Intersections of graphs

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Abstract

Let G and H be two graphs of order n. If we place copies of G and H on a common vertex set, how much or little can they be made to overlap? The aim of this paper is to provide some answers to this question, and to pose a number of related problems. Along the way, we solve a conjecture of Erdős, Goldberg, Pach and Spencer.

1 Introduction

Let G and H be two graphs of order n, with $e(G) = p\binom{n}{2}$ and $e(H) = q\binom{n}{2}$. If we place G and H at random onto the same vertex set, then we expect them to overlap in $pq\binom{n}{2}$ edges. How much or little can we make them overlap? Let us write $\mathrm{disc}^+(G,H)$ for the largest amount by which we can exceed $pq\binom{n}{2}$, and $\mathrm{disc}^-(G,H)$ for the largest amount less than $pq\binom{n}{2}$ that we can achieve (more formal definitions are given below). We shall refer to these quantities as the *positive* and *negative discrepancy of G with respect to H*, and write $\mathrm{disc}(G,H) = \max\{\mathrm{disc}^+(G,H), \mathrm{disc}^-(G,H)\}$.

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The main aim of this paper is to prove a result (Theorem 1) of form

$$\mathrm{disc}^+(G,H)\mathrm{disc}^-(G,H) \geq c(p,q)n^3.$$

In particular, this implies immediately that $\mathrm{disc}(G,H) \geq c'(p,q)n^{3/2}$.

By taking H to belong to specific families of graphs, we obtain results on the distribution of edges in G. For instance, taking H to be a complete graph of order $n/2$ together with $n/2$ isolated vertices, we obtain results on the *discrepancy of* G (see [10, 3], and the next two subsections). Taking $G = K_{n/2,n/2}$ gives a bound on the *bipartite discrepancy* of G and (Corollary 3) proves a conjecture of Erdős, Goldberg, Pach and Spencer.

The rest of the paper is organized as follows. In Section 1.1, we discuss the discrepancy of a single graph, while in Section 1.2, we talk about the discrepancy of one graph with respect to another; we finish Section 1 with a discussion of our notation and conventions. The proof of Theorem 1 depends on results from the following four sections. In Section 2, we give a lower bound on $\mathrm{disc}^+(G,H)\mathrm{disc}^-(G,H)$ in terms of another parameter Δ^{abs} of the pair (G,H). In order to bound Δ^{abs}, we look at certain subgraphs of G and H that we shall refer to as *good 4-cycles*. We discuss good 4-cycles in Section 3, and in Section 4 we give a lower bound on Δ^{abs} in terms of the number of good 4-cycles in G and the number of good 4-cycles in H (thus bounding a parameter of G and H jointly in terms of simple parameters of G and H separately). In Section 5, we put these results together and prove our main result, namely a lower bound on $\mathrm{disc}^+(G,H)\mathrm{disc}^-(G,H)$. In the final section we present a substantial number of conjectures and open questions.

We remark that there are various other notions of graph discrepancy. For general accounts of discrepancy, see Sós [17], Beck and Sós [1], Matoušek [15] and Chazelle [4]; for ideas related to negative discrepancy, see Erdős, Faudree, Rousseau and Schelp [8], Krivelevich [14] and Keevash and Sudakov [13].

1.1 The discrepancy of a graph

For a graph G with n vertices and $e(G) = p\binom{n}{2}$, we define the *discrepancy*[1] of G to be

$$\mathrm{disc}(G) = \max_{S \subset V(G)} |e(S) - p\binom{|S|}{2}|.$$

[1] The definition of discrepancy in [10], which we followed in [3], was $\max_{S\subset V(G)} |e(S) - \frac{1}{2}\binom{|S|}{2}|$. Here we follow [9], as it seems more natural for graphs with density $p \neq 1/2$.

Thus the discrepancy is the maximum difference between the number of edges in an induced subgraph and the average number of edges in subgraphs of that order. Erdős and Spencer [10] proved that every graph G of order n has a set $S \subset V(G)$ with $|e(S) - \frac{1}{2}\binom{|S|}{2}| \geq cn^{3/2}$; more generally, they showed that if H is a k-uniform hypergraph of order n then there is $S \subset V(H)$ with $|e(S) - \frac{1}{2}\binom{|S|}{k}| \geq cn^{(k+1)/2}$. Thus every graph of density 1/2 has discrepancy at least $cn^{3/2}$. Erdős, Goldberg, Pach and Spencer [9] extended this to graphs of arbitrary density, showing that if $e(G) = p\binom{n}{2}$, where $2/(n-1) < p < 1 - 2/(n-1)$, then

$$\mathrm{disc}(G) \geq c\sqrt{r}n^{3/2},$$

where $r = \min\{p, 1-p\}$ and c is an absolute constant.

A subset S of vertices with large discrepancy can have either more or fewer edges than $p\binom{|S|}{2}$. Let us define the *positive discrepancy* of a graph G with n vertices and $p\binom{n}{2}$ edges by

$$\mathrm{disc}^+(G) = \max_{S \subset V(G)} \left(e(S) - p\binom{|S|}{2} \right)$$

and the *negative discrepancy* by

$$\mathrm{disc}^-(G) = \max_{S \subset V(G)} \left(p\binom{|S|}{2} - e(S) \right),$$

so $\mathrm{disc}(G) = \max\{\mathrm{disc}^-(G), \mathrm{disc}^+(G)\}$. Note that disc^+ and disc^- are both nonnegative, since (for any k) choosing S uniformly at random from all k-sets we have $\mathbb{E}\, e(S) = p\binom{|S|}{2}$.

By considering random graphs $G \in \mathcal{G}(n, 1/2)$, it can be seen that $\mathrm{disc}(G)$ can be as small as $O(n^{3/2})$. However, it is possible to have smaller one-sided discrepancies: $K_{n/2,n/2}$ and its complement $2K_{n/2}$ each have discrepancy $O(n)$ on one side, although we pay by having discrepancy $\Omega(n^2)$ on the other side (see [3] and [9] for further discussion, and extremal results on one-sided discrepancy). The trade-off was quantified in [3], where it was shown that for every graph G with n vertices and $p\binom{n}{2}$ edges, with $p(1-p) \geq 1/n$,

$$\mathrm{disc}^+(G)\mathrm{disc}^-(G) \geq cp(1-p)n^3. \tag{1}$$

A similar result holds for k-uniform hypergraphs (see [3] and section 6.1 below). Note that this extends the results of Erdős and Spencer [10] and of Erdős, Goldberg, Pach and Spencer [9] mentioned above.

1.2 Discrepancy of G with respect to H

In this paper, we consider the discrepancy of a *pair* of graphs G, H. Given graphs G and H of order n, with $e(G) = p\binom{n}{2}$ and $e(H) = q\binom{n}{2}$, a random placement of both graphs onto the same vertex set has an overlap of expected size $pq\binom{n}{2}$. We define the *positive discrepancy of* G *with respect to* H by

$$\mathrm{disc}^+(G,H) = \max_{G' \cong G} |E(G') \cap E(H)| - pq\binom{n}{2} \tag{2}$$

and the *negative discrepancy of* G *with respect to* H by

$$\mathrm{disc}^-(G,H) = pq\binom{n}{2} - \min_{G' \cong G} |E(G') \cap E(H)|, \tag{3}$$

where in both cases the maximum/minimum is taken over graphs G' isomorphic to G and with the same vertex set as H. Clearly both discrepancies are nonnegative, and are symmetric in G and H. We define

$$\mathrm{disc}(G,H) = \max\{\mathrm{disc}^+(G,H), \mathrm{disc}^-(G,H)\}. \tag{4}$$

The discrepancy measures the maximum and minimum overlap of edges that we can get by defining G and H on a common vertex set. If G and H are graphs with m_G and m_H edges respectively, then the maximum possible value of $\mathrm{disc}^+(G,H)$ is attained when one of the graphs is isomorphic to a subgraph of the other, while the maximum possible value of $\mathrm{disc}^-(G,H)$ is attained when the two graphs have an edge-disjoint packing into the complete graph or can cover its edges (equivalently, one of $\overline{G}$ and H is a subgraph of the other).

By restricting H to particular families we can pick out various parameters of G. For instance, if H is a clique of order αn together with $(1-\alpha)n$ isolated vertices and G has $p\binom{n}{2}$ edges, then $pq\binom{n}{2} = p\binom{\alpha n}{2}$ and so $\mathrm{disc}(G,H)$ is related to the discrepancy of G, as

$$\mathrm{disc}^+(G,H) = \max_{S \in V(G)^{(\alpha n)}} \left(e_G(S) - p\binom{\alpha n}{2}\right) \leq \mathrm{disc}^+(G)$$

and

$$\mathrm{disc}^-(G,H) = \max_{S \in V(G)^{(\alpha n)}} \left(p\binom{\alpha n}{2} - e_G(S)\right) \leq \mathrm{disc}^-(G).$$

Indeed,

$$\mathrm{disc}(G) = \max_i \mathrm{disc}(G, K_i \cup E_{n-i}),$$

where $K_i \cup E_{n-i}$ is the graph obtained by adding $n - i$ isolated vertices to K_i.

