

# Sparse graphs without linear anticomplete pairs

Maria Chudnovsky<sup>1</sup>  
Princeton University, Princeton, NJ 08544

Alex Scott<sup>2</sup>  
Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour<sup>3</sup> and Sophie Spirkl  
Princeton University, Princeton, NJ 08544

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### **Abstract**

We prove that for every graph  $H$  there exists  $c > 0$  such that, for every graph  $G$ , if  $G$  has no clique or stable set of size  $|G|^c$ , then either  $G$  or its complement contains an induced subdivision of  $H$ . This follows from a stronger result for sparse graphs which says, roughly, that every sparse graph contains either an induced subdivision of  $H$  or two linear-sized sets of vertices with no edges between them. These results are related to the Erdős-Hajnal conjecture.

# 1 Introduction

For a graph  $G$ , we write  $\omega(G), \alpha(G)$  for the cardinalities of the largest clique and largest stable set in  $G$  respectively. The number of vertices of  $G$  is denoted by  $|G|$ , and  $\overline{G}$  denotes the complement graph of  $G$ . If  $v \in V(G)$ ,  $N(v)$  denotes the set of neighbours of  $v$ . Disjoint subsets  $A, B$  of  $V(G)$  are *complete* if every vertex of  $A$  is adjacent to every vertex of  $B$ , and *anticomplete* if no vertex in  $A$  has a neighbour in  $B$ . For graphs  $G, H$ , we say  $G$  is  *$H$ -free* if no induced subgraph of  $G$  is isomorphic to  $H$ . (All graphs in this paper are finite and have no loops or parallel edges.) An *ideal* of graphs is a class of graphs closed under isomorphism and under taking induced subgraphs; and an ideal is *proper* if it is not the class of all graphs.

It is well-known from Ramsey theory [14] that every graph on  $n$  vertices contains a clique or stable set of size at least  $\frac{1}{2} \log n$ . On the other hand, there are graphs with no clique or stable set of size more than  $2 \log n$  [11] (in fact, most graphs have this property). The celebrated Erdős-Hajnal conjecture asserts that  $H$ -free graphs have *much* larger cliques or stable sets. Let us say that an ideal  $\mathcal{I}$  has the *Erdős-Hajnal property* if there is some  $\epsilon > 0$  such that every graph  $G \in \mathcal{I}$  has a clique or stable set of size at least  $|G|^\epsilon$ . The Erdős-Hajnal conjecture [12, 13] is the following:

**1.1 Conjecture:** *For every graph  $H$ , the ideal of  $H$ -free graphs has the Erdős-Hajnal property.*

A related but stronger property for an ideal is that every graph in the ideal contains a pair of linear-sized sets that are either complete or anticomplete. More formally, let us say that an ideal  $\mathcal{I}$  has the *strong Erdős-Hajnal property* if there is some  $\epsilon > 0$  such that every graph  $G \in \mathcal{I}$  with at least two vertices contains disjoint sets  $A, B$  that have size at least  $\epsilon|G|$  and are either complete or anticomplete. It is easy to show that if an ideal has the strong Erdős-Hajnal property then it has the Erdős-Hajnal property (see [1, 16]; or section 3 below). But the reverse implication does not hold. In fact, if the ideal of  $H$ -free graphs has the strong Erdős-Hajnal property then both  $H$  and  $\overline{H}$  are forests (to show that  $H$  must be a forest, suppose that  $H$  contains a cycle of length  $k$  and consider a random graph  $G \in \mathcal{G}(n, p)$ , where  $p$  is chosen so that  $np \rightarrow \infty$  and  $(np)^k = o(n)$  and delete one vertex from each cycle of length  $k$ ; for  $\overline{H}$ , take complements). Thus the ideal of  $H$ -free graphs does not have the strong Erdős-Hajnal property for any graph  $H$  with more than four vertices. In this paper, we are interested in which ideals do have the strong Erdős-Hajnal property.

