots \cup L_{k-1}$. For $0 \leq i \leq c$, let V_i be the set of vertices $u \in V$ such that the shortest path of $G[V]$ between u, p_{h+1} has length i. Then $(V_0, \ldots, V_c)$ is a levelling. Moreover, $V_c = X_c$, and so no vertex in L_k has a neighbour in $V_0, \ldots, V_{c-1}$. Define $V_{c+1} = Z_2$; then also $(V_0, \ldots, V_{c+1})$ is a levelling.

Now no neighbour of p_{k-1} belongs to Z_0, and hence there are no edges between $\{p_{h+2}, \ldots, p_{k-1}\}$ and $V_1 \cup \cdots \cup V_{c+1}$. Since $G[L_k]$ is connected and p_{k-1} has a neighbour in L_k, there is a path $G[L_k]$ between a vertex adjacent to p_{k-1} and a vertex with a neighbour in $Z_2 = V_{c+1}$. Choose a minimal such path, D, and let s be its end adjacent to p_{k-1}. Then $(V_0, \ldots, V_c, V_{c+1} \cup V(D), s)$ is a shower, since $G[Z_2]$ is connected and hence so is $G[V_{c+1} \cup V(D)]$; and its floor includes Z_2 and hence has chromatic number more than τ; and p_{h+1}-p_{h+2}-$\cdots$-p_{k-1}-s is a recirculator for it. This proves 5.3. ∎

Let $\mathcal{S}$ be a shower with head z_0 drain s and vertex set V. An induced path of $G[V]$ between z_0, s is called a *jet* of $\mathcal{S}$. The set of all lengths of jets of $\mathcal{S}$ is called the *jetset* of $\mathcal{S}$. If $\mathcal{A}$ is a subset of the jetset of $\mathcal{S}$, then for each $a \in \mathcal{A}$ there is a jet J_a with length a, and we say the set of jets $\{J_a : a \in \mathcal{A}\}$ *realizes* $\mathcal{A}$. For $\nu \geq 2$, we say a shower $\mathcal{S}$ is ν*-complete* if there are ν consecutive integers in its jetset, and ν*-incomplete* otherwise. (Later we shall give a meaning to "1-complete", but at this stage it is not needed.) We deduce:

5.4 *Let $\tau, \kappa \geq 0$ and $\nu \geq 2$ be integers. Let G be a graph such that*

- *$\chi(N_G^8[v]) \leq \kappa$ for every vertex v;*
- *$\chi(G) > 44\tau + 4\kappa$; and*
- *G admits no hole ν-sequence.*

Then there is a ν-incomplete shower in G with floor of chromatic number more than τ.

Proof. By 5.2 there is a levelling $(L_0, \ldots, L_k)$ with chromatic number more than $22\tau + 2\kappa$. By 5.3, there is a shower $\mathcal{S}$, with a recirculator, and with floor of chromatic number more than τ. Since the union of the recirculator with any jet is a hole, and G admits no hole ν-sequence, it follows that $\mathcal{S}$ is not ν-complete. This proves 5.4. ∎

Thus, in order to prove 5.1, it suffices to show that if ν, κ, G are as in the hypothesis of 5.1 then the floor of every ν-incomplete shower in G has bounded chromatic number, and this is what we shall do.

6 Stabilizing a shower

A levelling $(L_0, \ldots, L_k)$ or shower $(L_0, \ldots, L_k, s)$ is *stable* if $L_0, \ldots, L_{k-1}$ are stable; and for $\lambda \geq 0$ an integer, it is λ*-stable* if $k \geq \lambda$ and L_i is stable for $k - \lambda \leq i \leq k-1$. We would like to prove that there exists a stable shower (still with base of large χ, but not as large as before), by converting the shower given by 5.4. This will take several steps. First we show how to convert a ν-incomplete shower into a ν-incomplete λ-stable shower (for any fixed λ).

6.1 *Let $\tau, \lambda \geq 0$ and $\nu \geq 2$ be integers, and let $\mu = (\lambda+1)(\nu-1)+1$. Let G be a triangle-free graph, and let $\mathcal{S}$ be a ν-incomplete shower in G, with floor of chromatic number more than $\nu\tau^{1+\mu}$, and with levels $L_0, \ldots, L_k$, where $k \geq \mu$. Then there is a λ-stable ν-incomplete shower with floor of chromatic number more than τ, and with levels $L'_0, \ldots, L'_h$, such that $0 \leq k - h \leq \mu - \lambda - 1$ and $L'_i \subseteq L_i$ for $0 \leq i \leq h$.*

Proof. We may assume that for $0 \leq i < k$, every vertex in L_i has a neighbour in L_{i+1}; for a vertex in L_i without this property could be deleted. Let $z_0 \in L_0$. For $1 \leq j \leq \nu$, let $h_j = k-1-(\lambda+1)(\nu-j)$; and for $1 \leq j < \nu$, let $I_j = \{i : h_j < i < h_{j+1}\}$. (Thus the sets I_j have cardinality λ, and there is an integer h_j between I_{j-1} and I_j that belongs to neither, that we use as insulation). For $1 \leq j \leq \nu$, let T_j be the set of vertices $v \in L_{h_j}$ such that there are j induced paths between v and z_0, each with interior in $L_1 \cup \cdots \cup L_{h_j-1}$, of lengths $h_j, h_j+1, \ldots, h_j+j-1$.

(1) $T_\nu = \emptyset$.

Because suppose that $v \in T_\nu$. Then there are ν induced paths between v and z_0, each with interior in $L_1 \cup \cdots \cup L_{k-2}$, of lengths $k-1, k, \ldots, k+\nu-2$, say $R_1, \ldots, R_\nu$. Let s be the drain of $\mathcal{S}$; and choose a minimal path Q between s, v with interior in L_k. Then for $1 \le i \le \nu$, the union of Q and R_i is a jet, contradicting that the shower is ν-incomplete. This proves (1).

If $X \subseteq L_0 \cup \cdots \cup L_k$, we denote by $\theta(X)$ the chromatic number of the set of all descendants in L_k of members of X. Since $T_1 = L_{h_1}$ it follows that

$$\theta(T_1) > \nu\tau^{1+\mu} \ge \tau^{k+1-h_2} + \tau^{k+1-h_3} + \cdots + \tau^{k+1-h_\nu},$$

and so there exists $j \in \{1, \ldots, \nu\}$ maximum such that

$$\theta(T_j) > \tau^{k+1-h_{j+1}} + \tau^{k+1-h_{j+2}} + \cdots + \tau^{k+1-h_\nu};$$

and $j < \nu$ by (1). From the maximality of j it follows that $\theta(T_j) - \theta(T_{j+1}) > \tau^{k+1-h_{j+1}}$. Let S_{j+1} be the set of vertices in $L_{h_{j+1}} \setminus T_{j+1}$ that have ancestors in T_j. For $h_j < i < h_{j+1}$ let M_i be the set of vertices in L_i with an ancestor in T_j and a descendant in S_{j+1}.

(2) *M_i is stable for $h_j < i < h_{j+1}$.*

For suppose that $x, y \in M_i$ are adjacent. Let $x', y' \in T_j$ be ancestors of x, y respectively (possibly equal). Let $z \in S_{j+1}$ be a descendant of x. Now there are induced paths from y' to z_0 with interior in $L_1 \cup \cdots \cup L_{h_j-1}$, of lengths $h_j, h_j+1, \ldots, h_j+j-1$. For each of these paths, its union with a path of length $i-h_j$ between y and y', a path of length $h_{j+1}-i$ between z and x, and the edge xy, makes an induced path between z, z_0, of lengths $h_{j+1}+1, \ldots, h_{j+1}+j$. But also there is an induced path between z, z_0 of length h_{j+1}, since $z \in L_{h_{j+1}}$; and so $z \in T_{j+1}$, a contradiction. This proves (2).

Now every vertex in L_k with an ancestor in T_j has an ancestor in $S_{j+1} \cup T_{j+1}$. Since $\theta(T_j) - \theta(T_{j+1}) > \tau^{k+1-h_{j+1}}$, it follows that $\theta(S_{j+1}) > \tau^{k+1-h_{j+1}}$. By setting $h = h_{j+1}$ and $M_h = S_{j+1}$, we have shown that:

(3) *There exist h with $0 \le k-h \le \mu-\lambda-1$, and subsets $M_i \subseteq L_i$ for $h-\lambda \le i \le h$, with the following properties:*

- *$\theta(M_h) > \tau^{k+1-h}$;*
- *M_i is stable for $h-\lambda \le i < h$; and*
- *every vertex in M_{i+1} has a neighbour in M_i for $h-\lambda \le i < h$.*

Choose such a value of h, maximal. Suppose first that $\chi(M_h) \le \tau$. Since

$$\theta(M_h) > \tau^{k-h+1} \ge \tau \ge \chi(M_h)$$

it follows that $h \neq k$. Take a partition of M_h into τ stable sets; then for one of these sets, say M'_h, $\theta(M'_h) \geq \theta(M_h)/\tau > \tau^{k-h}$. Let M_{h+1} be the set of vertices in L_{h+1} with a neighbour in M_h; then $\theta(M_{h+1}) = \theta(M'_h) > \tau^{k-h}$, contrary to the maximality of h. This proves that $\chi(M_h) > \tau$.

Let $Z = L_h \cup \cdots \cup L_k$; then $G[Z]$ is connected since $G[L_k]$ is connected and for $0 \leq i < k$, every vertex in L_i has a neighbour in L_{i+1}. Consequently

$$(L_0, \ldots, L_{h-\lambda-1}, M_{h-\lambda}, \ldots, M_{h-1}, Z, s)$$

is a shower $\mathcal{S}'$ say. Its floor includes M_h and so has chromatic number more than τ. Moreover, every jet for $\mathcal{S}'$ is also a jet for $\mathcal{S}$; and so $\mathcal{S}'$ is ν-incomplete. This proves 6.1. ∎

7 U-bends

For $\nu \geq 2$, a shower $(L_0, \ldots, L_k, s)$ is a *ν-sprinkler* if

- $G[L_k]$ is a path with one end s and with at least ν vertices; let its vertices be v_1-$\cdots$-v_n in order, where $v_1 = s$ and $n \geq \nu$;
- for $i = 1, \ldots, n - \nu$, no vertex in L_{k-1} is adjacent to v_i; and
- for $i = n - \nu + 1, \ldots, n$, some vertex in L_{k-1} is adjacent to v_i and to no other vertex in L_k.

Every ν-sprinkler is therefore ν-complete. We call $\{v_i : n - \nu + 1 \leq i \leq n\}$ its *floor*.

We need another object, a "u-bend", which is not exactly a shower; and also something which is partway to a u-bend, which we call a "w-bend". We start with the latter. Let $(L_0, \ldots, L_k)$ be a levelling in G with vertex set V, and let U be an induced path of G. Suppose that

- $G[L_k]$ is an induced path;
- $V \cap V(U) = \emptyset$;
- U has ends w, s, and there is at least one vertex in L_{k-1} adjacent to w and to a vertex in L_k; and
- there are no edges between $V(U)$ and L_k, and no vertex in L_{k-1} has a neighbour in L_k and a neighbour in $V(U) \setminus \{w\}$.

In this case, we call $(L_0, \ldots, L_k, U)$ a *w-bend*, and call s its *drain*; and any induced path of $G[V \cup V(U)]$ between the vertex in L_0 and the drain is called a *jet* of the w-bend. We call L_k its *floor*. Let $G[L_k]$ have ends v_1, v_2; then $d_G(v_1, v_2)$ is called the *size* of the w-bend. If in addition:

- w has a unique neighbour in L_{k-1}, say v;
- v has a unique neighbour in L_k, and this neighbour is an end of the path $G[L_k]$; and
- every vertex in L_{k-1} has a neighbour in L_k;

then we call $(L_0, \ldots, L_k, U)$ a *u-bend*. We need a containment relation for these objects:

- Let $\mathcal{S} = (L_0, \dots, L_k, s)$ and $\mathcal{S}' = (L'_0, \dots, L'_k, s')$ be showers. We say that $\mathcal{S}'$ is *contained in* $\mathcal{S}$ if they have the same drain, and $L'_i \subseteq L_i$ for $0 \le i \le k$.
- Let $\mathcal{S} = (L_0, \dots, L_k, s)$ be a shower, and let $\mathcal{S}' = (L'_0, \dots, L'_k, U)$ be a w-bend. We say that $\mathcal{S}'$ is *contained in* $\mathcal{S}$ if they have the same drain, and $L'_i \subseteq L_i$ for $0 \le i \le k$, and $V(U) \subseteq L_k$.
- Let $\mathcal{S} = (L_0, \dots, L_k, W)$ be a w-bend, and let $\mathcal{S}' = (L'_0, \dots, L'_k, U)$ be a u-bend. We say that $\mathcal{S}'$ is *contained in* $\mathcal{S}$ if they have the same drain, and $L'_i \subseteq L_i$ for $0 \le i \le k$, and $V(U) \subseteq L_k \cup V(W)$.

In all three cases, every jet of $\mathcal{S}'$ is a jet of $\mathcal{S}$.

We need to show that certain showers contain u-bends, and it is easier to show that they contain w-bends. Let us see first that that is enough, because a w-bend contains a u-bend (and containment is clearly transitive).