The main aim of this paper is to prove a lower bound in the product form (1) for the discrepancy of pairs of graphs. We shall show the following.

Theorem 1. *Let G and H be graphs of order n, and suppose that $e(G) = p\binom{n}{2}$ and $e(H) = q\binom{n}{2}$, where $16/n \le p, q \le 1 - 16/n$. Then*

$$\mathrm{disc}^+(G,H)\mathrm{disc}^-(G,H) \ge p^4(1-p)^4q^4(1-q)^4n^3/10^{20}.$$

By taking H to be a complete graph of order $n/2$ together with $n/2$ isolated vertices and G to be a random graph we see that (as in (1)) the n^3 bound is sharp, although the dependence on p and q can probably be improved (see Section 6.2 below for further discussion). Some upper and lower bound on p and q is necessary, however: if we take $H = K_{1,n-1}$ and G to be any regular graph of order n, then $\mathrm{disc}^+(G,H) = \mathrm{disc}^-(G,H) = \mathrm{disc}(G,H) = 0$.

The following corollary of Theorem 1 is immediate.

Corollary 2. *Let G and H be graphs of order n, and suppose that $e(G) = p\binom{n}{2}$ and $e(H) = q\binom{n}{2}$, where $16/n \le p, q \le 1 - 16/n$. Then*

$$\mathrm{disc}(G,H) \ge p^2(1-p)^2q^2(1-q)^2n^{3/2}/10^{10}.$$

If $H = K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ then $\mathrm{disc}(G,H)$ is the *bipartite discrepancy* $\mathrm{bdis}(G)$ defined by Erdős, Goldberg, Pach and Spencer [9], who conjectured that if $\frac{1}{2}\binom{n}{2} \le e(G) \le (1-\epsilon)\binom{n}{2}$ then the bipartite discrepancy of G

$$\mathrm{bdisc}(G) \ge \delta n^{3/2} \tag{5}$$

for some $\delta = \delta(\epsilon)$. The conjecture of Erdős, Goldberg, Pach and Spencer follows as an immediate consequence of Theorem 1

Corollary 3. *Let G be a graph of order n. Then suppose that $e(G) = p\binom{n}{2}$ where $16/n \le p \le 1 - 16/n$. Then*

$$\mathrm{bdis}(G) \ge p^2(1-p)^2n^{3/2}/10^{12}.$$

1.3 Definitions and notation

Throughout the rest of the paper, we will assume that G and H are graphs of order n with vertex set $V = [n]$. We define an action of the symmetric group S_n on G by

$$\pi(G) \equiv G_\pi = (V, E_\pi), \tag{6}$$

where

$$E_\pi = \{\pi(i)\pi(j) : ij \in E(G)\}. \tag{7}$$

Consider the vector space $\mathbb{R}^{\binom{n}{2}}$, where standard coordinate directions are indexed by elements of $V^{(2)} = [n]^{(2)}$ (in any canonical manner). We identify G with the vector $\boldsymbol{g} \in \mathbb{R}^{\binom{n}{2}}$ with coordinates

$$\boldsymbol{g}(ij) = \mathbf{1}_G(ij) - e(G)/\binom{n}{2}, \tag{8}$$

where $\mathbf{1}_G(ij) = 1$ if $ij \in E(G)$ and 0 otherwise. Similarly, we identify G_π with the corresponding vector $\boldsymbol{g}_\pi$. Writing $\mathbf{1} = (1, 1, \ldots, 1) = \mathbf{1}_{K_n}$, we get

$$\langle \boldsymbol{g}_\pi, \mathbf{1} \rangle = \langle \boldsymbol{g}, \mathbf{1} \rangle = \sum_{1 \le i < j \le n} \boldsymbol{g}(ij) = 0, \tag{9}$$

which is the reason for the second term on the right of the definition (8). Note that this implicitly defines an action of S_n on $\mathbb{R}^{\binom{n}{2}}$, by $\pi(\boldsymbol{v})(ij) = \boldsymbol{v}(\pi^{-1}(i)\pi^{-1}(j))$. In particular, $\boldsymbol{g}_\pi(e) = \boldsymbol{g}(\pi^{-1}(e))$.

For $\boldsymbol{\phi} \in \mathbb{R}^{\binom{n}{2}}$ and $\pi \in S_n$, we can consider the inner product $\langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle$. If π is chosen uniformly at random from S_n, it follows from (9) that, we have

$$\mathbb{E} \langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle = 0. \tag{10}$$

We define the *positive discrepancy of G with respect to ϕ* by

$$\operatorname{disc}^+(G, \boldsymbol{\phi}) = \max_{\pi \in S_n} \langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle$$

and the *negative discrepancy of G with respect to ϕ*

$$\operatorname{disc}^+(G, \boldsymbol{\phi}) = \max_{\pi \in S_n} \langle -\boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle = -\min_{\pi \in S_n} \langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle.$$

We also define the *discrepancy of G with respect to $\boldsymbol{\phi}$* by

$$\operatorname{disc}(G, \boldsymbol{\phi}) = \max\{\operatorname{disc}^+(G, \boldsymbol{\phi}), \operatorname{disc}^-(G, \boldsymbol{\phi})\}.$$

Note that both positive and negative discrepancies are nonnegative, by (10).

For a graph H, with corresponding vector $\boldsymbol{h}$ in $\mathbb{R}^{\binom{n}{2}}$, we have

$$\begin{aligned}\langle \boldsymbol{g}_\pi, \boldsymbol{h}\rangle &= \langle \mathbf{1}_{E(G_\pi)} - e(G_\pi)\mathbf{1}/\binom{n}{2}, \mathbf{1}_{E(H)} - e(H)\mathbf{1}/\binom{n}{2}\rangle \\ &= \langle \mathbf{1}_{E(G_\pi)}, \mathbf{1}_{E(H)}\rangle - e(G)e(H)/\binom{n}{2} \\ &= |E(G_\pi) \cap E(H)| - e(G)e(H)/\binom{n}{2},\end{aligned}$$

where we have used the fact that $\langle \mathbf{1}_{E(G_\pi)}, \mathbf{1}\rangle = e(G)$ and $\langle \mathbf{1}, \mathbf{1}_{E(H)}\rangle = e(H)$. In particular,

$$\mathrm{disc}^+(G, \boldsymbol{h}) = \max_{\pi\in S_n}\left(|E(G_\pi) \cap E(H)| - e(G)e(H)/\binom{n}{2}\right) = \mathrm{disc}^+(G, H)$$

and

$$\mathrm{disc}^-(G, \boldsymbol{h}) = \max_{\pi\in S_n}\left(e(G)e(H)/\binom{n}{2} - |E(G_\pi) \cap E(H)|\right) = \mathrm{disc}^-(G, H).$$

Thus our two definitions of discrepancy are consistent.

It will be useful later to consider inner products on certain subspaces of $\mathbb{R}^{\binom{n}{2}}$. Given vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, and a subset $E \subset V^{(2)}$, we define

$$\langle \boldsymbol{g}, \boldsymbol{h}\rangle_E = \sum_{e\in E} \boldsymbol{g}(e)\boldsymbol{h}(e). \tag{11}$$

Similarly, for $S \subset V$, we write

$$\langle \boldsymbol{g}, \boldsymbol{h}\rangle_S = \sum_{e\in S^{(2)}} \boldsymbol{g}(e)\boldsymbol{h}(e). \tag{12}$$

Note that for permutations π and σ, $\langle \boldsymbol{g}_\pi, \boldsymbol{h}\rangle = \langle \boldsymbol{g}, \boldsymbol{h}_{\pi^{-1}}\rangle$, and $\boldsymbol{g}_{\sigma\pi} = (\boldsymbol{g}_\pi)_\sigma$.

2 A bound on discrepancy

In this section we give a bound on $\mathrm{disc}(G, \boldsymbol{\phi})$ in terms of another parameter Δ^{abs} of G and $\boldsymbol{\phi}$. The most important case is when $\boldsymbol{\phi}$ corresponds to a graph

H as in (8); however, it costs nothing to prove a more general result for arbitrary vectors $\boldsymbol{\phi} \in \mathbb{R}^{\binom{n}{2}}$.

Let us fix $\boldsymbol{\phi} \in \mathbb{R}^{\binom{n}{2}}$, and recall that the coordinates of $\boldsymbol{\phi}$ are indexed by $V^{(2)}$, where $V = V(G) = [n]$. In particular, G and $\boldsymbol{\phi}$ are defined with reference to the same vertex set V, and V has a canonical ordering.