An ideal is characterized by the minimal induced subgraphs that it does not contain. If an ideal is defined by a finite number of excluded induced subgraphs, then the random graph construction shows that one of them must be a forest and one of them must be the complement of a forest. An important result of this type is due to Bousquet, Lagoutte and Thomassé [3]. Improving on earlier work of Chudnovsky and Zwols [10] and Chudnovsky and Seymour [9], they showed that for every path  $P$ , the ideal of graphs with no induced  $P$  or  $\overline{P}$  has the strong Erdős-Hajnal property. More recently, Liebenau, Pilipczuk, Seymour and Spirkl [18] (improving on an earlier result of Choromanski, Falik, Patel and Pilipczuk [4]) showed that if  $T$  is a subdivision of a caterpillar then the ideal of graphs with no induced  $T$  or  $\overline{T}$  has the strong Erdős-Hajnal property. They further conjecture that the same statement holds for any tree  $T$ . This conjecture is proved by the current authors in [8], completing the classification for ideals defined by a finite number of excluded induced subgraphs.

What about ideals that are not defined by a finite number of excluded induced subgraphs? A breakthrough result in this direction was proved by Bonamy, Bousquet and Thomassé [2], who showed that for every  $k$  the ideal of graphs  $G$  such that neither  $G$  nor  $\overline{G}$  contains an induced cycle of length

at least  $k$  has the strong Erdős-Hajnal property. In other words, we exclude induced subdivisions of the cycle  $C_k$  from both  $G$  and  $\overline{G}$ . In this paper, we prove a very substantial extension of this result.

**1.2** *For every graph  $H$ , the ideal of graphs  $G$  such that neither  $G$  nor  $\overline{G}$  contains an induced subdivision of  $H$  has the strong Erdős-Hajnal property.*

As an immediate corollary we obtain the following.

**1.3** *For every graph  $H$ , there exists  $c > 0$  such that for every graph  $G$ , one of the following holds:*

- $G$  or its complement contains an induced subdivision of  $H$ ;
- $G$  contains a clique or stable set of size at least  $|G|^c$ .

We can say a little more than 1.2. If an ideal satisfies the strong Erdős-Hajnal property then we know that every graph in the ideal has a pair of large sets that are either complete or anticomplete, but we may not be able to choose which (for instance, consider the ideal consisting of all vertex-disjoint unions of cliques). However, a theorem of Rödl [20] (discussed in section 3) allows us to assume that our graph is either quite sparse or quite dense. We can then deduce 1.2 from the following significantly stronger ‘one-sided’ result.

**1.4** *For every graph  $H$ , there exists  $c > 0$  such that every graph  $G$  with at least two vertices and at most  $c|G|^2$  edges satisfies one of the following:*

- $G$  contains an induced subdivision of  $H$ ;
- $G$  contains disjoint, anticomplete sets  $A, B$  of size at least  $c|G|$ .

In fact, we will prove an even more general result: we will show it is enough to consider induced subdivisions where the edges in a specified path are not subdivided; and we will prove a version of the result (stated as 2.4) that works when the graph is weighted. We introduce the necessary definitions and state results formally in the next section. We discuss Rödl’s theorem and its application in section 3, and then give the proof of 2.4 over the next four sections. An important feature of the proof is to divide the problem into two cases: in one case, we may assume that all small balls have small mass; in the other, we may assume that a significant mass is always concentrated in a small ball. After some initial work, these cases are handled separately in sections 6 and 7. We conclude, in the final section, with some applications, and a discussion of the relationship between the Erdős-Hajnal conjecture and questions about  $\chi$ -boundedness.

## 2 Statement of results

Every proper ideal is contained in the ideal of  $H$ -free graphs for some  $H$ . Thus 1.1 can be reformulated as:

**2.1 Conjecture:** *For every proper ideal  $\mathcal{I}$ , there exists  $c > 0$  such that every graph  $G \in \mathcal{I}$  satisfies  $\omega(G)\alpha(G) \geq |G|^c$ .*

For  $\epsilon > 0$ , let us say a graph  $G$  is  $\epsilon$ -coherent if

- $|G| \geq 2$ ;
- $|N(v)| < \epsilon|G|$  for each  $v \in V(G)$ ; and
- $\min(|A|, |B|) < \epsilon|G|$ , for every two anticomplete sets  $A, B \subseteq V(G)$ .