7.1 *Let $(L_0, \dots, L_k, W)$ be a w-bend in a triangle-free graph G, with size at least $2p + 4$. Then it contains a u-bend with size at least p.*

Proof. Let W have ends w, s where s is the drain. Let $G[L_k]$ have vertices $v_0\text{-}\cdots\text{-}v_n$ say, in order. Since $d_G(v_0, v_n) \ge 2p + 4$, we may assume by exchanging v_0, v_n if necessary that $d_G(w, v_0) \ge p + 2$. Let Y be the set of vertices in L_{k-1} adjacent to w and to a vertex in L_k. By hypothesis, $Y \ne \emptyset$. Choose $i \le n$ minimum such that v_i has a neighbour in Y, say v. Since $d_G(w, v_0) \ge p + 2$, and $d_G(w, v_i) = 2$, it follows that $d_G(v_i, v_0) \ge p$. Let L'_{k-1} consist of all vertices in L_k with a neighbour in $\{v_0, \dots, v_{i-1}\}$, together with v. Then v is the unique neighbour of w in L'_{k-1}; and so

$$(L_0, \dots, L_{k-2}, L'_{k-1}, \{v_0, \dots, v_i\}, W)$$

is a u-bend contained in $(L_0, \dots, L_k, W)$, and its size is at least p. This proves 7.1. ∎

7.2 *Let $\nu \ge 2$ be an integer, and let $\mu \ge 1$. Let $\mathcal{S}$ be a shower in a triangle-free graph G. Let P be an induced path of G with $V(P)$ a subset of the floor of $\mathcal{S}$, with ends w_1, w_2 such that $d_G(w_1, w_2) \ge 2(\mu + \nu)$. Then $\mathcal{S}$ contains either:*

- *a ν-sprinkler with floor a subset of $V(P)$, or*
- *a u-bend with size at least μ and with floor a subset of $V(P)$.*

Proof. Let $\mathcal{S} = (L_0, \dots, L_k, s)$, and let L^1_{k-1} be the set of vertices in L_{k-1} with a neighbour in $V(P)$. If $s \in V(P)$, let $u = s$ and let D be the one-vertex path with vertex s. If $s \notin V(P)$, then since $G[L_k]$ is connected, there is an induced path D of $G[L_k]$ between s and a vertex with a neighbour in $V(P)$; choose a minimal such path D, with ends s, u say. From the minimality of D, no vertex in $D \setminus \{u\}$ has a neighbour in $V(P)$.

Suppose that some vertex of $D \setminus \{u\}$ has a neighbour in L^1_{k-1}; and choose such a vertex, w say, such that the subpath D' of D between w, s is minimal. Then

$$(L_0, \dots, L_{k-2}, L^1_{k-1}, V(P), D')$$

is a w-bend contained in $\mathcal{S}$, of size at least $2(\mu+2)$, and the result follows from 7.1. We may therefore assume that there are no edges between $D \setminus \{u\}$ and L^1_{k-1}.

Let Y be the set of vertices in L^1_{k-1} that are adjacent to u. Now no vertex of D except possibly u has a neighbour in L^1_{k-1}; and u has at least one neighbour in $V(P) \cup Y$. Let $G[L_k]$ have vertices $v_0\text{-}\cdots\text{-}v_n$ in order. By hypothesis, $d_G(v_0, v_n) \geq 2(\mu + \nu)$, so by exchanging v_0, v_n if necessary, we may assume that $d_G(u, v_0) \geq \mu + \nu$. Choose i minimum such that v_i has a neighbour in $Y \cup \{u\}$.

Suppose first that v_i has a neighbour in Y. Choose such a neighbour v say, and let L^2_{k-1} be the set of vertices in L_{k-1} with a neighbour in $\{v_0, \ldots, v_{i-1}\}$, together with v. Now v_i is not adjacent to u (since G is triangle-free); and $d_G(v_0, v_i) \geq d_G(v_0, u) - 2 \geq \mu$; so

$$(L_0, \ldots, L_{k-2}, L^2_{k-1}, \{v_0, \ldots, v_i\}, D)$$

is a u-bend contained in $\mathcal{S}$ with size at least μ, as required.

We may assume then that v_i has no neighbour in Y, and therefore v_i is adjacent to u. In summary, no vertex in L_{k-1} has a neighbour in $V(D)$ and a neighbour in $\{v_0, \ldots, v_i\}$; and there are no edges between $V(D)$ and $\{v_0, \ldots, v_i\}$ except the edge uv_i. Since $d_G(v_0, u) \geq \mu + \nu$, it follows that $i \geq \mu + \nu - 1$, and so $i - \nu + 1 \geq \mu$.

Suppose next that there exists a vertex in L_{k-1} adjacent to at least two of $v_{i-\nu+1}, \ldots, v_i$. Choose j with $i - \nu + 3 \leq j \leq i$ maximum such that some vertex in L_{k-1} is adjacent to v_j and to one of $v_0, \ldots, v_{j-2}$; choose h with $0 \leq h \leq j - 2$ minimum such that some vertex in L_{k-1} is adjacent to v_h, v_j; and choose $v \in L_{k-1}$ adjacent to v_h, v_j. Let L^3_{k-1} be the set of vertices in L_{k-1} with a neighbour in $\{v_0, \ldots, v_{h-1}\}$, together with v. Then since there is a path between u, v_h (via v) of length $j - i + 3 \leq \nu$, it follows that $d_G(u, v_h) \leq \nu$, and so

$$d_G(v_h, v_0) \geq d_G(u, v_0) - \nu \geq \mu.$$

Let D_2 be the path formed by the union of D and the path $u\text{-}v_i\text{-}\cdots\text{-}v_j$. Then

$$(L_0, \ldots, L_{k-2}, L^3_{k-1}, \{v_0, \ldots, v_h\}, D_2)$$

is a u-bend contained in $\mathcal{S}$, of size at least μ, as required.

We may therefore assume that no vertex in L_{k-1} is adjacent to more than one of $v_{i-\nu+1}, \ldots, v_i$. Let L^4_{k-1} be the set of vertices in L_{k-1} with a neighbour in $\{v_{i-\nu+1}, \ldots, v_i\}$. Every vertex in $\{v_{i-\nu+1}, \ldots, v_i\}$ has a neighbour in L^4_{k-1}, and u has no neighbour in L^4_{k-1}, so

$$(L_0, \ldots, L_{k-2}, L^4_{k-1}, V(D) \cup \{v_{i-\nu+1}, \ldots, v_i\}, s)$$

is a ν-sprinkler contained in $\mathcal{S}$. This proves 7.2. ∎

8 Jets of a shower

Let $L_0, \ldots, L_k$ be the levels of a shower or w-bend, and let J be a jet. Then at least one vertex of J belongs to L_{k-1}; and we define the *tail* of J to be the minimal subpath of J between L_{k-1} and the drain. For $\lambda \geq 0$, we say that J is λ-*monotone* if $\lambda \leq k$, and J contains exactly one vertex of L_i for $0 \leq i < k - \lambda$. In every jet J, at least $k - 1$ edges do not belong to its tail and have an end not in L_k. We say the *waste* of J is μ if there are $k - 1 + \mu$ edges of J that do not belong to its tail and have an end not in L_k; and J is μ-*wasteful* if its waste is at most μ. Thus the waste is nonnegative.

A set of integers $\mathcal{A}$ is *dense* if for all $a_1, a_2 \in \mathcal{A}$ with $a_1 < a_2$, there does not exist b with $a_1 < b < a_2$ such that $b, b+1 \notin \mathcal{A}$; that is, there are no two consecutive numbers both missing from $\mathcal{A}$ between the first and last members of $\mathcal{A}$. If $\mathcal{A}, \mathcal{B}$ are sets of integers, we define $\mathcal{A}+\mathcal{B} = \{a+b : a \in \mathcal{A}, b \in \mathcal{B}\}$. Thus if $\mathcal{A}$ is dense, then for any integer t, $\mathcal{A} + \{t, t+1\}$ is a set of consecutive integers of cardinality at least $|\mathcal{A}| + 1$.

Any subset of the floor of a shower is called a *mat*, and the only mat for a w-bend is its floor. The *size* of a mat M is the maximum of $d_G(w_1, w_2)$ over all pairs w_1, w_2 of vertices in the same component of $G[M]$. If M is a mat for a shower or w-bend $\mathcal{S}$, a jet J is an *M-jet* if there is no edge of J with an end in $L_k \setminus M$ and an end in L_{k-1}. We define the *M-jetset* as the set of all lengths of M-jets. A w-bend $(L_0, \ldots, L_k, U)$ is *λ-stable* if $k \geq \lambda$ and L_i is stable for $k - \lambda \leq i \leq k-1$. In this section we prove the following.

8.1 *Let $\nu \geq 2$ be an integer, and let G be a triangle-free graph. If $\mathcal{S}$ is a ν-stable shower or w-bend in G, and M is a mat for $\mathcal{S}$ of size at least $3^{\nu+2}$, then there is a set $\mathcal{A}$ of integers, realized by a set of $(\nu+1)$-monotone, $3\nu^2$-wasteful M-jets, such that $|\mathcal{A}| \leq \nu + 1$, and $\mathcal{A}$ includes a dense subset of cardinality ν, and two members of $\mathcal{A}$ differ by 1 or 3.*

Proof. We proceed by induction on ν. Thus we assume that either $\nu = 2$ or the result holds for $\nu - 1$. We claim we may assume:

(1) *There is a u-bend $\mathcal{S}_1 = (L_0, \ldots, L_k, U)$ contained in $\mathcal{S}$, and with $L_k \subseteq M$, of size at least $3^{\nu+2}/2 - \nu$.*

Assume first that $\mathcal{S}$ is a ν-stable shower in G, and M is a mat of size at least $3^{\nu+2}$. Let P be an induced path of $G[M]$ with ends w_1, w_2, where $d_G(w_1, w_2) \geq 3^{\nu+2}$. If $\mathcal{S}$ contains a ν-sprinkler with floor a subset of $V(P)$, then the theorem holds, so we assume not. By 7.2 with $\mu = 3^{\nu+2}/2 - \nu$, it follows that $\mathcal{S}$ contains a u-bend as in the claim. Next we assume that $\mathcal{S}$ is a w-bend, of size at least $3^{\nu+2}$; then the claim follows from 7.1. This proves (1).

Let $\mathcal{S}_1 = (L_0, \ldots, L_k, U)$ as in (1). Let U have ends u, s where s is the drain. Let q_0 be the unique neighbour of u in L_{k-1}; and let D be the path formed by adding the edge uq_0 to U. There is an induced path q_0-q_1-$\cdots$-q_n such that $\{q_1, \ldots, q_n\} = L_k$; and every vertex in L_{k-1} has a neighbour in L_k. Also, $d_G(q_1, q_n) \geq 3^{\nu+2}/2 - \nu$, and so $d_G(q_0, q_n) \geq 3^{\nu+2}/2 - \nu - 1$. We may assume that for $0 \leq i \leq k-1$ every vertex in L_i has a neighbour in L_{i+1} (because any other vertex could be removed). Let $V = L_0 \cup \cdots \cup L_k$.

We recall that for $v \in V$, its *height* $h(v) = k - i$ where $v \in L_i$; and we define the *reach* of v to be the maximum i such that q_i is a descendant of v. (Since every vertex in V has a descendant in $V(Q)$, this is well-defined.) We may assume that:

(2) *For $1 \leq m \leq n$ there do not exist induced paths R_1, R_2 of $G[V]$ between q_0 and q_m with the following properties:*

- *$|E(R_1)| + 1 = |E(R_2)| \leq 2\nu + 2$; and*
- *for all j with $m < j \leq n$, q_j has no neighbour in $V(R_1 \cup R_2) \setminus \{q_m\}$.*

For suppose that such m, R_1, R_2 exist. Since R_1, R_2 both have length at most $2\nu + 2$ and have ends in L_k and L_{k-1}, it follows that every vertex of $R_1 \cup R_2$ has height at most $\nu + 1$. Indeed, if $y \in V(R_1 \cup R_2)$ then there is a subpath of one of R_1, R_2 between y and q_m of length at least $h(y)$, and since R_1, R_2 both have length at most $2\nu+2$, it follows that $d_G(y, q_0) \le 2\nu+2-h(y)$. Consequently, if $x \in V$ has a neighbour (say y) in $R_1 \cup R_2$ then

$$d_G(x, q_0) \le d_G(y, q_0) + 1 \le 2\nu - h(y) + 3 \le 2\nu - h(x) + 4.$$

It follows that for every descendant in L_k of such a vertex x, its distance from q_0 is at most $d_G(x, q_0)+h(x) \le 2\nu + 4$. Since

$$d_G(q_0, q_n) \ge 3^{\nu+2}/2 - \nu - 1 > 2\nu + 4,$$

there exists $m' < n$ such that $d_G(q_0, q_{m'}) = 2\nu+4$, and $d_G(q_0, q_j) > 2\nu+4$ for all j with $m' < j \le n$. Since q_{m+1} has a neighbour in R_1, it follows that $d_G(q_{m+1}, q_0)) \le 2\nu + 4$, and so $m' \ge m + 1$. For $0 \le i < k$ let L'_i be the set of all vertices in L_i with a descendant in $\{q_j : \ m' < j \le n\}$. It follows that

$$(L'_0, \dots, L'_{k-1}, \{q_j : \ m \le j \le n\}, q_m)$$

is a shower $\mathcal{S}'$ say. It is ν-stable, since $L'_i \subseteq L_i$ for $0 \le i < k$. (It is not contained in $\mathcal{S}$ since the drain is different.) Let its vertex set be V'. If $v \in V' \setminus \{q_m\}$, and $v \in L_k$, then v has no neighbour in $V(R_1 \cup R_2) \setminus \{q_m\}$ from the properties of R_1, R_2; and if $v \notin L_k$, then v has a descendant in $\{q_j : \ m' < j \le n\}$, which therefore has distance in G more than $2\nu + 4$ from q_0, and again v has no neighbour in $R_1 \cup R_2$. Thus there are no edges between $V' \setminus \{q_m\}$ and $V(R_1 \cup R_2)$ except the edge $q_m q_{m+1}$.