We will consider $\langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle$ for different permutations $\pi \in S_n$. Our aim is to prove that $\langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle$ varies quite a lot, taking both relatively large positive values and relatively large negative values. In order to do this, we consider a more restricted family of permutations, generated by a set of disjoint transpositions.

Let $v_1, w_1, \ldots, v_{\lfloor n/2 \rfloor}, w_{\lfloor n/2 \rfloor}$ be the first $2\lfloor n/2 \rfloor$ vertices in $V = V(G)$. For $1 \leq i \leq \lfloor n/2 \rfloor$, let $t^i = (v_i w_i)$ be the transposition that switches the ith pair of vertices. For a permutation π, and $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, \lfloor n/2 \rfloor\}$, we write

$$\pi^I = t^I \pi = t^{i_1} \cdots t^{i_r} \pi$$

for the permutation obtained by switching the r pairs of vertices corresponding to I. If $I = \{i\}$ we sometimes write π^i for π^I. Note that

$$\boldsymbol{g}_{\pi^I}(e) = \boldsymbol{g}((\pi^I)^{-1} e) = \boldsymbol{g}(\pi^{-1} t^I e) = \boldsymbol{g}_\pi(t^I e).$$

We will consider the effect of individual transpositions t^i on $\langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle$. We therefore define

$$\Delta_\pi^i = \langle \boldsymbol{g}_{\pi^i}, \boldsymbol{\phi} \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle. \tag{13}$$

We set

$$\begin{aligned} \Delta_\pi^+ &= \sum_{i=1}^{\lfloor n/2 \rfloor} \max\{\Delta_\pi^i, 0\}, \qquad (14) \\ \Delta_\pi^- &= \sum_{i=1}^{\lfloor n/2 \rfloor} -\min\{\Delta_\pi^i, 0\}, \end{aligned}$$

and

$$\Delta_\pi^{\text{abs}} = \sum_{i=1}^{\lfloor n/2 \rfloor} |\Delta_\pi^i| = \Delta_\pi^+ + \Delta_\pi^-. \tag{15}$$

Note that Δ_π^+ and Δ_π^- are both nonnegative. Finally, we define

$$\Delta^{\text{abs}}(G, \boldsymbol{\phi}) = \mathbb{E}\Delta_\pi^{\text{abs}}, \tag{16}$$

where the expectation is taken over π chosen uniformly at random from S_n. Note that $\Delta_\pi^{\mathrm{abs}}$ depends on the canonical ordering of V: although π acts on G, so that its position is symmetrized in (16), the transpositions t^i and the placement of $\boldsymbol{\phi}$ are fixed, so that $\Delta^{\mathrm{abs}}(G, \boldsymbol{\phi})$ is not symmetric in the coordinates of $\boldsymbol{\phi}$ (this will be apparent in (21) below).

Theorem 4. *For any G and any $\boldsymbol{\phi}$,*

$$\mathrm{disc}^+(G, \boldsymbol{\phi})\mathrm{disc}^-(G, \boldsymbol{\phi}) \geq \Delta^{\mathrm{abs}}(G, \boldsymbol{\phi})^2/100. \tag{17}$$

Proof. Let $\Delta = \Delta^{\mathrm{abs}}(G, \boldsymbol{\phi})$. We may clearly assume that $\mathrm{disc}^+(G, \boldsymbol{\phi}) \leq \mathrm{disc}^-(G, \boldsymbol{\phi})$, or else replace $\boldsymbol{\phi}$ by $-\boldsymbol{\phi}$. (This leaves both sides of (17) unchanged, except that positive and negative discrepancies are exchanged.) If (17) is false, then we have

$$\mathrm{disc}^+(G, \boldsymbol{\phi}) = \frac{\Delta}{5\alpha}, \tag{18}$$

for some $\alpha \geq 2$. It is therefore enough to show that

$$\mathrm{disc}^-(G, \boldsymbol{\phi}) \geq \frac{\alpha\Delta}{20}, \tag{19}$$

and (17) will follow immediately.

How can we prove (19)? Suppose that there is a permutation π for which $\langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle$ and Δ_π^+ are both close to their expectation, so (as we shall see) $\langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle$ is close to 0 and Δ_π^+ is about $\Delta/2$. If the effects of applying individual transpositions were independent, then we could take $I = \{i : \Delta_\pi^i > 0\}$ and get $\langle \boldsymbol{g}_{\pi^I}, \boldsymbol{\phi} \rangle = \langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle + \Delta_\pi^+ \approx \Delta/2$, which would contradict (18). But the transpositions do not behave independently: for instance, comparing the effects of performing t^i and t^j separately with the effect of performing them together shows that the edges between $\{v_i, w_i\}$ and $\{v_j, w_j\}$ can make different contributions in the two cases. In order to satisfy (18), the interactions between transpositions must therefore tend to decrease the inner product. In fact, we will show that performing the set of all transpositions $\{t^i : i \in I\}$ simultaneously must decrease the inner product enough to satisfy (19).

Of course, in order to make such an argument work, we must find a suitable permutation π. However, we cannot expect to find π such that $\langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle$ and Δ_π^+ are both close to their expected values simultaneously. Instead, we will work with a linear combination of the two quantities. (A similar approach, in a rather simpler setting, is used in [3].) We now proceed with the argument.

Let $\pi \in S_n$ be chosen uniformly at random. Then, for $1 \leq i \leq \lfloor n/2 \rfloor$,

$$\mathbb{E}\Delta_\pi^i = \mathbb{E}(\langle \boldsymbol{g}_{\pi^i}, \boldsymbol{\phi} \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle) = 0,$$

as π^i is also uniformly distributed over S_n. Summing over i, we get

$$\mathbb{E}(\Delta_\pi^+ - \Delta_\pi^-) = 0.$$

It follows from (15) and (16) that

$$\mathbb{E}\Delta_\pi^+ = \mathbb{E}\Delta_\pi^- = \Delta/2.$$

Since $\mathbb{E}\langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle = 0$, we have

$$\mathbb{E}((\alpha+1)\langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle + \Delta_\pi^+) = \frac{\Delta}{2},$$

and so we can choose π such that

$$(\alpha+1)\langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle + \Delta_\pi^+ \geq \frac{\Delta}{2}. \tag{20}$$

Let

$$I = \{i : \Delta_\pi^i > 0\}$$

and

$$V^+ = \bigcup_{i \in I} \{v_i, w_i\}.$$

Thus V^+ is the union of pairs of vertices whose reversal *on their own* increases the inner product with $\boldsymbol{\phi}$.

We decompose Δ_π^i by expanding the inner products on the right of (13) and regrouping. Since $\Delta_\pi^i = \sum_e (\boldsymbol{g}_{\pi^i}(e) - \boldsymbol{g}_\pi(e))\boldsymbol{\phi}(e)$ and $\boldsymbol{g}_{\pi^i}(e) = \boldsymbol{g}_\pi(e)$ unless e contains exactly one vertex from $\{\pi(v_i), \pi(w_i)\}$, we have

$$\begin{aligned} \Delta_\pi^i &= \sum_{\substack{x \notin \{v_i, w_i\} \\ y \in \{v_i, w_i\}}} (\boldsymbol{g}_{\pi^i}(xy) - \boldsymbol{g}_\pi(xy))\boldsymbol{\phi}(xy) \\ &= \sum_{\substack{x \notin \{v_i, w_i\} \\ y \in \{v_i, w_i\}}} (\boldsymbol{g}_\pi(xt^i(y)) - \boldsymbol{g}_\pi(xy))\boldsymbol{\phi}(xy) \\ &= \sum_{x \notin \{v_i, w_i\}} \delta^i(x), \end{aligned}$$

where

$$\delta^i(x) = (\boldsymbol{g}_\pi(w_i x) - \boldsymbol{g}_\pi(v_i x))\boldsymbol{\phi}(v_i x) + (\boldsymbol{g}_\pi(v_i x) - \boldsymbol{g}_\pi(w_i x))\boldsymbol{\phi}(w_i x)$$
$$= (\boldsymbol{g}_\pi(w_i x) - \boldsymbol{g}_\pi(v_i x))(\boldsymbol{\phi}(v_i x) - \boldsymbol{\phi}(w_i x)) \quad (21)$$

is the contribution to Δ^i_π of the pairs $\{x, v_i\}$ and $\{x, w_i\}$. We write $\Delta^i_\pi = \Delta^i_{\text{in}} + \Delta^i_{\text{out}}$, where

$$\Delta^i_{\text{in}} = \sum_{x \in V^+ \setminus \{v_i, w_i\}} \delta^i(x)$$

and

$$\Delta^i_{\text{out}} = \sum_{x \in V \setminus V^+} \delta^i(x).$$

We define the sum over I of these contributions to be

$$\Delta_{\text{in}} = \sum_{i \in I} \Delta^i_{\text{in}}$$

and

$$\Delta_{\text{out}} = \sum_{i \in I} \Delta^i_{\text{out}}.$$

So

$$\Delta^+_\pi = \Delta_{\text{in}} + \Delta_{\text{out}}. \quad (22)$$

Informally, Δ_{in} measures the contribution to Δ^+_π from edges inside V^+ and Δ_{out} measures the contribution from edges between V^+ and $V \setminus V^+$ (note that edges outside V^+ do not contribute anything).