As we explain in section 3, 2.1 is equivalent to the following:

**2.2 Conjecture:** *For every proper ideal  $\mathcal{I}$  there exist  $\epsilon > 0$  and  $c > 0$ , such that every  $\epsilon$ -coherent graph  $G \in \mathcal{I}$  satisfies  $\omega(G)\alpha(G) \geq |G|^c$ .*

Let us say an ideal is *incoherent* if for some  $\epsilon > 0$ , no member of  $\mathcal{I}$  is  $\epsilon$ -coherent; and *coherent* if there is no such  $\epsilon$ .

Let  $H$  be a graph and let  $P$  be a path of  $H$ . Let  $J$  be a graph obtained from  $H$  by subdividing at least once every edge of  $H$  not in  $E(P)$ , and not subdividing the edges in  $E(P)$ . We call such a graph  $J$  (and graphs isomorphic to it) a *P-filleting* of  $H$ . Our main result states:

**2.3** *Let  $H$  be a graph and let  $P$  be a path of  $H$ . Then every coherent ideal contains a P-filleting of  $H$ .*

For every subset  $X \subseteq V(G)$ , let  $\mu(X)$  be some real number, satisfying

- $\mu(\emptyset) = 0$  and  $\mu(V(G)) = 1$ , and  $\mu(X) \leq \mu(Y)$  for all  $X, Y$  with  $X \subseteq Y$ ; and
- $\mu(X \cup Y) \leq \mu(X) + \mu(Y)$  for all disjoint sets  $X, Y$ .

We call such a function  $\mu$  a *mass* on  $G$ , and we call the pair  $(G, \mu)$  a *massed graph*. For instance, we could take  $\mu(X) = |X|/|G|$ , or  $\mu(X) = \chi(G[X])/\chi(G)$ , where  $\chi$  denotes chromatic number. (It is sometimes convenient to speak of the ‘mass’ of a set  $X$ , meaning  $\mu(X)$ .) For  $\epsilon > 0$  let us say a massed graph  $(G, \mu)$  is  *$\epsilon$ -coherent* if

- $\mu(\{v\}) < \epsilon$  for every vertex  $v$ ;
- $\mu(N(v)) < \epsilon$  for every vertex  $v$ ; and
- $\min(\mu(A), \mu(B)) < \epsilon$  for every two anticomplete sets of vertices  $A, B$ .

In [18] it was found that the main theorem of that paper could be extended to massed graphs: that for every caterpillar subdivision  $T$ , there exists  $\epsilon > 0$  such that for every  $\epsilon$ -coherent massed graph  $(G, \mu)$ , some induced subgraph of  $G$  is isomorphic to  $T$ . We will show that 2.3 admits the same extension to masses. We will prove:

**2.4** *For every graph  $H$  and path  $P$  of  $H$ , there exists  $\epsilon > 0$  such that for every  $\epsilon$ -coherent massed graph  $(G, \mu)$ , some induced subgraph of  $G$  is a P-filleting of  $H$ .*

### 3 Rödl's theorem

Before we go on, let us prove the equivalence of 2.1 and 2.2. This is routine, and no doubt well-known to those familiar with the field, but we give the proof anyway. Certainly 2.1 implies 2.2, and for the converse we use an invaluable tool due to Rödl [20], the following.