Now

$$d_G(q_n, q_{m'+1}) \ge d_G(q_n, q_0) - (2\nu + 5) \ge 3^{\nu+2}/2 - \nu - 1 - (2\nu + 5) \ge 3^{\nu+1}.$$

If $\nu > 2$, then from the inductive hypothesis on ν, applied to $\mathcal{S}'$ and the mat $M' = \{q_{m'+1}, \dots, q_n\}$, we deduce that there is a dense subset $\mathcal{A}$ of the M'-jetset of $\mathcal{S}'$ of cardinality $\nu - 1$, realized by a set of M'-jets of $\mathcal{S}'$ that are (ν)-monotone and $3(\nu - 1)^2$-wasteful. If $\nu = 2$, let $\mathcal{A}$ be a singleton set containing the length of a 0-monotone, 0-wasteful M'-jet of $\mathcal{S}'$. In either case, let J be an M'-jet in this set. Its tail has exactly one edge not in the path q_m-q_{m+1}-$\cdots$-$q_{m'}$, and so at most $3(\nu - 1)^2 + 1$ edges of J have an end not in L_k. Moreover, both $J \cup R_1 \cup D$ and $J \cup R_2 \cup D$ are jets of $\mathcal{S}_1$, and they are both $(\nu + 1)$-monotone (since every vertex of $R_1 \cup R_2$ has height at most $\nu + 1$). Since R_1, R_2 have length at most $2\nu + 2$, it follows that these two jets both have waste at most at most $3(\nu - 1)^2 + 1 + 2\nu + 2 \le 3\nu^2$. Let $|E(R_1)| + |E(D)| = t$; then $|E(R_2)| + |E(D)| = t + 1$, so for each $a \in \mathcal{A}$, both $a+t, a+t+1$ belong to the jetset of $\mathcal{S}_1$, and so $\mathcal{A} + \{t, t+1\}$ is a subset of the jetset of $\mathcal{S}_1$, and hence of the M-jetset of $\mathcal{S}$, and this is a set of at least ν consecutive integers. And this set is realized by M-jets of $\mathcal{S}$ that are $(\nu + 1)$-monotone and have waste at most $3\nu^2$. Thus in this case the theorem holds. Consequently we may assume that no such m, R_1, R_2 exist. This proves (2).

For each vertex $v \in V$ with reach $r < n$, let $f(v) \in V$ be defined as follows. There is a monotone path between v and q_r; let X be the set of all vertices x such that x is adjacent to a vertex in a monotone path between v and q_r. Consequently $q_{r+1} \in X$, and so there exists $x \in X$ with reach greater than r. Choose such a vertex x with maximum reach, and define $f(v) = x$. If v has reach n let $f(v) = v$.

Let $v_1 = q_0$, and for $1 \le i \le \nu - 1$ let $v_{i+1} = f(v_i)$. We need to establish several properties of this sequence. Let $t \le \nu$ be maximum such that $v_t \ne v_{t-1}$. Thus either $t = \nu$ or v_t has reach n. For $1 \le i \le t$, r_i be the reach of v_i; then $r_1 = 1$, and $r_i < r_{i+1}$ for $1 \le i < t$. For $1 \le i \le t$ let P_i be a monotone path between v_i and q_{r_i} such that if $i < t$ then v_{i+1} has a neighbour in P_i. The paths $P_1, \dots, P_t$ are pairwise vertex-disjoint, because the reach of every vertex in P_i is precisely r_i, and $r_1, \dots, r_t$ are all different. For $1 \le i < t$ let B_i be an induced path of $G[V(P_i) \cup \{v_{i+1}\}]$ between v_i and v_{i+1}. Thus for $1 \le i \le t$, $B_1 \cup B_2 \cup \dots \cup B_{i-1} \cup P_i$ is a path, say C_i, between v_1 and q_{r_i}. Then B_i has length at least one; let y_i be the vertex of B_i adjacent to v_{i+1}. For $1 \le i \le t$, let $\epsilon_i = 1$ if $v_{i+1}, y_i \in L_k$, and 2 otherwise.

(3) *$t = \nu$; for $1 \le i < \nu$, B_i has length $h(v_i) - h(v_{i+1}) + \epsilon_i$; for $1 \le i \le \nu$, C_i has length*

$$\sum_{1 \le j < i} \epsilon_j + 1 - h(v_i);$$

and for $1 \le i \le \nu$, C_i is an induced path.

Let $1 \le i < t$. Since $h(y_i) \le h(v_i)$, and $h(v_{i+1}) \le h(y_i) + 1$, it follows that $h(v_{i+1}) \le h(v_i) + 1$; and since $h(v_1) = 1$, it follows inductively that $h(v_i) \le i$ for $1 \le i \le t$. Consequently for $1 \le i < t$, y_i has height at most $\nu - 1$; and since the levelling is ν-stable, it follows that y_i, v_{i+1} do not have the same height unless they both have height zero. Moreover, v_{i+1} is not a child of y_i, since the reach of v_{i+1} is greater than the reach of y_i; so we have proved that either v_{i+1} is a parent of y_i, or v_{i+1}, y_i both have height zero. It follows that the length of B_i equals $h(v_i) - h(v_{i+1}) + \epsilon_i$, for all $i < t$.

For $1 \le i \le t$, the path $B_1 \cup B_2 \cup \dots \cup B_{i-1}$ therefore has length

$$\sum_{1 \le j < i} \epsilon_j + 1 - h(v_i),$$

and since P_i has length $h(v_i)$, it follows that C_i has length $1 + \sum_{1 \le j < i} \epsilon_j$. Since this quantity is less than 2ν, and $d_G(u, q_n) \ge 3^{\nu+2} > 2\nu$, it follows that $r_i < n$. In particular, $r_t < n$, and so $t = \nu$.

We claim that for $1 \le i \le \nu$, the path C_i is induced; and prove this by induction on i. Certainly C_1 is induced, so we may assume inductively that $i < \nu$ and C_i is induced, and we prove that C_{i+1} is induced. Now C_{i+1} is obtained from a subpath of C_i by adding the edge $y_i v_{i+1}$ and the path P_{i+1}; so it suffices to check that there are no edges between $B_1 \cup B_2 \cup \dots \cup B_i$ and P_{i+1} except the edge $y_i v_{i+1}$. Suppose then that $y \in V(B_j)$ for some $j \le i$, and $x \in V(P_{i+1})$, and xy is an edge. Since the reach of x equals r_{i+1}, it follows that x has no neighbour in any of $P_1, \dots, P_{i-1}$, and so $y \in V(P_i)$. Since also $y \in V(B_j)$ for some $j \le i$, it follows that $y \in V(B_i \cap P_i)$. Since B_i is induced and we may assume that $(x, y) \ne (v_{i+1}, y_i)$, it follows that $x \ne v_{i+1}$, and so $h(v_{i+1}) > 0$ and $h(x) < h(v_{i+1})$. Since $h(v_{i+1}) > 0$, also v_{i+1} is a parent of y_i, and so $h(x) \le h(y_i)$. But $h(y) \ge h(y_i)$, and since the levelling is ν-stable and xy is an edge, it follows that y is a parent of x. But this is impossible since the reach of x is greater than the reach of y. This proves that each C_i is induced, and so completes the proof of (3).

For $1 \le j \le n$, let A_j be a monotone path between q_j and the shower head z_0. Thus A_j has length k. For $1 \le i \le \nu$, the reach of every vertex in A_{r_i+1} is at least r_i+1, and so is greater than the reach of every vertex in C_i; and so there is a path J_i formed by the union of D, C_i, the edge $q_{r_i} q_{r_i+1}$, and A_{r_i+1}.

(4) *For $1 \le i \le \nu$ the path J_i is induced.*

Suppose that some J_t is not induced, where $1 \le t \le \nu$. Consequently some vertex x of A_{r_t+1} is adjacent to some vertex y of C_t, and $(x, y) \neq (q_{r_t}, q_{r_t+1})$. Choose such a pair x, y with x of minimum height. Since y has height at most ν, it follows that $h(x) \neq h(y)$; and x is not a child of y since the reach of x is greater than the reach of y. Thus x is a parent of y. Let $y \in V(P_j)$ where $j \le t$. Since x has a neighbour in P_j, it follows that the reach of x is at most r_{j+1}; and so $r_t < r_{j+1}$. Consequently $t < j + 1$, and since $j \le t$ it follows that it follows that $j = t$, and so $y \in V(P_t)$. Let a be the vertex of A_{r_t+1} of height 1. Now there are two cases. First suppose that a is nonadjacent to q_j for $r_t + 2 \le j \le n$. Let $i_1 = r_t + 1$, and let R_1 be the path formed by the union of C_t and the edge $q_{r_t}q_{r_t+1}$, and let R_2 be the path formed by the union of the subpath of C_t between u, y, the edge xy, and the subpath of A_{r_t+1} between x, q_{r_t+1}. Note that R_1 is induced by (2), and R_2 is induced since we chose xy with x of minimum height. Also R_1 has length at most 2ν, and R_2 has length one more. This is therefore impossible by (2). Consequently there exists $j > r_t + 1$ adjacent to a; choose such a value of j, maximum, and let $i_1 = j$. Let R_2 be the path formed by the union of C_t and the path q_{r_t}-q_{r_t+1}-a-q_j, and let R_1 be the path formed by the union of the subpath of C_t between u, y, the edge xy, the subpath of A_{r_t+1} between x, a, and the edge aq_j. In this case R_2 has length at most $2\nu + 2$, and R_1 has length one less. Since $j \ge r_t + 3$ (because G is triangle-free) it follows that both paths are induced, and again this contradicts (2). Thus there is no such t. This proves (4).

Since each J_i is induced, it is therefore a jet for the u-bend $\mathcal{S}_1$ (and hence an M-jet for $\mathcal{S}$), of length $k + 2 + \sum_{1 \le j < i} \epsilon_j + |V(D)|$, and with tail the path D; and since $J_i \setminus V(D)$ has length at most $k + 2\nu$, and all vertices of B_i have height at most ν, it follows that J_i is ν-monotone and 2ν-wasteful (and hence $3\nu^2$-wasteful). The shortest of these jets is J_1, and it has length $k + 1 + |V(D)|$. Let A_0 be a monotone path between v_1 and z_0; then also there is an M-jet formed by the union of D, the edge uv_1, and A_0, of length $k - 2 + |V(D)|$ (so, three less than the length of J_1). Consequently these M-jets realize a subset of the M-jetset satisfying the theorem. This proves 8.1. ∎

The previous result will have several applications later in the paper. First, let us use it to convert a λ-stable shower into a stable shower.

8.2 *Let $\kappa, \tau \ge 0$ and $\nu \ge 2$ be integers, and let $\rho = 3^{\nu+2}$. Let G be a triangle-free graph such that G has no hole ν-sequence, and $\chi(N^\rho[v]) \le \kappa$, for every vertex $v \in V(G)$. If G admits a ν-incomplete $(\nu+2)$-stable shower with floor of chromatic number more than $\kappa+\tau$, then G admits a ν-incomplete stable shower with floor of chromatic number more than τ.*

Proof. Let $\mathcal{S}$ be a ν-incomplete $(\nu + 2)$-stable shower $(L_0, \dots, L_k, s)$ in G. Thus $k \ge \nu + 2$. Let $j = k - \nu - 2$; then L_i is stable for $j \le i < k$. Let $L_0 = \{z_0\}$. Let X be the set of all vertices $v \in L_j$ such that there is an induced path P_v of G between v, z_0 with length $j + 1$, such that every vertex in P_v different from v belongs to one of $L_0, \dots, L_{j-1}$. Let $Y = L_j \setminus X$. Let X' be the set of vertices in L_k with an ancestor in X, and Y' the set of vertices in L_k with an ancestor in Y. Thus $X' \cup Y' = L_k$.

Suppose that $\chi(G[X']) > \kappa$. For $j \le i \le k - 1$, let L'_i be the set of vertices in L_i with an ancestor in X. Then

$$(L_0, \dots, L_{j-1}, L'_j, \dots, L'_{k-1}, L_k, s)$$

is a ν-stable shower $\mathcal{S}_1$ say, and its floor includes X'. This is contained in $\mathcal{S}$, so $\mathcal{S}_1$ is ν-incomplete. Since $\chi(C) > \kappa$, there exist $w_1, w_2 \in X'$, in the same component of $G[X']$, with $d_G(w_1, w_2) > \rho \geq 3^{\nu+2}$. By 8.1 there is a dense subset $\mathcal{A}$ of the jetset of $\mathcal{S}_1$ of cardinality ν, and a set $\{J_a : a \in \mathcal{A}\}$ of $(\nu+1)$-monotone jets for $\mathcal{S}_1$ realizing $\mathcal{A}$. Thus for each $a \in \mathcal{A}$, J_a contains exactly one vertex of L'_i for $0 \leq i \leq j$. In particular, J_a contains exactly one vertex in $L'_j = X$, say x. The subpath of J_a between x, z_0 has length j, and so the subpath R_a say of J_a between x, s has length $|E(J_a)| - j$. By definition of X, the path P_x exists and has length $j+1$; and since both R_a, P_x have exactly one vertex in L_j, their union $R_a \cup P_x$ is an induced path between s, z_0 of length exactly one more than the length of J_a. Now both J_a and $R_a \cup P_x$ are jets of $\mathcal{S}_1$ and hence of $\mathcal{S}$. Thus $\mathcal{A} + \{0, 1\}$ is a subset of the jetset of $\mathcal{S}$. But this set consist of at least $\nu + 1$ consecutive integers, since $\mathcal{A}$ is dense of cardinality ν; and this is impossible since $\mathcal{S}$ is not ν-complete. This proves that $\chi(G[X']) \leq \kappa$.