We shall prove (19) by flipping a random subset of pairs from V^+, so we shall also have to consider the interactions when we perform more than one transposition. Let

$$\Delta_{\text{all}} = \langle \boldsymbol{g}_{\pi^I}, \boldsymbol{\phi} \rangle_{V^+} - \langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle_{V^+},$$

where the inner product uses the notation defined at (12). Thus Δ_{all} measures the effects (inside V^+) of applying all the transpositions $\{t^i : i \in I\}$ simultaneously, whereas Δ_{in} measures the effect (inside V^+) of applying them individually. We have

$$\Delta_{\text{all}} = \langle \boldsymbol{g}_{\pi^I}, \boldsymbol{\phi} \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{\phi} \rangle - \Delta_{\text{out}}, \quad (23)$$

since π^I and π agree on vertices outside V^+, and (a short calculation shows that) Δ_{out} is the sum of $\boldsymbol{g}_{\pi^I}(e)\boldsymbol{\phi}(e) - \boldsymbol{g}_\pi(e)\boldsymbol{\phi}(e)$ over edges between V^+ and its complement.

After all this notation, we can now complete the argument. Let $J \subset I$ be a random subset, where each $i \in I$ is chosen independently with probability $1/\alpha$. Note that if vw is an edge outside V^+ then π^J and π agree on vw; if vw is an edge from V^+ to $V \setminus V^+$ then it is shifted with probability $1/\alpha$; if vw is inside V^+ (and is not of form $v_i w_i$) then it has each end shifted on its own with probability $\frac{1}{\alpha}(1-\frac{1}{\alpha})$ and both ends shifted (leading to an interaction between transpositions) with probability $1/\alpha^2$. Grouping terms together, we get

$$\begin{aligned}
\mathbb{E}\langle \boldsymbol{g}_{\pi^J}, \boldsymbol{\phi}\rangle &= \langle \boldsymbol{g}_\pi, \boldsymbol{\phi}\rangle + \frac{1}{\alpha}\sum_{i\in I}\Delta_{\text{out}}^i + \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)\sum_{i\in I}\Delta_{\text{in}}^i + \frac{1}{\alpha^2}\Delta_{\text{all}} \\
&= \langle \boldsymbol{g}_\pi, \boldsymbol{\phi}\rangle + \frac{1}{\alpha}\Delta_{\text{out}} + \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)\Delta_{\text{in}} + \frac{1}{\alpha^2}\Delta_{\text{all}} \\
&= \langle \boldsymbol{g}_\pi, \boldsymbol{\phi}\rangle + \frac{1}{\alpha}\Delta_\pi^+ + \frac{1}{\alpha^2}(\Delta_{\text{all}} - \Delta_{\text{in}}). \qquad (24)
\end{aligned}$$

Since, by (18), $\langle \boldsymbol{g}_{\pi^J}, \boldsymbol{\phi}\rangle \le \Delta/5\alpha$ for all π, we have $\mathbb{E}\langle \boldsymbol{g}_{\pi^J}, \boldsymbol{\phi}\rangle \le \Delta/5\alpha$ and so, by (24),

$$\Delta_{\text{all}} \le \frac{\alpha\Delta}{5} - \alpha^2\langle \boldsymbol{g}_\pi, \boldsymbol{\phi}\rangle - \alpha\Delta_\pi^+ + \Delta_{\text{in}}. \qquad (25)$$

It follows from (23), (25) and (22) that

$$\begin{aligned}
\langle \boldsymbol{g}_{\pi^I}, \boldsymbol{\phi}\rangle &= \langle \boldsymbol{g}_\pi, \boldsymbol{\phi}\rangle + \Delta_{\text{out}} + \Delta_{\text{all}} \\
&\le \frac{\alpha\Delta}{5} + (1-\alpha^2)\langle \boldsymbol{g}_\pi, \boldsymbol{\phi}\rangle - \alpha\Delta_\pi^+ + \Delta_{\text{in}} + \Delta_{\text{out}} \\
&= \frac{\alpha\Delta}{5} + (1-\alpha^2)\langle \boldsymbol{g}_\pi, \boldsymbol{\phi}\rangle - (\alpha-1)\Delta_\pi^+. \qquad (26)
\end{aligned}$$

By (20) we have

$$(\alpha^2-1)\langle \boldsymbol{g}_\pi, \boldsymbol{\phi}\rangle + (\alpha-1)\Delta_\pi^+ \ge (\alpha-1)\frac{\Delta}{2}$$

and so by (26)

$$\langle \boldsymbol{g}_{\pi^I}, \boldsymbol{\phi}\rangle \le \frac{\alpha\Delta}{5} - (\alpha-1)\frac{\Delta}{2} \le -\frac{\alpha\Delta}{20},$$

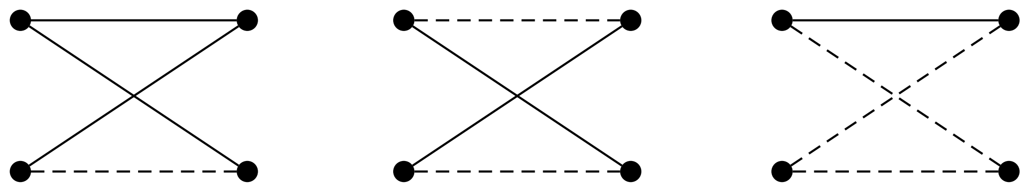
since $\alpha \ge 2$. □

In order to apply Theorem 4, we need some lower bound on $\Delta^{\text{abs}}(G, \boldsymbol{\phi})$, when $\boldsymbol{\phi}$ corresponds to a graph H. This is our next task.

3 Good 4-cycles

In order to give a bound on $\Delta^{\text{abs}}(G, H)$, we will need a lemma about graphs.

Given a graph G with vertex set V, and a 4-cycle C in the complete graph K_V, we say that C is *good (in G)* if either $|E(G) \cap E(C)|$ is odd, or $E(G) \cap E(C)$ is a pair of vertex-disjoint edges. (So the bad cases are when $E(C) \subset E(G)$ or $E(C) \cap E(G) = \emptyset$ or $E(G) \cap E(C)$ is a path of length 2.)



Good 4-cycles

Lemma 5. *Suppose that G has n vertices and $p\binom{n}{2}$ edges, where $\min\{p, 1 - p\} \geq 16/n$. Then the probability that a 4-cycle chosen uniformly at random is good is at least $p^2(1-p)^2/5040$.*

Proof. Note that $n \geq 32$ by the condition on p and $1 - p$. We may assume $p \leq 1/2$, since goodness is invariant under taking complements. We first consider a random 8-tuple $X = \{x_1, \dots, x_8\}$. Then with probability at least $p^2(1-p)^2/24$ we have $x_1x_2, x_3x_4 \in E(G)$ and $x_5x_6, x_7x_8 \notin E(G)$ (since $x_1x_2 \in E(G)$ with probability p, and $e(G - x_1 - x_2) \geq e(G) - 2n + 3$, so $\mathbb{P}(x_3x_4 \in E(G)|x_1x_2 \in E(G) \geq p - 5/n)$, and so on). If X does not contain a good C_4 then considering the 4-cycles $x_1x_2x_3x_4$ and $x_1x_2x_4x_3$ shows that $\{x_1, x_2, x_3, x_4\}$ induces a complete graph; similarly, considering $x_5x_6x_7x_8$ and $x_5x_6x_8x_7$ shows that $\{x_5, x_6, x_7, x_8\}$ is an independent set. We can now find two vertex-disjoint edges or two vertex-disjoint non-edges between $\{x_1, x_2, x_3, x_4\}$ and $\{x_5, x_6, x_7, x_8\}$, say x_1x_5 and x_2x_6. Then $x_1x_5x_6x_2$ is a good C_4. Thus X contains a good C_4 with probability at least $p^2(1-p)^2/24$.

Since each good C_4 is contained in $\binom{n-4}{4}$ 8-sets, the number of good C_4s must be at least

$$\frac{p^2(1-p)^2}{24} \cdot \frac{\binom{n}{8}}{\binom{n-4}{4}} = \frac{p^2(1-p)^2}{8!} n(n-1)(n-2)(n-3).$$

There are $3\binom{n}{4}$ copies of C_4 in total, so a random C_4 is good with probability at least

$$\frac{p^2(1-p)^2}{8!} \cdot \frac{4!}{3} = p^2(1-p)^2/5040.$$

□

The constants in Lemma 5 could probably be substantially improved. However, note that a lower bound on p of form $\Omega(1/n)$ is necessary, as the star $K_{1,n}$ has $p \sim 2/n$ and no good cycles.