**3.1** *For every graph  $H$  and all  $\epsilon > 0$  there exists  $\delta > 0$  such that for every  $H$ -free graph  $G$ , there exists  $X \subseteq V(G)$  with  $|X| \geq \delta|G|$  such that in one of  $G[X], \overline{G}[X]$ , every vertex in  $X$  has degree at most  $\epsilon|X|$ .*

This implies:

**3.2** *Let  $\mathcal{I}$  be a proper ideal, let  $\epsilon, c_0 > 0$ , and suppose that for every  $G \in \mathcal{I}$ , if one of  $G, \overline{G}$  is  $\epsilon$ -coherent then  $\omega(G)\alpha(G) \geq |G|^{c_0}$ . Then  $\mathcal{I}$  satisfies 2.1.*

**Proof.** Let  $\mathcal{I}_2$  be the ideal of complements of members of  $\mathcal{I}$ . Then every  $\epsilon$ -coherent graph  $G \in \mathcal{I} \cup \mathcal{I}_2$  satisfies  $\omega(G)\alpha(G) \geq |G|^{c_0}$ . Choose  $H \notin \mathcal{I}$ ; choose  $\delta$  such that 3.1 holds; and choose  $c$  with  $0 < c \leq c_0/2$  such that  $\delta^{2c} \geq 1/2$  and  $(\epsilon\delta)^c \geq 1/2$ . We prove by induction on  $|G|$  that for every graph  $G \in \mathcal{I} \cup \mathcal{I}_2$ , we have  $\omega(G)\alpha(G) \geq |G|^c$ . If  $|G| \leq 1$  the claim is trivial, and if  $2 \leq |G| \leq \delta^{-2}$  then the claim holds, since

$$\omega(G)\alpha(G) \geq 2 \geq \delta^{-2c} \geq |G|^c.$$

Thus we may assume that  $|G| > \delta^{-2}$ . By 3.1, since one of  $G, \overline{G}$  is  $H$ -free, there exists  $X \subseteq V(G)$  with  $|X| \geq \delta|G|$  such that in one of  $G[X], \overline{G}[X]$ , every vertex in  $X$  has degree at most  $\epsilon|X|$ . By replacing  $G$  by  $\overline{G}$  if necessary, we may assume that every vertex in  $X$  has degree at most  $\epsilon|X|$  in  $G[X]$ .

Now  $(\delta|G|)^{c_0} \geq (\delta|G|)^{2c} \geq |G|^c$ , since  $c_0 \geq 2c$  and  $|G| > \delta^{-2}$ . Thus if  $G[X]$  is  $\epsilon$ -coherent, then

$$\omega(G[X])\alpha(G[X]) \geq |X|^{c_0} \geq (\delta|G|)^{c_0} \geq |G|^c$$

as required. If  $G[X]$  is not  $\epsilon$ -coherent, there exist two anticomplete subsets  $A, B$  of  $X$  such that  $|A|, |B| \geq \epsilon|X|$ . By the inductive hypothesis,  $\omega(G[A])\alpha(G[A]) \geq |A|^c$ , and the same for  $B$ , and since  $\alpha(G) \geq \alpha(G[A]) + \alpha(G[B])$  and  $\omega(G) \geq \omega(G[A]), \omega(G[B])$ , it follows that

$$\omega(G)\alpha(G) \geq \omega(G[A])\alpha(G[A]) + \omega(G[B])\alpha(G[B]) \geq |A|^c + |B|^c \geq 2(\epsilon|X|)^c \geq 2(\epsilon\delta|G|)^c \geq |G|^c.$$

This proves 3.2. ■

Now, suppose that 2.2 holds, and let us prove 2.1. Let  $\mathcal{I}$  be a proper ideal, and let  $\mathcal{I}_2$  be the ideal of complements of members of  $\mathcal{I}$ . By applying 2.2 to  $\mathcal{I}$  and to  $\mathcal{I}_2$ , there exist  $\epsilon, c_0$  such that every  $\epsilon$ -coherent graph  $G \in \mathcal{I}$  satisfies  $\omega(G)\alpha(G) \geq |G|^{c_0}$ , and the same for  $\mathcal{I}_2$ . But then the result follows from 3.2.