Consequently $\chi(G[Y']) > \tau$. For $0 \leq i \leq j$, let L'_i be the set of vertices in L_i with a descendant in Y, and for $j+1 \leq i \leq k$, let L'_i be the set of vertices in L_i with an ancestor in Y. Then $(L'_0, \ldots, L'_{k-1}, L_k, s)$ is a shower $\mathcal{S}'$ say, with floor of chromatic number more than τ since its floor includes Y'. This is contained in $\mathcal{S}$, so $\mathcal{S}'$ is ν-incomplete. We claim that $\mathcal{S}'$ is stable. For certainly $L_j, \ldots, L_{k-1}$ are stable, since $\mathcal{S}$ is $(\nu+2)$-stable. Suppose that $0 \leq i \leq j-1$ and $y, y' \in L'_i$ are adjacent. Since y has a descendant in Y, there is a path between y and Y of length $j - i$; and since $y' \in L_i$, there is a path between y', z_0 of length i. Since G is triangle-free, yy' is the only edge between these two paths; and so their union, together with this edge, is an induced path between y, z_0 of length $j+1$, contradicting that $y \notin X$. This proves that $\mathcal{S}'$ is stable; and so the theorem holds. This proves 8.2. ∎

We deduce:

8.3 *Let $\tau \geq 0$ and $\nu \geq 2$ be integers, and let $\rho = 3^{\nu+2}$. Let G be a triangle-free graph, such that G has no hole ν-sequence, and $\chi(N^\rho[v]) \leq \kappa$ for every vertex $v \in V(G)$. If $\chi(G) > 44\nu(\kappa+\tau)^{(\nu+1)^2} + 4\kappa$, then G admits a ν-incomplete stable shower with floor of chromatic number more than τ.*

Proof. By 5.4, there is a ν-incomplete shower $(L_0, \ldots, L_k, s)$ in G, with floor of chromatic number more than $\nu(\kappa+\tau)^{(\nu+1)^2}$. Then $k > \rho$, since $\nu(\kappa+\tau)^{(\nu+1)^2} \geq \kappa$. Since $\rho \geq (\nu+3)(\nu-1)+1$, 6.1 (with $\lambda = \nu+2$) implies that there is a $(\nu+2)$-stable ν-incomplete shower in G with floor of chromatic number more than $\kappa + \tau$, so the result follows from 8.2. This proves 8.3. ∎

The reason for controlling the waste of the jets that are output by 8.1 is that a jet with bounded waste can be covered by a bounded number of monotone paths. More precisely:

8.4 *Let $\mathcal{S} = (L_0, \ldots, L_k, s)$ be a shower in a graph G, and let J be a μ-wasteful jet of $\mathcal{S}$. Then there is a set of at most $\mu+1$ monotone paths of $\mathcal{S}$ such that every vertex of J in $L_0 \cup \cdots \cup L_{k-1}$ belongs to one of these paths.*

Proof. Choose $d \in V(J)$ such the tail T of J has ends d, s. Then no vertex of T belongs to $L_0 \cup \cdots \cup L_{k-1}$ except d. Let P be the subpath of J between z_0, d, where $z_0 \in L_0$. At most $k-1+\mu$ edges of P have an end not in L_k, since the waste of J is at most μ. Let us say the *height* of an edge uv of P is the maximum of the heights of u, v. Thus at most $k-1+\mu$ edges of P have nonzero height. As P is traversed starting from d, the number of edges in it that have height at least 2 and different from the heights of all previous edges is at least $k-1$, since the difference of the heights of z_0, d is $k-1$; and so there are at most μ edges of P that have height 1 or the same nonzero height as some earlier edge. By removing all such edges, we decompose P into at most $\mu+1$ paths each of which is either monotone or a path of $G[L_k]$; and every vertex of P in $L_0 \cup \cdots \cup L_{k-1}$ belongs to one of these monotone paths. This proves 8.4. ∎

9 Stable showers

From now on, there is no need to consider general showers; we might as well just concern ourselves with stable showers, in view of 8.3. To complete the proof of 5.1, we only need to show that if ν, κ, G satisfy the hypotheses of 5.1 then every ν-incomplete stable shower in G has floor with bounded χ, and that is the goal of the remainder of the paper.

Here is a useful fact about stable showers, that will help to control them. If $\mathcal{S} = (L_0, \ldots, L_k)$ is a levelling, and $X \subseteq L_0 \cup \cdots \cup L_k$, we denote by $\Theta(X)$ or $\Theta_{\mathcal{S}}(X)$ the set of vertices in L_k that have an ancestor in X, and by $\theta(X)$ or $\theta_{\mathcal{S}}(X)$ the chromatic number of $\Theta(X)$. (We use the same notation if $L_0, \ldots, L_k$ are the levels of a shower.)

We are concerned with a triangle-free graph which admits no hole ν-sequence; and we will not need to use induction on ν any more; so from now on we shall fix $\nu \geq 2$, to avoid having to carry it along. We might as well also set $\rho = 3^{\nu+2} + 4$, for the remainder of the paper, and fix $\kappa \geq 0$. Let us say a graph G is a *candidate* if G is triangle-free, and admits no hole ν-sequence, and $\chi(N^{\rho}[v]) \leq \kappa$ for every vertex v. Our eventual goal is to prove that every stable shower in every candidate has floor of bounded χ.

Let $\mathcal{S}$ be a stable shower, with vertex set V, and let M be a mat. For $X \subseteq V$, we denote the set of vertices in M with an ancestor in X by $M(X)$, or $M_{\mathcal{S}}(X)$ if there is danger of ambiguity (and we write $M(v)$ for $M(\{v\})$.)

We already defined "containment" for showers, but now we need a slightly different inclusion relation. Let $\mathcal{S} = (L_0, \ldots, L_k, s)$ be a stable shower, and let $\mathcal{S}' = (L'_i, \ldots, L'_{k'}, s')$ be a shower, both in a graph G. We say that $\mathcal{S}'$ is a *subshower* of $\mathcal{S}$ if

- $s = s'$
- $k' \leq k$; let $h = k - k'$; and
- $L'_i \subseteq L_{i+h}$ for $0 \leq i \leq k'$.

In particular, let M be a mat for $\mathcal{S}$, and let $z_1 \in L_h$, where $0 \leq h < k$; then we define the *subshower of $\mathcal{S}$ under z_1 and over M* to be $(L'_h, \ldots, L'_{k-1}, L_k, s)$, where L'_i is the set of all descendants of z_1 in L_i that have descendants in M.

If $\mathcal{S} = (L_0, \ldots, L_k, s)$ is a shower, we define its *union* $U(\mathcal{S})$ to be $L_0 \cup \cdots \cup L_{k-1}$. (Note that this is different from the vertex set, as we do not include L_k.)

9.1 *Let G be a candidate. Let $\mathcal{S} = (L_0, \ldots, L_k, s)$ be a stable shower in G, and let $z_1, z_2 \in U(\mathcal{S})$, either equal or nonadjacent. For $i = 1, 2$, let $\mathcal{S}_i$ be a subshower with union V_i and head z_i respectively, and let M_i be a mat for $\mathcal{S}_i$. Suppose that $V_1 \cap V_2 = \{z_1\} \cap \{z_2\}$, and $\chi(M_1) > \kappa$. Let X be the set of* *vertices in $V_2 \setminus \{z_1\}$ with a neighbour in $V_1 \setminus \{z_2\}$; and for every monotone path R in $G[V_1]$ between z_1 and M_1, let $X(R)$ denote the set of vertices in $V_2 \setminus \{z_1\}$ with a neighbour in $V(R) \setminus \{z_2\}$. Then either*

- *$z_1 \neq z_2$, and there are ν induced paths $Q_0, \dots, Q_{\nu-1}$ of $G[V_1 \cup V_2 \cup L_k]$ between z_1, z_2, such that $|E(Q_i)| = |E(Q_0)| + i$ for $0 \leq i < \nu$; or*
- *$\chi(M_2 \setminus \Theta(X)) \leq 2\kappa$, and for all $\tau \geq 0$, if $\chi(M_2) > 2\kappa + (\nu+1)(3\nu^2+1)\tau$, then there is a monotone path R of $G[V_1]$ between z_1 and M_1 such that $\chi(M_2(X(R))) > \tau$.*

Proof. Choose a component of $G[M_1]$ with maximum chromatic number; and since this chromatic number is larger than κ, it follows that there are two vertices of this component with distance more than ρ. Consequently there is a path P_1 with $V(P_1) \subseteq M_1$ joining two vertices with distance at least $3^{\nu+2}$ (in G). Choose a minimal such path P_1, and let w_1 be one of its ends. From the minimality of P_1 it follows that $d_G(w_1, v) \leq 3^{\nu+2}$ for every vertex v of P_1.

Let C_2 be a connected induced subgraph of $G[M_2 \setminus N^\rho[w_1]]$ with $\chi(C_2) > \kappa$ (if there is no such subgraph, then $\chi(M_2) \leq 2\kappa$ and the theorem holds). In addition, choose C_2 with $V(C_2) \cap \Theta_{\mathcal{S}_2}(X) = \emptyset$ if possible. Every path of G between C_2 and P_1 has length at least 3, since $\rho \geq 3^{\nu+2}+3$.

Let $(L^1_{h_1}, \dots, L^1_{k-1}, L_k, s)$ be the subshower $\mathcal{S}'_1$ of $\mathcal{S}_1$ above P_1, and let its union be V'_1. Since $G[L_k]$ is connected, there is a path of $G[L_k]$ between $V(P_1)$ and C_2; let D be a minimal path of $G[L_k]$ such that one end (say d_1) has a neighbour in $V(P_1) \cup L^1_{k-1}$ and the other (say d_2) has a neighbour in C_2.

(1) *There is a set $\mathcal{A}_1$ of integers, of cardinality at most $\nu+1$, including a dense subset of cardinality ν, and containing two integers x, y with $y - x \in \{1, 3\}$. For each $a \in \mathcal{A}_1$ there is an induced path J_a of G between d_1, z_1 of length a, such that*

- *$V(J_a) \subseteq V'_1 \cup \{d_1\}$; and*
- *there is a set of $3\nu^2+1$ monotone paths of $G[V'_1]$ between $V(P_1)$ and z_1, such that every vertex of $V(J_a) \setminus (V(P_1) \cup \{d_1\})$ belongs to one of these paths.*

Let D_1 be the one-vertex path with vertex d_1. If d_1 has no neighbour in $V(P_1)$, then

$$(L^1_{h_1}, \dots, L^1_{k-1}, V(P_1), D_1)$$

is a w-bend $\mathcal{S}'_1$ of size at least $3^{\nu+2}$; and otherwise $(L^1_{h_1}, \dots, L^1_{k-1}, V(P_1) \cup \{d_1\}, d_1)$ is a shower $\mathcal{S}'_1$. In either case we can apply 8.1 to $\mathcal{S}'_1$, and deduce that there is a subset $\mathcal{A}_1$ of the jetset of $\mathcal{S}'_1$, of size at most $\nu+1$, including a dense subset of cardinality ν, and containing two integers x, y with $y - x \in \{1, 3\}$; and realized by a set of jets of $\mathcal{S}'_1$ that are $3\nu^2$-wasteful. By 8.4, this proves (1).

Since $|\mathcal{A}_1| \leq \nu+1$, there is a set of at most $(\nu+1)(3\nu^2+1)$ monotone paths of $G[V'_1]$ between $V(P_1)$ and z_1 such that, if Y denotes the set of vertices in these paths, then $V(J_a) \subseteq Y \cup V(P_1) \cup \{d_1\}$ for each $a \in \mathcal{A}_1$. Let X' denote the set of vertices in $V_2 \setminus \{z_1\}$ with a neighbour in $Y \setminus \{z_2\}$. Let $z_2 \in L_{h_2}$, and for $h_2 \leq i \leq k$ let L^2_i be the set of vertices in L_i such that there is a monotone path of $G[V_2 \setminus X']$ between v, z_2. It follows that no vertex in $Y \setminus \{z_2\}$ has a neighbour in $(L^2_{h_2} \cup \dots \cup L^2_k) \setminus \{z_1\}$.

(2) *If $\chi(L_k^2 \cap V(C_2)) > \kappa$ then the theorem holds.*

For then there exists an induced path P_2 of $G[L_k^2 \cap V(C_2)]$ with ends at distance at least ρ. Since d_2 has a neighbour in C_2, it follows that $G[V(C_2) \cup V(D)]$ is connected. Thus

$$(\{z_1\}, L_{h+1}^2, \ldots, L_{k-1}^2, V(C_2) \cup V(D), d_1)$$

is a shower $\mathcal{S}_2'$, and $L_k^2 \cap V(C_2)$ is a mat M say; and by 8.1, there is a dense subset $\mathcal{A}_2$ of the M-jetset of $\mathcal{S}_2'$ of cardinality ν. We claim that $\mathcal{A}_1 + \mathcal{A}_2$ contains a set $\mathcal{B}$ of ν consecutive integers. To see this, suppose first that there are two consecutive integers $a, a+1 \in \mathcal{A}_2$. Let $\mathcal{A}'$ be a dense subset of $\mathcal{A}_1$ of cardinality ν; then $\mathcal{A}' + \{a, a+1\}$ consists of at least $\nu + 1$ consecutive integers, all contained in $\mathcal{A}_1 + \mathcal{A}_2$ as required. We may assume that that no two members of $\mathcal{A}_2$ are consecutive. Since $\mathcal{A}_2$ is dense of cardinality ν, there exists s such that $s, s+2, s+4, \ldots, s+2(\nu-1) \in \mathcal{A}_2$. But there exist $x, y \in \mathcal{A}_1$ with $y - x \in \{1, 3\}$; and then

$$\{s, s+2, s+4, \ldots, s+2(\nu-1)\} + \{x, y\}$$

contains ν consecutive integers (indeed, almost 2ν). This proves that $\mathcal{B}$ exists.

If $z_1 = z_2$ then for every J_a ($a \in \mathcal{A}_1$) and every M-jet of $\mathcal{S}_2'$, their union is a hole; and so G has holes of every length in $\mathcal{B}$, and so has a hole ν-sequence, which is impossible since G is a candidate. Thus $z_1 \neq z_2$, and so they are nonadjacent; but then for every J_a ($a \in \mathcal{A}_1$) and every M-jet of $\mathcal{S}_2'$, their union is an induced path between z_1, z_2 and the theorem holds. This proves (2).

We may therefore assume that $\chi(L_k^2 \cap V(C_2)) \leq \kappa$. Consequently, $V(C_2) \not\subseteq L_k^2$, and therefore $V(C_2) \cap \Theta_{\mathcal{S}_2}(X') \neq \emptyset$. From the choice of C_2, it follows that $\chi(M_2 \setminus \Theta(X)) \leq 2\kappa$ (for otherwise we would have chosen C_2 with $V(C_2) \subseteq M_2 \setminus (\Theta(X) \cup N^\rho[w_1])$). This proves the first statement of the theorem.