4 Inner products and good cycles

Let G and H be graphs, and $\boldsymbol{g}, \boldsymbol{h}$ the associated vectors defined by (8). We are interested in the inner product $\langle \boldsymbol{g}_\pi, \boldsymbol{h} \rangle$ for various choices of the permutation π. In particular, in order to bound Δ^{abs} in the next section, we shall want a lower bound on expressions of form

$$\mathbb{E}\, |\langle \boldsymbol{g}_{t\pi}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{h} \rangle|, \tag{27}$$

where t is a given transposition and π is chosen uniformly at random. We shall approach (27) by considering the effect of replacing $\boldsymbol{h}$ by $\boldsymbol{h}_\sigma$ for various permutations σ, and decomposing $\langle \boldsymbol{g}_{t\pi}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{h} \rangle$ into a sum of simpler quantities.

Given a 4-cycle $C = uvwx$, we define $\langle \boldsymbol{g}, \boldsymbol{h} \rangle_C = \langle \boldsymbol{g}, \boldsymbol{h} \rangle_{E(C)}$, so

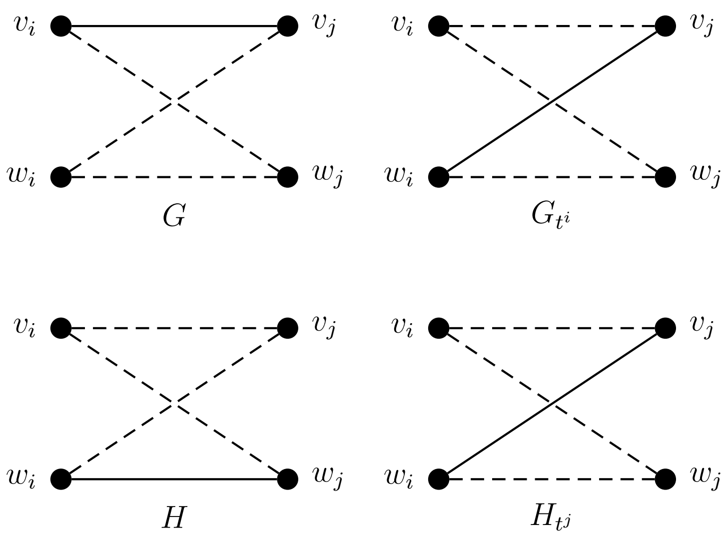
$$\langle \boldsymbol{g}, \boldsymbol{h} \rangle_C = \boldsymbol{g}(uv)\boldsymbol{h}(uv) + \boldsymbol{g}(vw)\boldsymbol{h}(vw) + \boldsymbol{g}(wx)\boldsymbol{h}(wx) + \boldsymbol{g}(xu)\boldsymbol{h}(xu).$$

The following lemma is crucial.

Lemma 6. *If the 4-cycle $C = v_i v_j w_i w_j$ is good in both G and H, and t^i and t^j are the transpositions $(v_i w_i)$ and $(v_j w_j)$ respectively, then*

$$|(\langle \boldsymbol{g}_{t^i}, \boldsymbol{h} \rangle_C - \langle \boldsymbol{g}, \boldsymbol{h} \rangle_C) - (\langle \boldsymbol{g}_{t^i}, \boldsymbol{h}_{t^j} \rangle_C - \langle \boldsymbol{g}, \boldsymbol{h}_{t^j} \rangle_C)| \geq 1. \tag{28}$$

Proof. This is a straightforward case analysis. For instance, consider the example in the diagram.



Then $\langle \boldsymbol{g}, \boldsymbol{h} \rangle_C$, $\langle \boldsymbol{g}_{t^i}, \boldsymbol{h} \rangle_C$ and $\langle \boldsymbol{g}, \boldsymbol{h}_{t^j} \rangle_C$ are all equal to

$$(1-p)(-q) + (1-q)(-p) + 2(-p)(-q) = 4pq - p - q,$$

and

$$\langle \boldsymbol{g}_{t^i}, \boldsymbol{h}_{t^j} \rangle_C = (1-p)(1-q) + 3(-p)(-q) = 1 - p - q + 4pq,$$

so $\langle \boldsymbol{g}_{t^i}, \boldsymbol{h} \rangle_C - \langle \boldsymbol{g}, \boldsymbol{h} \rangle_C = 0$, while

$$\langle \boldsymbol{g}_{t^i}, \boldsymbol{h}_{t^j} \rangle_C - \langle \boldsymbol{g}, \boldsymbol{h}_{t^j} \rangle_C = (1 - p - q + 4pq) - (4pq - p - q) = 1.$$

The other cases are similar. □

We can now give a lower bound on (27) in terms of the numbers of good cycles in G and H.

Lemma 7. *Let G and H be graphs with vertex set V with $|V| = n \geq 15$. Suppose that the pair $\{v, w\}$ belongs to $r_h \cdot \binom{n-2}{2}$ good 4-cycles $vxwy$ in H and that G contains $r_g \cdot 3\binom{n}{4}$ good 4-cycles. Let τ be the transposition (vw), and suppose that $\pi \in S_n$ is chosen uniformly at random. Then*

$$\mathbb{E}|\langle \boldsymbol{g}_{\tau\pi}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_{\pi}, \boldsymbol{h} \rangle| \geq r_g r_h \sqrt{n}/10.$$

Proof. A 4-cycle $vxwy$ containing v and w at distance 2, and with $\{x, y\} \subset V \setminus \{v, w\}$ chosen uniformly at random, is good in H with probability r_h. Thus there is a sequence $x_1, y_1, \ldots, x_{\lfloor (n-2)/2 \rfloor}, y_{\lfloor (n-2)/2 \rfloor}$ of distinct vertices in $V \setminus \{v, w\}$ such that at least $r_h \lfloor (n-2)/2 \rfloor \geq r_h(n-3)/2$ of the 4-cycles vx_iwy_i are good (consider a random choice).

We generate our random permutation π in two steps: let $\rho \in S_n$ be chosen uniformly at random, and let σ be the product of a random subset of the transpositions $\{(x_i y_i) : 1 \leq i \leq \lfloor (n-2)/2 \rfloor\}$, taking each transposition independently with probability 1/2. Thus

$$\sigma = (x_1 y_1)^{\epsilon_1} \cdots (x_{\lfloor n/2 \rfloor}, y_{\lfloor (n-2)/2 \rfloor})^{\epsilon_{\lfloor (n-2)/2 \rfloor}},$$

where $\epsilon_i \in \{0, 1\}$ for each i. We let $\pi = \sigma\rho$, so that π has uniform distribution in S_n as required.

For any $\boldsymbol{g}'$, and any $\gamma \in S_n$, we have $\langle \boldsymbol{g}'_\gamma, \boldsymbol{h}_\gamma \rangle = \langle \boldsymbol{g}', \boldsymbol{h} \rangle$. Since $\sigma = \sigma^{-1}$, we have $\langle \boldsymbol{g}'_\sigma, \boldsymbol{h} \rangle = \langle \boldsymbol{g}', \boldsymbol{h}'_{\sigma^{-1}} \rangle = \langle \boldsymbol{g}', \boldsymbol{h}'_\sigma \rangle$. Furthermore, σ and τ commute, so

$$\langle \boldsymbol{g}_{\tau\pi}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{h} \rangle = \langle \boldsymbol{g}_{\tau\sigma\rho}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_{\sigma\rho}, \boldsymbol{h} \rangle = \langle \boldsymbol{g}_{\tau\rho}, \boldsymbol{h}_\sigma \rangle - \langle \boldsymbol{g}_\rho, \boldsymbol{h}_\sigma \rangle. \tag{29}$$

(Note that $\boldsymbol{g}_{\tau\sigma\rho} = \boldsymbol{g}_{\sigma\tau\rho} = (\boldsymbol{g}_{\tau\rho})_\sigma$.)