Another useful consequence of 3.2 is the following:

**3.3** *If  $\mathcal{I}$  is an ideal such that both  $\mathcal{I}$  and the ideal of complements of members of  $\mathcal{I}$  are incoherent, then  $\mathcal{I}$  satisfies 2.1.*

**Proof.** Let  $\mathcal{I}_2$  be the ideal of complements of members of  $\mathcal{I}$ , and choose  $\epsilon > 0$  such that no member of  $\mathcal{I} \cup \mathcal{I}_2$  is  $\epsilon$ -coherent. Then the result follows from 3.2. ■

## 4 Some preliminaries

In order to prove 2.4, we might as well assume that  $P$  is a Hamilton path of  $H$ . To see this, let  $P$  have vertices  $v_1, \dots, v_k$  in order and let the remaining vertices of  $H$  be  $v_{k+1}, \dots, v_n$ . Add new vertices  $u_{k+1}, \dots, u_n$  to  $H$ , where each  $u_i$  is adjacent to  $v_{i-1}$  and  $v_i$ ; let the new graph be  $H'$  and let  $P'$  be the path with vertices

$$v_1, \dots, v_k, u_{k+1}, v_{k+1}, u_{k+2}, \dots, u_n, v_n.$$

Then  $P'$  is a Hamilton path of  $H'$ , and if the theorem holds for  $(H', P')$  then it holds for  $(H, P)$ . We will therefore assume that  $P$  is a Hamilton path of  $H$ . In order to find a  $P$ -filleting of  $H$  as an induced subgraph of  $G$ , we need to find an induced path  $Q$  say of  $G$ , with the same number of vertices as  $P$ , such that certain pairs of vertices of  $Q$  are joined by induced paths in  $G$ , pairwise disjoint and disjoint from  $Q$  (except for their ends), such that their union with  $Q$  is induced in  $G$ .

A *pairing*  $\Pi$  in a graph  $G$  is a set of pairwise disjoint subsets of  $V(G)$ , each of cardinality one or two; and let  $V(\Pi)$  be the union of the members of  $\Pi$ . If  $X \subseteq V(G)$ , a *pairing of  $X$*  means a pairing  $\Pi$  with  $V(\Pi) = X$ . A pairing  $\Pi$  of  $X$  is *feasible* in  $G$  if for each  $e \in \Pi$  with  $|e| = 2$  there is an induced path  $P_e$  of  $G$  joining the two members of  $e$ , and for each  $e \in \Pi$  with  $|e| = 1$ ,  $P_e$  is the one-vertex path with vertex set  $e$ , such that for all distinct  $e, f \in \Pi$ , the sets  $V(P_e), V(P_f)$  are anticomplete.

An induced subgraph of  $G$  isomorphic to  $T$  is called a *copy* of  $T$  in  $G$ . Let us say a *caterpillar* is a tree in which some path contains all vertices with degree more than one. Its *leaves* are its vertices of degree one. A *caterpillar in  $G$*  means an induced subgraph of  $G$  that is a caterpillar. A *leaf-pairing* of  $T$  means a pairing  $\Pi$  of the set of leaves of  $T$ . Let  $T$  be a caterpillar in  $G$ , and let  $\Pi$  be a leaf-pairing of  $T$ . Let  $X$  be the set of all vertices in  $V(G) \setminus V(T)$  with no neighbours in  $V(T) \setminus V(\Pi)$ . The pairing  $\Pi$  is *feasible in  $G$  relative to  $T$*  if  $\Pi$  is feasible in  $G[X \cup V(\Pi)]$ . Thus, another way to pose the problem of 2.4 is to say that we are given a caterpillar  $T$  with a leaf-pairing, and we are searching for a copy  $T'$  of  $T$  in  $G$  such that the corresponding leaf-pairing of  $T'$  is feasible in  $G$  relative to  $T'$ .

Let  $T$  be a caterpillar in  $G$ . We say  $T$  is *versatile* in  $G$  if every leaf-pairing of  $T$  is feasible in  $G$  relative to  $T$ . In order to prove 2.4 it therefore suffices to prove the following strengthening.