Now let $\tau \geq 0$, with $\chi(M_2) > 2\kappa + (\nu+1)(3\nu^2+2)\tau$; then we may choose C_2 with $\chi(C_2) > \kappa + (\nu+1)(3\nu^2+2)\tau$. Since $\chi(C_2 \setminus \Theta_{\mathcal{S}_2}(X)) \leq \kappa$, it follows that $\chi(M_2(X)) > (\nu+1)(3\nu^2+1)\tau$. Thus, one of the $(\nu+1)(3\nu^2+1)$ monotone paths satisfies the theorem. This proves 9.1. ∎

There is a special case of 9.1 that we often need, and we extract it to make application easier.

9.2 *Let G be a candidate. Let $(L_0, \ldots, L_k, s)$ be a stable shower in G, and let $z_1 \in U(\mathcal{S})$. Let A be a set of children of z_1, and let $b \notin A$ be another child of z_1. Let M be a mat for $\mathcal{S}$, and suppose that $\chi(M(b) \setminus M(A)) > \kappa$. Then $\chi(M(A) \setminus M(b)) \leq 2\kappa$, and for all $\tau \geq 0$, if $\chi(M(A)) > 2\kappa + (\nu+1)(3\nu^2+1)\tau$, then there is a monotone path R between $b, M(b) \setminus M(A)$ such that $\chi(M(X)) > \tau$, where X denotes the set of vertices with a parent in $V(R)$ and an ancestor in A and a descendant in $M(A)$.*

Proof. Let $\mathcal{S}_1$ be the maximal subshower with head z_1 such that no vertex of its union has an ancestor in A, and every vertex of its union except z_1 is a descendant of b; and let $\mathcal{S}_2$ be the maximal subshower with head z_1 such that every vertex of its union except z_1 has an ancestor in A. Let their unions be V_1, V_2 respectively. Then $V_1 \cap V_2 = \{z_1\}$, and no vertex in V_1 has a parent in v_2. Also $M(b) \setminus M(A)$ is a mat for $\mathcal{S}_1$, and $M(A)$ for $\mathcal{S}_2$. By 9.1, the result follows. This proves 9.2. ∎

9.3 *Let G be a candidate. Let $(L_0, \dots, L_k, s)$ be a stable shower in G, with union V, let $z_1 \in V$, let Y be a subset of the set of children of z_1, and let $M \subseteq \Theta(Y)$. For $\tau \geq 0$, if*

$$\chi(M) > \tau + ((\nu + 1)(3\nu^2 + 1) + 7)\kappa,$$

then there exists $z_2 \in Y$ such that $\chi(M(z_2)) > \tau$.

Proof. Choose $A \subseteq Y$ minimal such that $\chi(M(A)) > 2\kappa + \tau$. Suppose that $\chi(M(A)) > 3\kappa + \tau$, and choose $z_2 \in A$; then from the minimality of A, $\chi(M(A \setminus \{z_2\})) \leq 2\kappa + \tau$, and so $\chi(M(z_2) \setminus M(A \setminus \{z_2\})) > \kappa$. By 9.2 applied to $A \setminus \{z_2\}$ and $\{z_2\}$, it follows that $\chi(M(A) \setminus M(z_2)) \leq 2\kappa$; and since $\chi(M(A)) > 2\kappa + \tau$, it follows that $\chi(M(z_2)) > \tau$, as required.

We may assume therefore that $\chi(M(A)) \leq 3\kappa + \tau$. Let $n = (\nu + 1)(3\nu^2 + 1)\kappa$; then

$$\chi(M) > \tau + n + 7\kappa \geq \chi(M(A)) + n + 4\kappa,$$

so we may choose $B \subseteq Y$ with $A \subseteq B$, minimal such that $\chi(M(B)) > \chi(M(A)) + n + 2\kappa$. Again, by the same argument, we may assume that

$$\chi(M(B)) \leq \chi(M(A)) + n + 3\kappa \leq \tau + n + 6\kappa;$$

and since $\chi(M) > \tau + n + 7\kappa$, it follows that $\chi(M \setminus M(B)) > \kappa$. By 9.2 applied to the mat $M \setminus M(A)$ and the sets $B, Y \setminus B$ (with τ replaced by κ), there exists $z_2 \in Y \setminus B$ such that

$$\chi((M \setminus M(A)) \cap M(z_2)) > \kappa.$$

By 9.2 again, applied to the mat M and the sets $A, \{z_2\}$, it follows that $\chi(M(A) \setminus M(z_2)) \leq 2\kappa$, and consequently $\chi(M(z_2)) > \tau$, as required. This proves 9.3. ∎

10 Shower completeness

To go further we use a global induction that we explain next. For $n \geq 2$, a set of integers is *n-solid* if some subset consists of n consecutive integers. It is *1-solid* if it contains two integers that differ by 1 or 3. A key observation is that if a set $\mathcal{A}$ of integers is n-solid where $n > 0$, then $\mathcal{A} + \{0, 2\}$ is $(n + 1)$-solid. Let us say a shower is *n-complete* over a mat M if its M-jetset is n-solid. (For $n \geq 2$ this agrees with our earlier definition.) Now 8.1 implies that in every candidate, all stable showers with a mat M of large enough chromatic number are 1-complete over M; and as we have seen, to finish the proof of our main theorem 5.1 we only need to show that all stable showers with a mat M of large enough chromatic number are ν-complete over M.

For $\sigma > 0$, let us say an integer $\zeta \geq 0$ is a *sidekick* for σ if for every candidate G, and every stable shower $\mathcal{S}$ in G, $\mathcal{S}$ is σ-complete over M for every mat M for $\mathcal{S}$ with chromatic number more than ζ.

Next we need a third inclusion relation for showers, as follows. Let $\mathcal{S} = (L_0, \dots, L_k, s)$ be a stable shower, and let $\mathcal{S}' = (L'_i, \dots, L'_{k'}, s')$ be a shower, both in a graph G; and let P be an induced path between L_0, L'_0. Suppose that

- $s = s'$;

- $L'_0, \dots, L'_{k'-1} \subseteq L_0 \cup \dots \cup L_{k-1}$;
- $L'_{k'} \subseteq L_k$; and
- no vertex of P belongs to $L'_1 \cup \dots \cup L'_{k'}$, and no vertex of P has a neighbour in this set except the vertex in L'_0.

In this situation we say that $\mathcal{S}'$ is *included* in $\mathcal{S}$, and P is a *pipe*. It follows that $\mathcal{S}'$ is a stable shower, because the subgraph induced on $L_0 \cup \dots \cup L_{k-1}$ is bipartite. We see that if $\mathcal{S}'$ is a subshower of $\mathcal{S}$ then it is included in $\mathcal{S}$, via a pipe consisting of a monotone path between the two shower heads.

Let $\mathcal{S}'$ be included in $\mathcal{S}$, with a pipe P. For every jet J' of $\mathcal{S}'$, $J \cup P$ is a jet of $\mathcal{S}$; and consequently, if the jetsets of the two showers are $\mathcal{A}, \mathcal{A}'$ respectively then $\mathcal{A}' + \{|E(P)|\} \subseteq \mathcal{A}$. Thus if $\mathcal{S}'$ is n-complete for some n, then so is $\mathcal{S}$. If M, M' are mats for $\mathcal{S}, \mathcal{S}'$ respectively, and $M' \subseteq M$, then for every M'-jet J of $\mathcal{S}'$, $J \cup P$ is an M-jet of $\mathcal{S}$; and so the same relation holds between the M- and M'-jetsets of the two showers. Note that the floor of $\mathcal{S}'$ is a subset of the floor of $\mathcal{S}$, but for an individual vertex v, there may be descendants of v in $\mathcal{S}'$ that are not descendants in $\mathcal{S}$. (This is not the case for subshowers, and for this reason some results will only work for subshowers.)

Let $\mathcal{S}'$ be included in $\mathcal{S}$. We say a *switch* for $\mathcal{S}'$ in $\mathcal{S}$ is a pair (P_1, P_2) of pipes such that $|E(P_2)| = |E(P_1)| + 2$.

10.1 *Let ζ be a sidekick for σ. Let $\mathcal{S}$ be a stable shower in a candidate G, and let $\mathcal{S}$ include a shower $\mathcal{S}'$. Let M, M' be mats for $\mathcal{S}$, $\mathcal{S}'$ respectively, with $M' \subseteq M$. If $\mathcal{S}$ is not $(\sigma + 1)$-complete over M, and $\chi(M') > \zeta$, then there is no switch for $\mathcal{S}'$ in $\mathcal{S}$.*

Proof. Let $\mathcal{S}$, $\mathcal{S}'$ have heads z_0, z_1 respectively, and suppose that (P_1, P_2) is a switch for $\mathcal{S}'$ in $\mathcal{S}$. Let $\mathcal{A}$ be the M-jetset of $\mathcal{S}$, and let $\mathcal{A}'$ be the M'-jetset of $\mathcal{S}'$. As we saw above,

$$\mathcal{A}' + \{|E(P_1)|, |E(P_1) + 2\} \subseteq \mathcal{A}.$$

Since $\chi(M') > \zeta$ and ζ is a sidekick for σ, it follows that $\mathcal{S}'$ is σ-complete over M'. Consequently $\mathcal{A}' + \{|E(Q)|, |E(Q) + 2\}$ is $(\sigma+1)$-complete, and hence so is $\mathcal{A}$, a contradiction. This proves 10.1. ∎

The first use of 10.1 is the following companion to 9.1:

10.2 *Let ζ be a sidekick for σ. Let G be a candidate, let $\mathcal{S} = (L_0, \dots, L_k, s)$ be a stable shower in G with a mat M, and suppose that $\mathcal{S}$ is not $(\sigma+1)$-complete over M. Let $z_1 \in L_h$, where $0 \leq h < k$, and let A be a set of children of z_1; and let $b \notin A$ be another child of z_1. Let P be a monotone path between b and $L_k \cap M$, such that some vertex of P has a child with an ancestor in A. Let P have vertices p_{h+1}-$\cdots$-p_k, where $p_{h+1} = b$ and $p_k \in L_k \cap M$. If $i \geq h+2$ and p_i has a child x which has an ancestor in A, then the set of descendants of p_i in M that are not descendants of x has chromatic number at most ζ.*

Proof. Let P_1 be a monotone path between z_0, z_1. Since x has an ancestor in A, there is a monotone path Q between x, z_0 containing a vertex in A. Consequently Q contains no descendant of b except x; and since $i > h+1$, the path P_2 formed by the union of Q and the edge xp_i is induced. Its length is $|E(P_1)|+2$. Let $\mathcal{S}'$ be the subshower of $\mathcal{S}$ between p_i and $M \setminus \Theta(x)$, and let M' be the intersection of its floor with $M \setminus \Theta(x)$. This is a stable shower included in $\mathcal{S}$, with a mat M', and (P_1, P_2) is a switch for it. By 10.1, $\chi(M') \leq \zeta$. This proves 10.2. ∎

11 The shadow of a wand

Let $\mathcal{S} = (L_0, \dots, L_k, s)$ be a stable shower. A *wand* $\mathcal{W}$ of length t in $\mathcal{S}$ P is a sequence $(W_0, \dots, W_t)$ with the following properties:

- $t \le k - 2$;
- $\emptyset \neq W_i \subseteq L_i$ for $0 \le i \le t$;
- every vertex in W_i is adjacent to every vertex in W_{i+1} for $0 \le i \le t-1$.

We define $V(\mathcal{W}) = W_0 \cup \cdots \cup W_t$.

Let $(W_0, \dots, W_t)$ be a wand $\mathcal{W}$ in $\mathcal{S}$. If $u \in W_i$ for some i, we say that a neighbour v of u is an *up-neighbour* of u if

- $v \notin V(\mathcal{W})$;
- $v \in L_{i-1}$ (and therefore $i \ge 2$); and
- every neighbour of v in $V(\mathcal{W})$ belongs to W_i (and therefore $i \ge 3$).

For $0 \le i \le t-1$, let T_i be the set of all vertices $v \in L_i$ such that v is an up-neighbour of some vertex in W_{i+1}. Let $T = T_0 \cup \cdots \cup T_{t-1}$. For $t \in T$, a *post* with *top* t is a monotone path between t and L_k such that no vertex of this path except t belongs to or has a neighbour in $V(\mathcal{W})$. For $0 \le i \le k$, let S_i be the set of all vertices $v \in L_i$ that belong to a post with top in T. (Thus $T_i \subseteq S_i \subseteq L_i \setminus V(\mathcal{W})$, and $S_0 = \emptyset$.) If M is a mat for $\mathcal{S}$, we call $M \cap S_k$ the *shadow* (over M) of the wand.

Showers in which no wand shadow has large χ are easier to work with than general showers. In this section we prove that their mats have bounded chromatic number. The proof requires several steps. We begin with:

11.1 *Let $\mathcal{S}$ be a stable shower with mat M in a candidate G, such that every wand in $\mathcal{S}$ has shadow over M with chromatic number at most τ. Let $z_1 \in U(\mathcal{S})$, and let A, B be disjoint sets of children of z_1. If $\chi(M(A)) > \kappa$ then $\chi(M(B) \setminus M(A)) \le (\nu+1)(3\nu^2+1)(\tau+\kappa) + 2\kappa$.*

Proof. Suppose not. Let $\mathcal{S}_1$ be the maximal subshower of $\mathcal{S}$ with head z_1 such that every vertex in $U(\mathcal{S}_1)$ except z_1 has an ancestor in A. Then $B \cap U(\mathcal{S}_1) = \emptyset$. Let $\mathcal{S}_2$ be the maximal subshower of $\mathcal{S}$ with head z_1 such that every vertex in $U(\mathcal{S}_2)$ except z_1 has an ancestor in B and has no ancestor in A. For $i = 1, 2$ let $V_i = U(\mathcal{S}_i)$. Thus $V_1 \cap V_2 = \{z_1\}$, and no vertex in V_2 has a parent in V_1. Moreover, $M(A)$ is a mat for $\mathcal{S}_1$, and $M(B) \setminus M(A)$ is a mat for $\mathcal{S}_2$. By 9.1, since $\chi(M(B) \setminus M(A)) > (\nu+1)(3\nu^2+1)(\tau+\kappa) + 2\kappa$, there is a monotone path R of $G[V_1]$ between z_1 and $M(A)$ such that, if X denotes the set of vertices in $V_2 \setminus \{z_1\}$ with a neighbour in $V(R) \setminus \{z_2\}$, then the set of vertices in $M(B) \setminus M(A)$ with an ancestor in X has chromatic number more than $\tau + \kappa$.