Since τ is fixed, and π, ρ are both uniformly distributed,

$$\mathbb{E}|\langle \boldsymbol{g}_{\tau\pi}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{h} \rangle| = \mathbb{E}|\langle \boldsymbol{g}_{\tau\rho}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_\rho, \boldsymbol{h} \rangle|.$$

Thus the triangle inequality gives

$$\begin{aligned}
\mathbb{E}|\langle \boldsymbol{g}_{\tau\pi}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{h} \rangle| &= \frac{1}{2}\mathbb{E}\left(|\langle \boldsymbol{g}_{\tau\pi}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{h} \rangle| + \mathbb{E}|\langle \boldsymbol{g}_{\tau\rho}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_\rho, \boldsymbol{h} \rangle|\right) \\
&\geq \frac{1}{2}\mathbb{E}|(\langle \boldsymbol{g}_{\tau\pi}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{h} \rangle) - (\langle \boldsymbol{g}_{\tau\rho}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_\rho, \boldsymbol{h} \rangle)| \\
&= \frac{1}{2}\mathbb{E}|\langle \boldsymbol{g}_{\tau\rho}, \boldsymbol{h}_\sigma \rangle - \langle \boldsymbol{g}_\rho, \boldsymbol{h}_\sigma \rangle + \langle \boldsymbol{g}_\rho, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_{\tau\rho}, \boldsymbol{h} \rangle|,
\end{aligned}$$

where the last line follows from (29) (expectations are over both σ and ρ).

But $\langle \boldsymbol{g}_{\tau\rho}, \boldsymbol{h}_\sigma \rangle - \langle \boldsymbol{g}_\rho, \boldsymbol{h}_\sigma \rangle + \langle \boldsymbol{g}_\rho, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_{\tau\rho}, \boldsymbol{h} \rangle$ can be decomposed as

$$\sum_{i=1}^{\lfloor (n-2)/2 \rfloor} \left(\langle \boldsymbol{g}_{\tau\rho}, \boldsymbol{h}_\sigma \rangle_{C_i} - \langle \boldsymbol{g}_\rho, \boldsymbol{h}_\sigma \rangle_{C_i} + \langle \boldsymbol{g}_\rho, \boldsymbol{h} \rangle_{C_i} - \langle \boldsymbol{g}_{\tau\rho}, \boldsymbol{h} \rangle_{C_i}\right) = \sum_{i=1}^{\lfloor (n-2)/2 \rfloor} \epsilon_i \lambda_i, \tag{30}$$

where $C_i = vx_iwy_i$, the λ_i depend only on ρ and, by Lemma 6, $|\lambda_i| \geq 1$ whenever C_i is good in both G_ρ and H. (Note that only edges between

$\{v, w\}$ and $\{x_1, y_1, \ldots, x_{\lfloor (n-2)/2 \rfloor}, y_{\lfloor (n-2)/2 \rfloor}\}$ contribute to the sum: the rest are cancelled out.)

Let $d = d(\rho)$ ($\leq \lfloor n/2 \rfloor$) be the number of C_i that are good in both G and H. Each ϵ_i independently in (30) is 0 or 1 with probability 1/2. Using basic facts about random sums (see, for instance, [3]) we get that, conditioning on a fixed ρ,

$$\begin{aligned}
\mathbb{E}_\sigma |\langle \boldsymbol{g}_{\tau\pi}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{h} \rangle| &\geq \frac{1}{2} \mathbb{E}_\sigma |\sum_{i=1}^{\lfloor (n-2)/2 \rfloor} \epsilon_i \lambda_i| \\
&\geq \frac{1}{2} \sqrt{\frac{d(\rho)}{8}} \\
&\geq \frac{d(\rho)}{4\sqrt{n}}.
\end{aligned}$$

But then, by the tower law for expectation,

$$\mathbb{E}_\pi |\langle \boldsymbol{g}_{\tau\pi}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{h} \rangle| \mathbb{E}_\rho \frac{d(\rho)}{4\sqrt{n}}.$$

Since at least $r_h(n-3)/2$ of the C_i are good in H, by our initial choice of $x_1, y_1, \ldots$, and a (random) 4-cycle is good in G with probability r_g, the expected number of 4-cycles C_i that are good in both G and H is at least $r_g r_h (n-3)/2$. It follows that

$$\mathbb{E}_\pi |\langle \boldsymbol{g}_{\tau\pi}, \boldsymbol{h} \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{h} \rangle| \geq r_g r_h (n-3)/8\sqrt{n} \geq r_g r_h \sqrt{n}/10.$$

□

5 The main result

In this short section, we put together the results we have proved in previous sections to give our main result.

Given graphs G and H, we know from Theorem 4 that we can bound $\mathrm{disc}^+(G, H)\mathrm{disc}^-(G, H)$ in terms of the quantity $\Delta^{\mathrm{abs}}(G, H)$ (or, more precisely, $\Delta^{\mathrm{abs}}(G, \boldsymbol{h})$) defined in (16). Because we defined this quantity by using a restricted set of permutations, and then took an expectation over permutations applied to G, we have $\Delta^{\mathrm{abs}}(G_\pi, H) = \Delta^{\mathrm{abs}}(G, H)$ for any permutation π. However, $\Delta^{\mathrm{abs}}(G, H_\pi)$ may be different from $\Delta^{\mathrm{abs}}(G, H)$. The idea of the proof is therefore to show that there is a placement H_ρ of H for which $\Delta^{\text{abs}}(G, H_\rho)$ is large and then apply Theorem 4.

We will use the following corollary of Lemma 7.

Corollary 8. *Suppose G and H are graphs of order $n \geq 15$, and that a random 4-cycle in G is good with probability p_g and a random 4-cycle in H is good with probability p_h. let $\rho \in S_n$ be chosen uniformly at random. Then*

$$\mathbb{E}\Delta^{\text{abs}}(G, H_\rho) \geq p_g p_h n^{3/2}/22.$$

Proof. Let us begin by fixing a permutation ρ and consider the graph H_ρ, with associated vector $\boldsymbol{h}_\rho \in \mathbb{R}^{\binom{n}{2}}$. Let $v_1, w_1, \ldots, v_{\lfloor n/2 \rfloor}, w_{\lfloor n/2 \rfloor}$ be the first $2\lfloor n/2 \rfloor$ vertices of $V(G)$ (recall that the vertex set has a canonical ordering), and define $r_1, \ldots, r_{\lfloor (n-2)/2 \rfloor}$ so that $r_i \binom{n-2}{2}$ is the number of good 4-cycles of form $v_i x w_i y$ in H. Set $R = \sum_{i=1}^{\lfloor n/2 \rfloor} r_i$. Note that R and the quantities r_i depend on ρ.

Writing $t^i = (v_i w_i)$, Lemma 7 implies that

$$\mathbb{E}_\pi \left(|\langle \boldsymbol{g}_{t^i \pi}, \boldsymbol{h}_\rho \rangle - \langle \boldsymbol{g}_\pi, \boldsymbol{h}_\rho \rangle| \right) \geq p_g r_i \sqrt{n}/10,$$

and so (using the notation defined at (15)),

$$\Delta^{\text{abs}}(G, H_\rho) = \mathbb{E}(\Delta_\pi^{\text{abs}}) \geq \sum_{i=1}^{\lfloor n/2 \rfloor} p_g r_i \sqrt{n}/10 = R p_g \sqrt{n}/10.$$

Now, choosing ρ uniformly at random, $\mathbb{E}R = \lfloor n/2 \rfloor \mathbb{E} r_1 = \lfloor n/2 \rfloor p_h$, so

$$\begin{aligned} \mathbb{E}\Delta^{\text{abs}}(G, H_\rho) &\geq \left\lfloor \frac{n}{2} \right\rfloor p_g p_h \sqrt{n}/10 \\ &\geq p_g p_h n^{3/2}/22. \end{aligned}$$

□

Lemma 5 allows us to convert this into an estimate in terms of $e(G)$ and $e(H)$.

Theorem 9. *Let G and H be graphs with a common vertex set of order n, with $e(G) = p\binom{n}{2}$ and $e(H) = q\binom{n}{2}$, where $16/n \leq p, q \leq 1 - 16/n$. There is a permutation ρ such that*

$$\Delta^{\text{abs}}(G, H_\rho) \geq p^2(1-p)^2 q^2 (1-q)^2 n^{3/2}/10^9. \tag{31}$$

Proof. Lemma 5 implies that $p_g \geq p^2(1-p)^2/5040$ and $p_h \geq q^2(1-q)^2/5040$. We are done by Corollary 8. $\square$

We can now complete the proof of our main result.

Proof of Theorem 1. This follows immediately from (17) and (31). $\square$

6 Discussion and open problems

A number of open questions remain, and we gather a selection of these together under various headings.

6.1 Extensions to hypergraphs

The definition of discrepancy extends naturally to r-uniform hypergraphs. If H is k-uniform, with n vertices and $p\binom{n}{k}$ edges, we define

$$\mathrm{disc}^+(H) = \max_{S\subset V(H)} \left(e(S) - p\binom{|S|}{k} \right),$$
$$\mathrm{disc}^-(H) = \max_{S\subset V(H)} \left(p\binom{|S|}{k} - e(S) \right)$$

and

$$\mathrm{disc}(H) = \max\{\mathrm{disc}^-(H), \mathrm{disc}^+(H)\}.$$

It was shown in [3] that if $p(1-p) \geq 1/n$ then

$$\mathrm{disc}^+(H)\mathrm{disc}^-(H) \geq c_k p(1-p)n^{k+1}.$$

Similarly, we can define $\mathrm{disc}^+(G,H)$, $\mathrm{disc}^-(G,H)$ and $\mathrm{disc}(G,H)$ for k-uniform hypergraphs G and H (just replace $\binom{n}{2}$ by $\binom{n}{k}$ in (2) and (3); (4) is unchanged).