**4.1** *For every caterpillar  $T$ , there exists  $\epsilon > 0$ , such that for every  $\epsilon$ -coherent massed graph  $(G, \mu)$ , there is a versatile copy of  $T$  in  $G$ .*

If  $u$  is a vertex of a graph  $H$ , we denote by  $N^r(u)$  the set of all vertices  $v$  of  $G$  such that the distance between  $u, v$  is exactly  $r$ , and  $N^r[u]$  the set of  $v$  such that this distance is at most  $r$ . For  $r \geq 1$  an integer, and  $\epsilon > 0$ , let us say a massed graph  $(G, \mu)$  is  $(\epsilon, r)$ -coherent if

- $\mu(N^r[v]) < \epsilon$  for every vertex  $v$ ; and
- $\min(\mu(A), \mu(B)) < \epsilon$  for every two anticomplete sets of vertices  $A, B$ .

(Thus,  $(\epsilon, 1)$ -coherent  $\Rightarrow$   $\epsilon$ -coherent  $\Rightarrow$   $(2\epsilon, 1)$ -coherent.)

The proof of 4.1 breaks into two parts. Given a caterpillar  $T$ , we will first prove the statement of 4.1 for massed graphs  $(G, \mu)$  that are  $(\epsilon, r)$ -coherent (for some appropriate value of  $r$  depending on  $T$  but not on  $G$ ); and then we will use this to prove 4.1 in general. The first part is more difficult, and carried out in section 6.

## 5 Finding a caterpillar

We need a result which is a modification of the main theorem (2.6) of [18]. The main idea of its proof is exactly that of [18], but we need several minor changes, and it seemed best to prove the whole thing again. First we need a lemma (also proved in [18]). We say  $X \subseteq V(G)$  is *connected* if  $G[X]$  is connected.

**5.1** *Let  $(G, \mu)$  be an  $\epsilon$ -coherent massed graph, and let  $Y \subseteq V(G)$  with  $\mu(Y) \geq 3\epsilon$ . Then there is a connected subset  $X \subseteq Y$  with  $\mu(X) > \mu(Y) - \epsilon$ .*

**Proof.** Let the vertex sets of the components of  $G[Y]$  be  $X_1, \dots, X_k$  say. Choose  $i \geq 1$  minimal such that  $\mu(X_1 \cup \dots \cup X_i) \geq \epsilon$ . Since the sets  $X_1 \cup \dots \cup X_i$  and  $X_{i+1} \cup \dots \cup X_n$  are anticomplete, it follows that  $\mu(X_{i+1} \cup \dots \cup X_n) < \epsilon$ ; and from the minimality of  $i$ ,  $\mu(X_1 \cup \dots \cup X_{i-1}) < \epsilon$ . But

$$\mu(X_1 \cup \dots \cup X_{i-1}) + \mu(X_i) + \mu(X_{i+1} \cup \dots \cup X_n) \geq \mu(Y) \geq 3\epsilon,$$

and so  $\mu(X_i) \geq \epsilon$ . Since the sets  $X_i$  and  $Y \setminus X_i$  are anticomplete, it follows that  $\mu(Y \setminus X_i) < \epsilon$ , and so  $\mu(X_i) > \mu(Y) - \epsilon$ . This proves 5.1. ■

A rooted caterpillar is a tree  $T$  with a distinguished vertex  $h$ , called its *head*, such that some path of  $T$  with one end  $h$  contains all the vertices with degree more than one. A rooted caterpillar  $T$  with more than one vertex has a unique *predecessor*  $T'$  (up to isomorphism), defined as follows. Let  $h$  be the head of  $T$ .

- If  $h$  is adjacent to some leaf  $u$  of  $T$ , let  $T'$  be the rooted caterpillar obtained from  $T$  by deleting  $u$ , with the same head  $h$ .
- If  $h$  has no neighbours that are leaves, then  $h$  is a leaf; let its neighbour be  $u$ , and let  $T'$  be the rooted caterpillar obtained by deleting  $v$ , with head  $u$ .

Thus, every rooted caterpillar can be grown in canonical one-vertex steps from a one-vertex rooted caterpillar. If  $T$  is a rooted caterpillar with  $n$  vertices, say, let  $T_1, \dots, T_n$  be the rooted caterpillars such that  $T_n = T$ , and  $|T_1| = 1$ , and  $T_{i-1}$  is the predecessor of  $T_i$  for  $2 \leq i \leq n$ . We call  $T_1, \dots, T_n$  the *ancestors* of  $T$ .