Now no vertex of R different from z_1 has a child in V_2, so every vertex in X has a child in $V(R)$. Let y be the vertex of R with height two, and let R' be the subpath of R between z_1, y. Let X_1 be the set of vertices in X with a child in R', and let X_2 be the set of vertices in X with a child in R with height at most one. Let P be the union of R and a monotone path between L_0 and z_1.

The vertices of P in order form a wand, and every vertex in X_1 is an up-neighbour of a vertex of this wand. Consequently the set of descendants in M of X_2 is a subset of the shadow of this wand, and so has chromatic number at most τ. But every vertex with an ancestor in X_2 is at distance at most three from the penultimate vertex of R, and in particular the set of descendants in M of X_2 has chromatic number at most κ. Consequently $\chi(M(X)) \leq \tau + \kappa$, a contradiction. This proves 11.1. ∎

Let $\mathcal{S}$ be a stable shower in a candidate G. A wand $\mathcal{W} = (W_0, \dots, W_t)$ is said to be ξ-*diagonal* if

- every vertex of $U(\mathcal{S})$ with a child in $V(\mathcal{W})$ belongs to $V(\mathcal{W})$; and
- for $0 \leq i \leq t$, the set of vertices in M that have an ancestor in W_i and no ancestor in W_{i+1} has chromatic number at most ξ (where $W_{t+1} = \emptyset$).

Next we need some results about showers that admits ξ-diagonal wands, where ξ is bounded. Before we do so, let us set up some notation for these things.

If $\mathcal{S} = (L_0, \dots, L_k, s)$ with mat M, and $\mathcal{W}$ is a ξ-diagonal wand $(W_0, \dots, W_t)$ in $\mathcal{S}$, then for every vertex v of $U(\mathcal{S}) \cup M$, there is a maximum $i \leq t$ such that W_i contains an ancestor of v. We call this number i the *reach* of v (with respect to $\mathcal{W}$). Let $V = U(\mathcal{S})$, and for $0 \leq i \leq t$ let M_i and V_i be the sets of all vertices in M and in V with reach i, respectively. A monotone path is *vertical* if for some i, all its vertices belong to $M_i \cup V_i$. Let $0 \leq h \leq t$, and let P be a monotone path between some vertex in M_h and some vertex in W_h. It follows that P is vertical. If $X \subseteq V \cup M$, the set of vertices in M joined to a vertex in X by a vertical path is denoted by $X \downarrow M$.

11.2 *Let ζ be a sidekick for σ. Let $\mathcal{S}$ be a stable shower with mat M in a candidate G, such that $\mathcal{S}$ is not $(\sigma+1)$-complete over M, and the shadow over M of every wand in $\mathcal{S}$ has chromatic number at most τ. Let $\mathcal{W}$ be a ξ-diagonal wand, and, with notation as above, let $0 \leq h \leq t$, and let P be a monotone path between M and W_h, with no vertex in $V(\mathcal{W})$ except its vertex in W_h. Let $N(P)$ be the set of vertices in $M \cup (V \setminus V(\mathcal{W}))$ with a neighbour in $V(P)$. Then*

$$\chi((N(P) \downarrow M) \setminus (M_h \cup M_{h+1})) \leq 2\zeta + 2\xi + \kappa + \tau.$$

Proof. Let P have vertices $p_h\text{-}\cdots\text{-}p_k$ in order, where $p_i \in L_i$ for $h \leq i \leq k$. Thus $p_h \in W_h$. For $h \leq i \leq k$, let X_i be the set of children of p_{i-1} in $M \cup (V \setminus V(\mathcal{W}))$ (taking $X_h = \emptyset$), and let Y_i be the set of parents of p_{i+1} in $V \setminus V(\mathcal{W})$ (taking $Y_k = \emptyset$). Let Z_1 be the set of vertices in $V \setminus V(\mathcal{W})$ with height at least three, with a child in $V(P)$ and with no parent in $V(P)$. It follows that $Z_1 \cap V(P) = \emptyset$.

(1) $\chi(Z_1 \downarrow M) \leq \tau$.

The sequence

$$(W_0, \dots, W_{h-1}, \{p_h\}, \{p_{h+1}\}, \dots, \{p_{k-2}\})$$

is a wand, and every vertex in Z_1 is an up-neighbour of a vertex in this wand, and so every vertex in $Z_1 \downarrow M$ belongs to the shadow of this wand over M. Since by hypothesis the shadow of every wand over M has chromatic number at most τ, this proves (1).

Let Z_2 be the set of vertices in $V \setminus V(\mathcal{W})$ adjacent to one of p_{k-2}, p_{k-1}, p_k.

(2) $\chi(Z_2 \downarrow M) \leq \kappa$.

This is immediate since every vertex in $Z_1 \downarrow M$ has distance at most four from p_{k-2}, and $\rho \geq 4$.

Let Z_3 be the set of vertices in $V \setminus V(\mathcal{W})$ with a parent in $\{p_h, \ldots, p_{k-3}\}$.

(3) $\chi((Z_3 \downarrow M) \setminus (M_h \cup M_{h+1})) \leq 2\zeta + 2\xi$.

To show this, we may therefore assume that $Z_3 \downarrow M \not\subseteq M_h \cup M_{h+1}$. Thus there exists $j \leq k-2$ and $i \geq h$ such that $j \neq h, h+1$ and and some child of p_i belongs to V_j. Since no descendant of p_h belongs to $V_{h'}$ for $h' < h$, it follows that $j \geq h+2$. Every vertex in V_j is a descendent of a vertex in W_j, and consequently $i \geq j \geq h+2$. We have shown then that there exists $i_1 \in \{h+2, \ldots, k-3\}$ such that p_i has a child in V_j for some $j \geq h+2$. Choose $i \geq h+2$ minimum with this property; and let $j_1 \in \{h+2, \ldots, k-2\}$ be maximum such that p_i has a child in V_{j_1}. If possible, let $j_2 \in \{h+2, \ldots, k-2\}$ be maximum such that p_{i+1} has a child in V_{j_2}, and otherwise let $j_2 = h$.

Let x_1 be a child of p_i in V_{j_1}. Let $\mathcal{S}_1$ be the maximal subshower of $\mathcal{S}$ with head p_i such that no child of x belongs to $U(\mathcal{S}_1)$, and let M^1 be the set of vertices $v \in M$ such that there is a monotone path of $\mathcal{S}$ between v, p_i containing no child of x. Thus, M^1 is a mat for $\mathcal{S}_1$. For $0 \leq a \leq k-2$, let $c_a \in W_a$, where $c_h = p_h$. Then c_0-$\cdots$-c_h-p_{h+1}-$\cdots$-p_i and c_0-c_j-Q-x-p_i are both induced paths (where Q is a vertical path between x and c_j), and so the pair form a switch for $\mathcal{S}_1$. From 10.1, it follows that $\chi(M^1) \leq \zeta$.

If $j_2 > h$ let x_2 be a child of p_{i+1} in V_{j_2}, and let M^2 be the set of vertices $v \in M$ such that there is a monotone path of $\mathcal{S}$ between v, p_{i+1} containing no child of x_2; then similarly, $\chi(M^2) \leq \zeta$. (If $j_2 = h$ let $M^2 = \emptyset$.)

Let M^3 be the set of all $v \in Z_3 \downarrow M$ such that $v \notin M^1 \cup M^2$. Let $v \in M^3$; since $v \in Z_3 \downarrow M$ there is a vertical path R between v and a child of $p_{i'}$ for some i' with $h \leq i' \leq k-3$. Let $v \in M_{j'}$; then R can be extended to a vertical path between v and $W_{j'}$. Since $v \notin M^1$, this vertical path contains a child of x. Every child of x belongs to $V_{j_1} \cup \cdots \cup V_{k-2}$, and so $j' \geq j_1$. Since there is a child of $p_{i'}$ in R and hence in $V_{j'}$, it follows from the choice of i that either $i' \geq i$ or $j' = h+1$. If $i' \geq i+2$ there is a monotone path between v, p_i containing no child of x, a contradiction; so $i' \leq i+1$. Hence either $i' = i$, or $i' = i+1$, or $j' = h+1$. If $i' = i$, then the choice of j_1 implies that $j' \leq j_1$; and since a child of x belongs to $V_{j'}$, it follows that $j' = j_1$. Similarly, if $i' = i+1$, then the choice of j_2 implies that $j' \leq j_2$; and since a child of x_2 belongs to $V_{j'}$, it follows that $j' = j_2$. We have shown then that j' is one of $j_1, j_2, h+1$. Consequently $M^3 \subseteq M_{h+1} \cup M_{j_1} \cup M_{j_2}$.

But $Z_3 \downarrow M \subseteq M^1 \cup M^2 \cup M^3$, and $\chi(M_{j_1} \cup M_{j_2}) \leq 2\xi$, so $\chi((Z_3 \downarrow M) \setminus (M_h \cup M_{h+1})) \leq 2\xi + 2\zeta$. This proves (3).

From (1), (2) and (3), since $N(P) = Z_1 \cup Z_2 \cup Z_3$, the result follows. This proves 11.2. ∎

11.3 *Let ζ be a sidekick for σ. Let $\mathcal{S}$ be a stable shower with mat M in a candidate G, such that $\mathcal{S}$ is not $(\sigma+1)$-complete over M, and the shadow over M of every wand in $\mathcal{S}$ has chromatic number at most τ. Let $\mathcal{W}$ be a ξ-diagonal wand. In the usual notation, let $h < j \leq t$, and let $A \subseteq \bigcup_{h<i<j} M_i$ such that $G[A]$ is connected and $\chi(A) > 2\zeta + 5\xi + 2\kappa + \tau$. Let $c_h \in W_h$, and $c_j \in W_j$. Then there is a set $\mathcal{A}$ of integers, and for each $a \in \mathcal{A}$ there is an induced path J_a of G between c_h, c_j, with the following properties:*

- *$\mathcal{A}$ has cardinality at most $\nu+1$, and includes a dense set of cardinality ν, and contains two integers x, y with $y - x \in \{1,3\}$;*
- *$|E(J_a)| = a$ for each $a \in \mathcal{A}$;*
- *for each $a \in \mathcal{A}$, $V(J_a) \setminus \{c_h, c_j\} \subseteq V_{h+1} \cup \cdots \cup V_{j-1} \cup A$;*
- *for each $a \in \mathcal{A}$, there is a set of at most $3\nu^2 + 2$ monotone paths between A and c_h, such that every vertex of $V(J_a) \setminus (A \cup V(\mathcal{W}))$ belongs to one of these paths.*

Proof. No vertex in A has an ancestor in W_j; choose $i < j$ maximum such that some $c_i \in W_i$ has a descendant in A. Let Q be a monotone path between c_i and A. By 11.2, there exists $A' \subseteq A$ such that $G[A']$ is connected,

$$\chi(A') \geq \chi(A) - (2\zeta + 5\xi + \kappa + \tau) \geq \kappa,$$

and no vertical path meets both $N(Q) \cup W_i \cup W_{i-1}$ and A'. Let $\mathcal{S}'$ be the maximal subshower of $\mathcal{S}$ with head c_h such that $U(\mathcal{S}')$ contains no vertex of $N(Q) \cup W_i \cup W_{i-1}$; then A' is a mat for $\mathcal{S}'$. Let $\mathcal{S}'$ be $L'_j, \ldots, L'_{k-1}, L_k, s$ say. Let L'_k be the union of A, $V(Q)$, and $W_{i+1} \cup W_{i+2} \cup \cdots \cup W_j$; then $G[L'_k]$ is connected, and every vertex of $U(\mathcal{S}')$ with a neighbour in L'_k belongs to L'_{k-1}. Thus $L'_j, \ldots, L'_{k-1}, L'_k, c_j$ is a shower, with mat A'; and the result follows from 8.1 and 8.4. (Note: 8.4 gives us $3\nu^2 + 1$ monotone paths containing all the vertices of J_a not in L'_k; but we also need to cover the vertices of J_a in $L'_k \setminus L_k$. One more monotone path will do this, namely Q.) This proves 11.3. ∎

11.4 *Let ζ be a sidekick for σ. Let $\mathcal{S}$ be a stable shower with mat M in a candidate G, such that $\mathcal{S}$ is not $(\sigma+1)$-complete over M, and the shadow over M of every wand in $\mathcal{S}$ has chromatic number at most τ. Let $\mathcal{W}$ be a ξ-diagonal wand. With the usual notation, let $j_0 < j_1 < j_2 \leq t$, and let $u_1 \in M_{j_0}$ and $u_2 \in M_{j_2}$ be adjacent. Let $M^1 \subseteq \bigcup_{j_0<j<j_1} M_j$ and $M^2 \subseteq \bigcup_{j_1<j<j_2} M_j$. If $\chi(M^1) > 6\zeta + 13\xi + 4\kappa + 3\tau$ and*

$$\chi(M^2) > (2\zeta + 4\xi + \kappa + \tau)(3 + (\nu+1)(3\nu^2+2)) + \xi + \kappa$$

then there is an edge between M^1, M^2.