An obvious question is whether Theorem 1 has an extension to hypergraphs. We conjecture that it does.

Conjecture 10. *Let G and H be k-uniform hypergraphs with vertex set $[n]$, and suppose $e(G) = p\binom{n}{k}$ and $e(H) = q\binom{n}{k}$, where $1/n \leq p, q \leq 1 - 1/n$. Then*

$$\mathrm{disc}^+(G,H)\mathrm{disc}^-(G,H) \geq c(p,q)n^{k+1}.$$

It seems likely that it should be possible to take $c(p, q)$ to be a polynomial in p and q.

More generally, it may be possible to let r grow with n. As a first step in this direction, we raise the following question.

Problem 11. *Consider the family $\mathcal{X}$ of $(n/2)$-uniform hypergraphs on n vertices, with $\frac{1}{2}\binom{n}{n/2}$ edges. What is*

$$\min_{G\in\mathcal{X}} \operatorname{disc}(G)?$$

Of course, there are many variants of this problem; and it would also be desirable to answer the product form of the question, that is to determine

$$\min_{G\in\mathcal{X}} \operatorname{disc}^+(G)\operatorname{disc}^-(G).$$

More ambitiously, it would be very interesting to know the values of

$$\min_{G,H\in\mathcal{X}} \operatorname{disc}^+(G,H)\operatorname{disc}^-(G,H). \qquad (32)$$

6.2 Sharpness of constants and random graphs

How good is the constant $p^4(1-p)^4q^4(1-q)^4$ in Theorem 1, as a function of p and q? For single graphs with density p, we know that $\operatorname{disc}^+(G)\operatorname{disc}^-(G) \geq cp(1-p)n^3$, which is much larger when p is small. A natural conjecture would be that when we have two graphs with densities p and q we should have a constant of form $cp(1-p)q(1-q)$. However, this is not correct. For $p < 1/2$, consider the graphs $G = K_{pn,(1-p)n}$ and $H = K_{n/2,n/2}$, so $e(G) = p(1-p)n^2$ and $e(H) = n^2/4$. The maximum overlap between G and H is obtained by putting the smaller vertex class from G on one side of H, while the minimum overlap is obtained by splitting it equally between the two sides. Thus

$$\max_{\pi} |E(G_\pi \cap E(H)| = pn \cdot \frac{n}{2}$$

and

$$\min_{\pi} |E(G_\pi \cap E(H)| = 2\frac{pn}{2}\left(\frac{n}{2} - \frac{pn}{2}\right) = p(1-p)n^2/2.$$

Since

$$\begin{aligned}
\mathbb{E}|E(G_\pi) \cap E(H)| &= p(1-p)n^2 \cdot \frac{n^2}{4}\Big/\binom{n}{2} \\
&= \frac{1}{2}p(1-p)n^3/(n-1) \\
&= p(1-p)n^2/2 + p(1-p)n/2 + O(1),
\end{aligned}$$

we have

$$\mathrm{disc}^+(G,H) \sim p^2n^2/2$$

and

$$\mathrm{disc}^-(G,H) \sim p(1-p)n/2,$$

which gives

$$\mathrm{disc}^+(G,H)\mathrm{disc}^-(G,H) \sim p^3(1-p)n^3/4.$$

So with $p \le 1/2$, noting that $(1-p)^3 \ge 1/8$, we obtain an upper bound of form $cp^3(1-p)^3$.

A different upper bound comes from considering random graphs. Fixing p and q, let $G \in \mathcal{G}(n,p)$ be a random graph and let H be a clique of order $\sqrt{q}n$ (plus some isolated vertices), so $e(H) \sim q\binom{n}{2}$. We shall use a version of Chernoff's inequality (see Janson, Łuczak and Rucinski [12]): if $X \sim B(N,p)$ and $t > 0$, then $\mathbb{P}(|X - Np| \ge t) \le 2e^{-t^2/(2Np+2t/3)}$. For a random permutation π, and $N = \binom{\sqrt{q}n}{2} \sim qn^2/2$, $|E(G_\pi) \cap E(H)|$ has distribution $B(N,p)$. Since $\mathbb{E}\,|E(G_\pi) \cap E(H)| = pN \sim pqn^2/2$, and there are

$$\binom{n}{\sqrt{q}n} = O\left(n^{-1/2}\left(\sqrt{q}^{\sqrt{q}}(1-\sqrt{q})^{1-\sqrt{q}}\right)^{-n}\right)$$

distinct ways for G_π to overlap with H, we have $\mathbb{P}(\mathrm{disc}(G,H) > \lambda) < 1/n$ provided

$$\binom{n}{\sqrt{q}n}\mathbb{P}(|B(N,p) - Np| > \lambda) < \frac{1}{n}. \tag{33}$$

The left hand side of (33) is at most

$$cn^{-1/2}\left(\sqrt{q}^{\sqrt{q}}(1-\sqrt{q})^{1-\sqrt{q}}\right)^{-n} e^{-\lambda^2/(2Np+2\lambda/3)},$$

which, assuming $\lambda = o(n^2)$ and $q < 1/2$, is smaller than $1/n$ provided

$$\begin{aligned} \lambda^2 &> (2Np + 2\lambda/3)\cdot n(-\sqrt{q}\log(\sqrt{q}) - (1-\sqrt{q})\log(1-\sqrt{q})) + O(\log n) \\ &= \Theta(pqn^2 \cdot n \cdot \sqrt{q}\log(1/\sqrt{q})). \end{aligned}$$

Since $e(G) = p\binom{n}{2}$ with probability at least $1/n$ (provided $p\binom{n}{2}$ is an integer), we deduce that (for $q \le 1/2$) there is some G with $p\binom{n}{2}$ edges and $\mathrm{disc}^+(G,H)\mathrm{disc}^-(G,H) < c'pq^{3/2}\log(1/q)n^3$.

We can also try working with both G and H random graphs. However, there is an additional problem. When we want to bound $\mathrm{disc}(G)$ for a single random graph G, and H is a clique plus isolated vertices, we only need to consider $O(2^n)$ ways of superimposing the two graphs (as H has a large automorphism group). But when considering a pair G, H of random graphs, there are $n!$ ways of placing the two graphs together, leading to a bound of form $c(p)n^{3/2}\sqrt{\log n}$ rather than $c(p)n^{3/2}$ (we have to avoid $n!$ bad events rather than 2^n, so we want the probability of each bad event to be at most $1/n!$ rather than at most 2^{-n}; the bound then follows by considering the tail of the binomial distribution). It would be interesting to know the correct value.

Problem 12. *Let G and H be random graphs chosen independently from $\mathcal{G}(n,p)$. What is*

$$\mathbb{E}\,\mathrm{disc}(G,H)?$$

From the remarks above, it is clear that $c(p)n^{3/2} < \mathbb{E}\,\mathrm{disc}(G,H) < c'(p)n^{3/2}\sqrt{\log n}$. However, it is not obvious where in this range the correct value lies, and it is equally unclear when G and H are random graphs with different densities.

What about $\mathbb{E}\,\mathrm{disc}^+(G,H)\mathrm{disc}^-(G,H)$? It turns out that this is closely related to $\mathrm{disc}(G,H)$. Indeed, consider the edge-exposure martingale obtained by revealing the edges of G and H one at a time, and taking the conditional expectation of $\mathrm{disc}^+(G,H)$. There are $2\binom{n}{2}$ edges, and differences are bounded by 1, so by the Azuma-Hoeffding inequality the probability we are more than λ from expectation is at most $2\exp(-\lambda^2/2n^2)$. It follows that the discrepancy is concentrated close to some value. If $\omega(n) \to \infty$ as $n \to \infty$ then, with high probability, $|\mathrm{disc}^+(G,H) - \mathbb{E}\,\mathrm{disc}^+(G,H)| \le n\omega(n)$ and, with probability $1-O(1/n^5)$, $|\mathrm{disc}^+(G,H) - \mathbb{E}\,\mathrm{disc}^+(G,H)| \le n\sqrt{10\log n}$. Applying the same argument to $\mathrm{disc}^-(G,H)$ and noting that $\mathrm{disc}(G,H) = O(n^2)$, we see that $\mathbb{E}\,\mathrm{disc}^+(G,H)\mathrm{disc}^-(G,H) = (1+o(1))(\mathbb{E}\,\mathrm{disc}(G,H))^2$. However, this does not help with finding the value of $\mathbb{E}\,\mathrm{disc}(G,H)$.

It would also be interesting to look at graphs (and hypergraphs) with low discrepancy. Graphs G with density $1/2$ and $\mathrm{disc}(G) = o(n^2)$ are known to be quasi-random, and therefore share certain properties with random graphs (see Thomason [18], Chung, Graham and Wilson [5] and Chung and Graham [6, 7]; for related results see Mubayi and Rödl [16]). What can we say about graphs for which $\mathrm{disc}^+(G)\mathrm{disc}^-(G)$ is small? For instance, we know from [3] that if G has density $1/2$ then $\mathrm{disc}^+(G)\mathrm{disc}^-(G) \ge cn^3$. This bound is achieved by random graphs in $\mathcal{G}(n,1/2)$, as well as the complete bipartite graph $K_{n/2,n/2}$ and its complement. Must graphs that are close to extremal look like these examples?

Problem 13. *Let G be a graph with density $1/2$ and $\mathrm{disc}^+(G)\mathrm{disc}^-(G) = O(n^3)$. Must G be close to one of $K_{n/2,n/2}$, $2K_{n/2}$ or a quasi-random graph?*

Of course, as stated the problem is a little imprecise, as a suitable measure of "closeness" needs to be specified. However, it seems likely that there should be some sort of "stability theory", saying that graphs that are "close" to extremal should also be "close" to graphs of one of a few different types. We could also demand a very small one-sided discrepancy: what do graphs G with $\mathrm{disc}(G) = O(n^{3/2})$ look like? Is there a stability theory for these? Perhaps there is a small family of examples that, together with graphs satisfying strong quasi-random properties, essentially characterize graphs with $\mathrm{disc}^+(G)\mathrm{disc}^-(G) = O(n^3)$. Or there may be a decomposition of G into a small number of such graphs. The weaker condition that $\mathrm{disc}^+(G)\mathrm{disc}^-(G) = o(n^4)$ is also interesting in this context. Does this imply that the graph is in some sense quasi-random? The focus here is on graphs with small one-sided discrepancy: do graphs G with $\mathrm{disc}^+(G) = o(n^2)$ have nice properties like quasirandomness? These questions, and the ones above, also arise for graphs (and hypergraphs) with densities $p \neq 1/2$. Of course, it could also be very interesting to understand the structure of pairs of graphs G and H for which $\mathrm{disc}(G, H)$ or $\mathrm{disc}^+(G, H)$ is small.

In a similar vein, it would be interesting to understand the relationship between $\mathrm{disc}(G)$ and the eigenvalues of G. For instance, what bounds can be given for $\mathrm{disc}(G)$ in terms of the density and spectral gap of G?

6.3 Other norms

Given graphs G and H, we can consider the vector

$$\boldsymbol{v}_{G,H} = (\langle \boldsymbol{g}_\pi, \boldsymbol{h} \rangle)_{\pi \in S_n} \in \mathbb{R}^{n!}.$$

Then $\mathrm{disc}^+(G, H)$ is the maximum entry of $\boldsymbol{v}_{G,H}$ and $\mathrm{disc}^-(G, H)$ is the minimum entry, while

$$\mathrm{disc}(G, H) = ||\boldsymbol{v}_{G,H}||_\infty.$$

It would be interesting to look at other norms. For instance,

$$||\boldsymbol{v}_{G,H}||_1 = n!\, \mathbb{E}|\langle \boldsymbol{g}_\pi, \boldsymbol{h} \rangle|,$$

where $\pi \in S_n$ is chosen with the uniform distribution. How large is $||\boldsymbol{v}_{G,H}||_1$? For random graphs it can be calculated precisely, but the extremal question seems to be less trivial.

Problem 14. *Let G and H be graphs with $e(G) = p\binom{n}{2}$ and $e(H) = q\binom{n}{2}$, and let $\pi \in S_n$ be chosen uniformly at random. What is the minimum possible value of*

$$\mathbb{E}\left| |E(G_\pi) \cap E(H)| - pq\binom{n}{2} \right| ?$$

Similarly, what happens if we consider the l_2 norm?

Another interesting problem is to find a fractional version of Theorem 1. Suppose that G is a real-valued edge-weighting of K_n with total weight zero and $||\boldsymbol{g}||_1 = p\binom{n}{2}$. For a graph H with $q\binom{n}{2}$ edges, what can we say about $\mathrm{disc}(G, H)$? What if H is also edge-weighted? Note that Theorem 4 goes through in this context; and perhaps there is a fractional version of Theorem 9. Of course, the same problem arises for pairs of hypergraphs (results for $\mathrm{disc}(H)$, where H is an edge-weighted hypergraph can be found in [3]).

6.4 Group actions and set systems

Finally, let us note that these problems can be raised in the more general context of a group G acting transitively on a set X. Given $A, B \subset X$, we define

$$\mathrm{disc}^+_G(A, B) = \max_{g \in G} \left(|g(A) \cap B| - \frac{|A| \cdot |B|}{|X|} \right)$$

and

$$\mathrm{disc}^-_G(A, B) = \max_{g \in G} \left(\frac{|A| \cdot |B|}{|X|} - |g(A) \cap B| \right).$$

There are many interesting examples of this, and we would expect bounds in general to involve both the size of G and X and the number of automorphisms of A and B.

The problem can also be stated in the infinite context, with an appropriate probability measures on G and on X. For instance, given subsets $A, B \subset S^n$, taking normalized Lebesgue measure on S_n and letting G be the group of isometries of S_n with normalized Haar measure, what can we say about $\mathrm{disc}^+(A, B)$ and $\mathrm{disc}^-(A, B)$? If $\lambda(A) = 1/2$ or $\lambda_B = 1/2$, then we could have zero discrepancy: for instance, if one set is a half-sphere and the other set is symmetric in the origin. Similarly, in S_1 we can construct pairs of sets with $\lambda(A) = r/s$, any $\lambda(B)$ and zero discrepancy by taking A to be an interval and demanding that B has a rotational symmetry of order s. More generally, we can generate pairs of sets in S^n with arbitrarily low discrepancy by taking A to be a spherical cap (or any reasonably nice set) and B to be a subset of spherical cap obtained by dividing it into sets of small diameter and throwing away half of each set. These examples suggest that the symmetry group of S^n is too small (or that the class of sets is too large) for us to generate atypical overlaps. However, it may be possible to obtain more interesting results if we restrict the sets A and B to be sufficiently "nice".

Returning to the finite context, we consider actions of S_n. There appears to be a trade-off between discrepancy and the size of the set on which S_n acts: when S_n acts on the singletons in $[n]$, we get the maximum possible discrepancy, as any two sets can be shifted to overlap as much or as little as possible. On the other hand, if we allow S_n to act on itself, we get very small discrepancy: let H be a subgroup of S_n and choose A to be a union of left cosets of H and B to be a set of left coset representatives. Then $|\pi(A) \cap B|$ takes the same value for every $\pi \in S_n$. There is a similar problem when we allow S_n to act on the cube $\{0,1\}^n$ by permuting coordinates. As in the case of S_n acting on itself, it is easy to construct pairs of sets with small discrepancy: for instance, let A be a face of the cube and let B be closed under pointwise complementation. However, the question becomes more challenging if we enlarge the group to the full isometry group T_n of the cube $\{0,1\}^n$, which has $2^n n!$ elements.

Problem 15. *For $p \in [0,1]$, let $Q_{p,n}$ be the collection of subsets of $\{0,1\}^n$ with $p2^n$ elements. What is*

$$\min_{A \in Q_{p,n}, B \in Q_{q,n}} \operatorname{disc}_{T_n}(A, B)?$$

Interesting questions arise if we consider the action of S_n on $\{0,1\}^n$ by permuting coordinates, and restrict our sets to be upsets. By the Harris-Kleitman inequality [11], if A and B are upsets then

$$|A \cap B| \geq \frac{|A| \cdot |B|}{2^n}.$$

How much better than this can we do if we can permute coordinates?

Problem 16. *What is the minimum of $\max_{\pi \in S_n} |\pi(A) \cap B|$ over upsets $A, B \subset \{0,1\}^n$ with $|A| = p2^n$ and $|B| = q2^n$?*

Could the pair (A, B) with $A = \{\mathbf{x} \in \{0,1\}^n : x_i = 1 \text{ for } i \leq k\}$ and $B = \{\mathbf{x} \in \{0,1\}^n : \sum x_i \geq l\}$ be extremal for this problem?

The same question arises for the discrepancy.

Problem 17. *What is the minimum of* $\mathrm{disc}_{S_n}(A, B)$ *over upsets* $A, B \subset \{0,1\}^n$ *with* $|A| = p2^n$ *and* $|B| = q2^n$*?*

In all the questions above, it is also natural to ask about bounds on the product form (32).

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