Proof. Let P_1 be a vertical path between u_1, W_{j_0}, and let P_2 be a vertical path between u_2, W_{j_2}. Let c_{j_0} be the end of P_1 in W_{j_0}, and let c_{j_2} be the end of P_2 in W_{j_2}. Since u_1, u_2 are adjacent, there is an induced path P between c_{j_0}, c_{j_2} with $V(P) \subseteq V(P_1 \cup P_2)$ For $i = 1, 2$, let $N(P_i)$ the set of vertices in $M \cup (V \setminus V(\mathcal{W}))$ with a neighbour in $V(P_i)$. By 11.2, for $i = 1, 2$, $\chi(N(P_i) \downarrow M) \leq 2\zeta + 4\xi + \kappa + \tau$, and so there exists $A_1 \subseteq M^1$ with $\chi(A_1) > 2\zeta + 5\xi + 2\kappa + \tau$, such that $G[A_1]$ is connected and no vertex in A_1 belongs to a vertical path that intersects $N(P_1) \cup N(P_2)$. Choose $c_{j_1} \in W_{j_1}$. By 11.3,

(1) *There is a set $\mathcal{A}$ of integers, and for each $a \in \mathcal{A}$ there is an induced path J_a of G between c_{j_0}, c_{j_1}, with the following properties:*

- *$\mathcal{A}$ has cardinality at most $\nu+1$, and includes a dense set of cardinality ν, and contains two integers x, y with $y - x \in \{1,3\}$;*

- $|E(J_a)| = a$ *for each* $a \in \mathcal{A}$*;*
- *for each* $a \in \mathcal{A}$, $V(J_a) \subseteq V_{j_0+1} \cup \cdots \cup V_{j_1-1} \cup A_1 \cup \{c_{j_0}, c_{j_1}$*;*
- *for each* $a \in \mathcal{A}$*, there is a set of at most* $3\nu^2 + 2$ *monotone paths between* A_1 *and* c_{j_0}*, such that every vertex of* $V(J_a) \setminus (A_1 \cup V(\mathcal{W}))$ *belongs to one of these paths.*

Now suppose that there are no edges between M^1, M^2. By $(\nu + 1)(3\nu^2 + 2) + 2$ applications of 11.3, there exists $A_2 \subseteq M_2$ with the following properties:

- $G[A_2]$ is connected, and no vertex in A_2 belongs to a vertical path that intersects $N(P_1) \cup N(P_2)$;
- for each $a \in \mathcal{A}$, no vertex in A_2 belongs to a vertical path that contains a vertex in $V(J_a)$ or a neighbour of such a vertex (here we use that there is no edge between A_1 and M^2)
- $\chi(A_2) > 2\zeta + 5\xi + 2\kappa + \tau$.

We apply 11.3 to A_2, and thereby obtain a set of paths joining c_{j_1} and c_{j_2}. But for each of these paths, say J, and each $a \in \mathcal{A}$, the union $J \cup J_a$ is an induced path between c_{j_0}, c_{j_2}; and it can be combined with the induced path P to form a hole. It follows as usual that G contains a hole ν-sequence, a contradiction. This proves 11.4. ∎

11.5 *Let* ζ *be a sidekick for* σ*. Let* $\mathcal{S}$ *be a stable shower with mat* M *in a candidate* G*, such that* $\mathcal{S}$ *is not* $(\sigma + 1)$*-complete over* M*, and the shadow over* M *of every wand in* $\mathcal{S}$ *has chromatic number at most* τ*. Let* $\mathcal{W}$ *be a* ξ*-diagonal wand. In the usual notation, let* $j_1 < j_2 \leq t$*, and let* $u_1 \in M_{j_1}$ *and* $u_2 \in M_{j_2}$ *be adjacent. Let* $M^1 \subseteq \bigcup_{j_1<j<j_2} M_j$ *and* $M^2 \subseteq \bigcup_{j_2<j\leq t} M_j$*. If*

$$\chi(M^1) > 2(2\zeta + 4\xi + \kappa + \tau) + \kappa$$

and

$$\chi(M^2) > 2(2\zeta + 4\xi + \kappa + \tau) + 2\kappa + (\nu + 1)(3\nu^2 + 1)\tau$$

then there exist $A_1 \subseteq M^1$ *and* $A_2 \subseteq M^2$ *such that* $\chi(A_i) \geq \chi(M^i) - 2(2\zeta + 4\xi + \kappa + \tau)$ *for* $i = 1, 2$*, such that there is no edge between* A_1, A_2*.*

Proof. For $i = 1, 2$, let P_i be a vertical path between u_i and some $c_{j_i} \in W_{j_i}$. Let P be an induced path between c_{j_1}, c_{j_2} with $V(P) \subseteq V(P_1 \cup P_2)$. For $i = 1, 2$, let $N(P_i)$ the set of vertices in $M \cup (V \setminus V(\mathcal{W}))$ with a neighbour in $V(P_i)$.

Let B be the set of all vertices that belong to a vertical path R between $M^1 \cup M^2$ and $V(\mathcal{W})$ such that no vertex of R belongs to $N(P_1) \cup N(P_2)$. Then there is a subshower $\mathcal{S}'$ of $\mathcal{S}$ with head c_{j_1} such that $U(\mathcal{S}') \setminus V(\mathcal{W}) = B \setminus L_k$. Let

$$\mathcal{S}' = (L'_{j_1}, L'_{j_1+1}, \ldots, L'_{k-1}, L_k, s).$$

By 11.2, for $i = 1, 2$, $\chi(N(P_i) \downarrow M) \leq 2\zeta + 4\xi + \kappa + \tau$, and so $\chi(B \cap M_1) > \chi(M_1) - 2(2\zeta + 4\xi + \kappa + \tau)$. Choose $A_1 \subseteq B \cap M_1$, such that $G[A_1]$ is connected and $\chi(A_1) = \chi(B \cap M_1)$. Similarly, we may choose $A_2 \cap B \cap M_2$ such that $G[A_2]$ is connected and $\chi(A_2) > \chi(M_2) - 2(2\zeta + 4\xi + \kappa + \tau)$.

Suppose that there is an edge between A_1, A_2. Then $G[A_1 \cup A_2]$ is connected, and so

$$(L'_{j_1}, L'_{j_1+1}, \dots, L'_{k-1}, A_1 \cup A_2, s_0)$$

is a shower $\mathcal{S}_0$ say (where $s_0 \in A_1 \cup A_2$ is arbitrary). Let $\mathcal{S}_1$ be the maximal subshower of $\mathcal{S}_0$ with head c_{j_1} such that every vertex in $U(\mathcal{S}_1) \setminus V(\mathcal{W})$ belongs to a vertical path of $\mathcal{S}$ with one end in M^1, and let $\mathcal{S}_2$ be the maximal subshower of $\mathcal{S}_0$ with head c_{j_2} such that every vertex in $U(\mathcal{S}_2) \setminus V(\mathcal{W})$ belongs to a vertical path of $\mathcal{S}$ with one end in M^2. It follows that A_i is a mat for $\mathcal{S}_i$ for $i = 1, 2$. Now there is no monotone path R of $G[V_2]$ between c_{j_2} and M_2 such that $\chi(M_1(X(R))) > \tau$, where $X(R)$ denotes the set of vertices in V_1 with a child in $V(R)$; and so by 9.1 (with M_1, M_2 replaced by A_2, A_1) since $\chi(A_2) > \kappa$, and $\chi(A_1) > 2\kappa + (\nu+1)(3\nu^2+1)\tau$, there are ν induced paths $Q_0, \dots, Q_{\nu-1}$ of $G[V_1 \cup V_2 \cup L_k]$ between c_{j_1}, c_{j_2}, such that $|E(Q_i)| = |E(Q_0)| + i$ for $0 \le i < \nu$. But each of these paths forms a hole when combined with P; and so G contains a hole ν-sequence, which is impossible.

It follows that there is no edge between A_1, A_2. This proves 11.5. ∎

We need the following lemma.

11.6 *Let G be a graph with chromatic number more than $4N$, and let $M_1, \dots, M_k$ be a partition of $V(G)$ such that $\chi(G[M_i]) \le N$ for $1 \le i \le k$. Then there exist $a < b < c < d < e \le k$ such that there is an edge of G between M_a and M_c, and an edge between M_a and M_e.*

Proof. Let J be the graph with vertex set $\{1, \dots, k\}$ in which i, j are adjacent if there is an edge of G between M_i and M_j. If J is 4-colourable, then $\chi(G) \le 4N$, a contradiction. So J is not 4-colourable, and consequently there exists $a \in \{1, \dots, k\}$ such that a is adjacent in J to at least four of $a+1, \dots, k$. Let a, b, c, d be the four neighbours, in order; then the theorem holds. This proves 11.6. ∎

11.7 *Let ζ be a sidekick for σ. Let $\mathcal{S}$ be a stable shower with mat M in a candidate G, such that $\mathcal{S}$ is not $(\sigma+1)$-complete over M, and the shadow over M of every wand in $\mathcal{S}$ has chromatic number at most τ. Let $\mathcal{W}$ be a ξ-diagonal wand. Let*

$$\eta = (2\zeta + 4\xi + \kappa + \tau)(5 + (\nu+1)(3\nu^2 + 2)) + \xi + \kappa.$$

Then $\chi(M) \le 4(\eta + \xi) + \eta$.

Proof. Suppose that $\chi(M) > 4(\eta + \xi) + \eta$. Let $j_0 = 0$, and define $j_1, j_2, \dots j_t$ and $M^1, \dots, M^{t-1}$ inductively as follows. Having defined $j_0, \dots, j_i$ and $M^0, \dots, M^{i-1}$, if $\chi(\bigcup_{j_i < j \le 2k-2} M_j) < \eta$ the sequence terminates; define $t = i$. Otherwise choose $j_{i+1} \le 2k-2$ minimum such that $\chi(\bigcup_{j_i < j \le j_{i+1}} M_j) \ge \eta$. Let $M^i = \bigcup_{j_i < j \le j_{i+1}} M_j$.

This completes the inductive definition. We see that the sets $M^1, \dots, M_{t-1}$ are disjoint, and their union has chromatic number at least $\chi(M) - \eta$; and each M_i has chromatic number at least η, and at most $\eta + \xi$ (from the minimality of j_{i+1}). Since $\chi(M) > 4(\eta + \xi) + \eta$, it follows from 11.6 that there exist $a < b < c < d < e \le t$ such that there is an edge of G between M^a and M^c, and an edge between M^a and M^e. Now

$$\eta \ge 2(2\zeta + 4\xi + \kappa + \tau) + 2\kappa + (\nu+1)(3\nu^2+1)\tau$$

so by 11.5 applied to M^b, M^d and the edge between M^a, M^c, there exist $A_1 \subseteq M^b$ and $A_2 \subseteq M^d$ such that $\chi(A_i) \geq \eta - 2(2\zeta + 4\xi + \kappa + \tau)$ for $i = 1, 2$, and there is no edge between A_1, A_2. But this contradicts 11.4 applied to A_1, A_2 and the edge between M^a, M^e. This completes the proof of 11.7. ∎

Now we can prove the objective of this section, the following.

11.8 *Let ζ be a sidekick for σ. Let $N = (3\nu^2 + 2)(\nu + 1) + 5$. Let $\tau \geq 0$, and let $\mathcal{S}$ be a stable shower with mat M in a candidate G, such that $\mathcal{S}$ is not $(\sigma + 1)$-complete over M, and the shadow over M of every wand in $\mathcal{S}$ has chromatic number at most τ. Then $\chi(M) \leq 40N\zeta + 80N^2(\tau + \kappa)$.*

Proof. Let

$$\xi = (\nu + 1)(3\nu^2 + 1)(\tau + \kappa) + 2\kappa,$$

and let

$$\eta = (2\zeta + 4\xi + \kappa + \tau)(5 + (\nu + 1)(3\nu^2 + 2)) + \xi + \kappa.$$

Let $z_0 \in L_0$, and recursively, having defined z_i, let z_{i+1} be a child of z_i chosen such that $\chi(M(z_{i+1})) > \kappa$ if there is such a child; otherwise the definition terminates.

(1) *For $0 \leq i < t$, $\chi(M(z_i) \setminus M(z_{i+1}) \leq \xi$, and $\chi(M(z_t)) \leq \xi$.*

For $0 \leq< t$, since $\chi(M(z_{i+1})) > \kappa$, 11.1 implies that

$$\chi(M(z_i) \setminus M(z_{i+1})) \leq (\nu + 1)(3\nu^2 + 1)(\tau + \kappa) + 2\kappa \leq \xi.$$

Every child z of z_t satisfies $\chi(M(z)) \leq \kappa$, and so by 9.3,

$$\chi(M(z_t)) \leq ((\nu + 1)(3\nu^2 + 1) + 8)\kappa \leq \xi.$$

This proves (1).

For each vertex $v \in M$, choose a monotone path R_v between v and some vertex x_v, such that x_v has a neighbour in $\{z_0, \dots, z_{k-1}\}$, and minimal with this property. Thus no vertex of R_v except x_v has a neighbour in $\{z_0, \dots, z_{k-1}\}$. Now x_v might have a parent in $\{z_0, \dots, z_{k-1}\}$, or a child, or both. Since $(\{z_0\}, \dots, \{z_t\})$ is a wand, it follows that the set of all v such that x_v has a child but no parent in $\{z_0, \dots, z_{k-1}\}$ has chromatic number at most τ. The set of all v such that x_v has a parent but no child in $\{z_0, \dots, z_{k-1}\}$ has chromatic number at most $4(\eta + \xi) + \eta$, by 11.5, applied to the subshower induced on the union of the vertex sets of the corresponding paths R_v, together with $\{z_0, \dots, z_{k-1}\}$. (Because in this subshower, the wand is ξ-diagonal).

It remains then to bound the chromatic number of the set M' of $v \in M$ such that x_v has both a parent and a child in $\{z_0, \dots, z_{k-1}\}$. Let $\mathcal{S}'$ be the subshower induced on the union of the vertex sets of the corresponding paths R_v, together with $\{z_0, \dots, z_{k-1}\}$. For $1 \leq i \leq t - 1$, let D_i be the set of all vertices of $U(\mathcal{S}')$ that are adjacent to both z_{i+1}, z_{i-1}, and let $D_0 = \{z_0\}$ and $D_t = \{z_t\}$. Thus $z_i \in D_i$ for all i. For $c = 0, 1, 2$, let $\mathcal{W}_c$ be the sequence $X_0, \dots, X_t$, where $X_i = D_i$ if $i = c$ modulo 3, and otherwise $X_i = \{z_i\}$. Thus each $\mathcal{W}_c$ is a wand. The set of $v \in M'$ such that some vertex in

$R_v \setminus V(\mathcal{W}_c)$ has a child in $V(\mathcal{W}_c)$ has chromatic number at most τ; and the set of $v \in M'$ such that x_v belongs to some set D_i where $i = c$ modulo 3, and no vertex of R_v has a child in $V(\mathcal{W}_c)$, has chromatic number at most $4(\eta + \xi) + \eta$, by 11.5. In total then, we have shown that

$$\chi(M) \leq \tau + (4(\eta + \xi) + \eta) + 3(\tau + (4(\eta + \xi) + \eta)) = 4\tau + 20\eta + 16\xi.$$

After some arithmetic, which we omit, this proves 11.8. ∎

12 Raising a wand

Now we turn to general showers, in which a wand shadow may have large chromatic number. We will prove that, if there is such a wand, then we can use it to construct a new shower, still with large χ, in which no wand shadow has large chromatic number. We begin with:

12.1 *Let $\mathcal{S} = (L_0, \dots, L_k, s)$ be a stable shower in a candidate G, and let $(W_0, \dots, W_t)$ be a wand $\mathcal{W}$ in $\mathcal{S}$. Let v be a vertex of some post, and let $v \in L_i$ say. Then there are two induced paths P_1, P_2 of G between v and L_0, such that $|E(P_2)| = |E(P_1)| + 2$, and for $j > i$ every vertex in L_j that belongs to or has a neighbour in either of these paths belongs to $W_{i+1} \cup W_{i+2} \cup \{v\}$.*

Proof. Let P be a post containing v, with top $t \in T_h$ say; thus $h \leq i$. Let P_0 be the subpath of P between v, t. Let $u \in W_{h+1}$ be adjacent to t. Let P_1 be the union of P_0 and a monotone path between t and L_0. P_2 be the union of P_0, the edge tu, and a path between u and W_0 with one vertex in each of $W_0, \dots, W_{h+1}$. This proves 12.1. ∎

12.2 *Let ζ be a sidekick for σ. Let $\mathcal{S} = (L_0, \dots, L_k, s)$ be a stable shower in a candidate G, with mat M, such that $\mathcal{S}$ is not $(\sigma + 1)$-complete over M. Let $(W_0, \dots, W_t)$ be a wand $\mathcal{W}$ in $\mathcal{S}$. Let $0 \leq i \leq t - 1$, and let T_i be the set of up-neighbours of vertices in W_{i+1}. Let M' be the set of all $v \in M$ that belong to a post with top in T_i. Then*

$$\chi(M') \leq \zeta + 2((\nu + 1)(3\nu^2 + 1) + 7)\kappa.$$

Proof. For $X \subseteq T_i$, and $j \in \{i, \dots, k\}$, let $L_j(X)$ be the set of all vertices in L_j that belong to a post with top in X. Then

$$(W_0, W_1, \dots, W_i, X, L_i(X), L_{i+1}(X), \dots, L_{k-1}(X), L_k, s)$$

is a stable shower $\mathcal{S}(X)$ included in $\mathcal{S}$ (with a one-vertex pipe); although it is not a subshower. Also $M' = M \cap L_k(T_i)$. By 9.3 applied to $\mathcal{S}(T_i)$ (taking $z_1 \in W_i$ and $Y = W_{i+1}$) there exists $u \in W_{i+1}$ such that

$$\chi(M \cap L_k(X)) \geq \chi(M') - ((\nu + 1)(3\nu^2 + 1) + 7)\kappa,$$

where X is the set of up-neighbours of u. By 9.3 applied to $\mathcal{S}(T(X)$ (taking $z_1 = u$, and $Y = X$) there exists $x \in X$ such that

$$\chi(M \cap L_k(x)) \geq \chi(M \cap L_k(X)) - ((\nu + 1)(3\nu^2 + 1) + 7)\kappa;$$

and so

$$\chi(M \cap L_k(x)) \geq \chi(M') - 2((\nu+1)(3\nu^2+1)+7)\kappa.$$

Now

$$(x, L_i(X), L_{i+1}(X), \dots, L_{k-1}(X), L_k, s)$$

is also a shower included in $\mathcal{S}$ (with pipe a monotone path between x and L_0), and $M \cap L_k(x)$ is a mat for it. But there is a switch for $\mathcal{S}(\{x\})$ in $\mathcal{S}$, by 12.1. From 10.1 it follows that $\chi(M \cap L_k(x)) \leq \zeta$. We deduce that

$$\chi(M \cap L_k(X)) \leq \zeta + 2((\nu+1)(3\nu^2+1)+7)\kappa.$$

This proves 12.2. ∎

Let $T_0, \dots, T_{t-1}, T$ be as before. For $0 \leq i \leq k$, let S_i be the set of all vertices $v \in L_i$ that belong to a post with top in T. (Thus $T_i \subseteq S_i \subseteq L_i \setminus V(\mathcal{W})$, and $S_0 = \emptyset$.) If M is a mat for $\mathcal{S}$, it follows (since $t \leq k-2$) that

$$(W_0, W_1, W_2, W_3 \cup S_1, W_4 \cup S_2, \dots, W_t \cup S_{t-2}, S_{t-1}, \dots, S_{k-2}, S_{k-1}, L_k, s)$$

is a stable shower $\mathcal{S}'$ included in $\mathcal{S}$; and we say that $\mathcal{S}'$ is obtained from $\mathcal{S}$ by *raising* the wand. Moreover, the shadow $M \cap S_k$ is a mat for $\mathcal{S}'$.

When we raise a wand, the new shower $\mathcal{S}'$ is included in $\mathcal{S}$, but it is not a subshower, and we must be cautious with concepts such as "child", "descendant", because they depend on which shower we are using. For clarity we temporarily replace them by expressions like "$\mathcal{S}$-child".

12.3 *Let $\mathcal{S} = (L_0, \dots, L_k, s)$ be a stable shower in a candidate G, and let $(W_0, \dots, W_t)$ be a wand in $\mathcal{S}$. Let $\mathcal{S}'$ be obtained from $\mathcal{S}$ by raising the wand. Then for $0 \leq i \leq t$, if $v \in W_i$ and v is an $\mathcal{S}'$-child of u then $i > 0$ and $u \in W_{i-1}$.*

Proof. In the notation given before, since $v \in W_i$ and v is an $\mathcal{S}'$-child of u, it follows that $i > 0$ and $u \in W_{i-1} \cup S_{i-3}$, where $S_{-1}, S_{-2} = \emptyset$. But $S_{i-3} \subseteq L_{i-3}$ and $v \in W_i \subseteq L_i$, so $u \notin S_{i-3}$, and hence $u \in W_{i-1}$. This proves 12.3. ∎

12.4 *Let ζ be a sidekick for σ. Let $\mathcal{S} = (L_0, \dots, L_k, s)$ be a stable shower in a candidate G, with mat M, such that $\mathcal{S}$ is not $(\sigma+1)$-complete over M. Suppose that $\mathcal{S}$ is obtained from some stable shower $\mathcal{S}_0$ in G with mat M_0 by raising some wand, and M is the shadow over M_0 of this wand. Let $\mathcal{W}$ be a wand in $\mathcal{S}$. Then the shadow M' of $\mathcal{W}$ over M has chromatic number at most*

$$3\zeta + 6((\nu+1)(3\nu^2+1)+7)\kappa.$$

Proof. Let $\mathcal{W} = (W_0, \dots, W_t)$, and for $0 \leq i \leq t-1$, let T_i be the set of up-neighbours of vertices in W_{i+1} and let $T = T_0 \cup \dots \cup T_{t-1}$. Thus M' is the set of all $v \in M$ that belong to a post with top in T. Choose h minimum such that $T_h \neq \emptyset$. Let M_1, M_2 be the sets of vertices in M that belong to posts with top in $T_h \cup T_{h+1}$ and with top in $T \setminus (T_h \cup T_{h+1})$ respectively. In view of 12.2 it suffices to bound $\chi(M_2)$. For $j = h+2, \dots, k$ let S_j be the set of vertices in L_j that belong to a post with top in $T \setminus (T_h \cup T_{h+1})$. Thus every vertex of every such post belongs to S_j for some j. Choose $u \in W_{h+1}$ with a neighbour $t \in T_h$. Consequently

$$(\{u\}, W_{h+2}, W_{h+3}, W_{h+4} \cup S_{h+2}, W_{h+5} \cup S_{h+3}, \ldots, W_t \cup S_{t-2}, S_{t-1}, \ldots, S_{k-1}, L_k, s)$$

is a shower $\mathcal{S}'$, and M_2 is a mat for it. Every vertex of $U(\mathcal{S}')$ belongs to L_j for some $j \geq h+2$, except u. We claim there is a switch for this shower; but in $\mathcal{S}_0$, not in $\mathcal{S}$.

Let $\mathcal{S}_0$ be $(M_0, \ldots, M_{k-3}, L_k, s)$. Now $\mathcal{S}$ is obtained from $\mathcal{S}_0$ by raising some wand $\mathcal{D}$ say, where M is the shadow of $\mathcal{D}$ on some mat M_0 for $\mathcal{S}_0$. Let $\mathcal{D}$ be $D_0, \ldots, D_r$, and set $D_i = \emptyset$ for $i > r$; then for $0 \leq i \leq t$, $L_i \subseteq D_i \cup (M_{i-2} \setminus V(\mathcal{D}))$ (where $M_{-1}, M_{-2} = \emptyset$). Suppose that $u \in V(\mathcal{D})$; then since $u \in L_{h+1}$, it follows that $u \in D_{h+1}$. Every vertex of W_{h-1} has distance two from u, and so $W_{h-1} \cap M_{h-3} = \emptyset$; so $W_{h-1} \subseteq D_{h-1}$, since

$$W_{h-1} \subseteq L_{h-1} \subseteq D_{h-1} \cup M_{h-3}.$$

Since t has no neighbour in W_{h-1}, and every vertex of D_h is adjacent to every vertex of D_{h-1}, it follows that $t \notin D_h$. But this contradicts 12.3, since t is an $\mathcal{S}$-parent of u.

This proves that $u \notin V(\mathcal{D})$. Since $u \in W_{h+1} \subseteq L_{h+1}$, it follows that $u \in M_{h-1}$. By 12.1 applied to $\mathcal{S}_0$, there are two induced paths P_1, P_2 of G between u and L_0, such that $|E(P_2)| = |E(P_1)| + 2$, and for $j > h-1$ every vertex in M_j that belongs to or has a neighbour in either of these paths belongs to $D_h \cup D_{h+1} \cup \{u\}$. Since every vertex of $U(\mathcal{S}')$ belongs to L_j for some $j \geq h+2$, except u, and L_j is disjoint from $D_h \cup D_{h+1}$ (because $L_j \subseteq D_j \cup (M_{j-2} \setminus V(\mathcal{D}))$), it follows that (P_1, P_2) is a switch for $\mathcal{S}'$ in $\mathcal{S}_0$. Hence by 10.1, $\chi(M_2) \leq \zeta$. Since two applications of 12.2 imply that

$$\chi(M_1) \leq 2\zeta + 4((\nu+1)(3\nu^2+1)+7)\kappa,$$

it follows that

$$\chi(M') \leq 3\zeta + 6((\nu+1)(3\nu^2+1)+7)\kappa.$$

This proves 12.4. ∎

12.5 *Let ζ be a sidekick for σ. Let $\mathcal{S} = (L_0, \ldots, L_k, s)$ be a stable shower in a candidate G, with mat M, such that $\mathcal{S}$ is not $(\sigma+1)$-complete over M. Let $N = (3\nu^2+2)(\nu+1)+5$. Then*

$$\chi(M) \leq 19201N^4\zeta + 38400N^5\kappa.$$

Proof. Let $\tau = 40N\zeta + 80N^2(3\zeta + 6(N-\nu+1)\kappa + \kappa)$. Let $\mathcal{W}$ be a wand in $\mathcal{S}$, let M' be its shadow over M, and let $\mathcal{S}'$ be obtained by raising $\mathcal{W}$. Every jet of $\mathcal{S}'$ is a jet of $\mathcal{S}$, and so $\mathcal{S}'$ is not $(\sigma+1)$-complete. By 12.4, the shadow over M' of every wand in $\mathcal{S}'$ has chromatic number at most

$$3\zeta + 6(N-\nu+1)\kappa.$$

By 11.8 applied to $\mathcal{S}'$, it follows that $\chi(M') \leq \tau$.

Thus every wand in $\mathcal{S}$ has shadow over M with chromatic number at most τ; and so the result follows from 11.8, since

$$\chi(M) \leq 40N\zeta + 80N^2(\tau+\kappa) \leq 19201N^4\zeta + 38400N^5\kappa.$$

This proves 12.5. ∎

Let us put these pieces together, to prove 5.1, in the following strengthened form.

12.6 *Let $\nu \geq 2$ and $\kappa \geq 0$ be integers. Let $N = (3\nu^2 + 2)(\nu + 1) + 5$, $\zeta_1 = \kappa$, and for $1 \leq \sigma < \nu$ define*

$$\zeta_{\sigma+1} = 19201N^4\zeta_\sigma + 38400N^5\kappa.$$

Let G be a triangle-free graph such that $\chi(N_G^\rho[v]) \leq \kappa$ for every vertex v, where $\rho = 3^{\nu+2} + 4$. If G admits no hole ν-sequence then $\chi(G) \leq 44\nu(\kappa + \zeta_\nu)^{(\nu+1)^2} + 4\kappa$.

Proof. By 8.1, ζ_1 is a sidekick for 1. We claim that for $1 \leq \sigma < \nu$, if ζ_σ is a sidekick for σ then $\zeta_{\sigma+1}$ is a sidekick for $\sigma + 1$. For let M be a mat for a stable shower $\mathcal{S}$ in a candidate G, such that $\mathcal{S}$ is not $(\sigma + 1)$-complete over M. By 12.5, $\chi(M) \leq \zeta_{\sigma+1}$. This proves the claim that $\zeta_{\sigma+1}$ is a sidekick for $\sigma + 1$. Consequently ζ_ν is a sidekick for ν, and in particular, for every candidate G, every ν-incomplete stable shower in G has floor of chromatic number at most ζ_ν. By 8.3, every candidate has chromatic number at most $44\nu(\kappa + \zeta_\nu)^{(\nu+1)^2} + 4\kappa$. This proves 12.6. ∎

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