Let  $\mathcal{Y}$  be a set of pairwise disjoint subsets of  $V(G)$ , where  $G$  is a graph. Let  $N$  be a graph, and for each  $v \in V(N)$  let  $X_v \subseteq V(G)$ . We say that the family  $X_v$  ( $v \in V(N)$ ) is  $\mathcal{Y}$ -*spread* if for each  $v \in V(N)$  there exists  $Y_v \in \mathcal{Y}$ , such that the sets  $Y_v$  ( $v \in V(N)$ ) are all different, and  $X_v \subseteq Y_v$  for each  $v \in V(N)$ .

If  $A, B \subseteq V(G)$  are disjoint, we say  $A$  *covers*  $B$  if every vertex in  $B$  has a neighbour in  $A$ . Let  $(G, \mu)$  be a massed graph. We say  $X \subseteq V(G)$  is  $\delta$ -*dominant* if  $\mu(X \cup \bigcup_{x \in X} N(x)) \geq \delta$ .

Let  $N$  be the disjoint union of one or more rooted caterpillars. (Thus each component of  $N$  has a head.) Let  $H$  be the set of heads of the components of  $N$ . Let  $(G, \mu)$  be a massed graph. A  $\delta$ -*realization* of  $N$  in  $G$  is an assignment of a subset  $X_v \subseteq V(G)$  to each vertex  $v \in V(N)$ , satisfying the following conditions:

- the sets  $X_v$  ( $v \in V(N)$ ) are pairwise disjoint;



- for every edge  $uv$  of  $N$ , if  $v$  lies on the path of  $N$  between  $u$  and the head of the component of  $N$  containing  $u$ , then  $X_u$  covers  $X_v$ ;
- for all distinct  $u, v \in V(N)$ , if  $u, v$  are nonadjacent in  $N$  then  $X_u, X_v$  are anticomplete; and
- for each  $v \in H$ ,  $\mu(X_v) \geq \delta$ , and for each  $v \in V(N) \setminus H$ ,  $X_v$  is connected and  $\delta$ -dominant.

**5.2** Let  $T$  be a rooted caterpillar, let  $\delta, \epsilon > 0$ , let  $(G, \mu)$  be an  $\epsilon$ -coherent massed graph, and let  $\mathcal{Y}$  be a set of disjoint subsets of  $V(G)$ , such that  $|\mathcal{Y}| = 2^{|T|}$  and  $\mu(Y) \geq 2^{2^{|T|}}(\delta + \epsilon)$  for each  $Y \in \mathcal{Y}$ . Then there is a  $\mathcal{Y}$ -spread  $\delta$ -realization of  $T$  in  $G$ .

**Proof.** Let  $p = 2^{|T|}$  and for  $0 \leq i \leq p$  let  $m_i = 2^i(\delta + \epsilon) - \epsilon$ . Thus  $m_0 = \delta$ , and  $m_{i+1} = 2m_i + \epsilon$  for  $0 \leq i < p$ .

If  $N$  is a disjoint union of rooted caterpillars, each isomorphic to an ancestor of  $T$ , we call  $N$  a *nursery*, and we define  $\phi(N)$  to be the sum of  $2^{|C|}$ , summed over all components  $C$  of  $N$ . Let  $N_p$  be the nursery with  $p$  components, each an isolated vertex. Thus  $\phi(N_p) = 2p$ , and since  $2^p(\delta + \epsilon) \geq m_p$ , there is a  $\mathcal{Y}$ -spread  $m_p$ -realization of  $N_p$  in  $G$ . Choose  $k \leq p$  minimum such that there is a nursery  $N_k$  with  $k$  components and with  $\phi(N_k) \geq 2p$ , and there is a  $\mathcal{Y}$ -spread  $m_k$ -realization of  $N_k$  in  $G$ . Since  $\phi(N_k) \geq 2p$  and each component of  $N_k$  is an ancestor of  $T$ , it follows that  $N_k$  has at least two components, and so  $k \geq 2$ . Suppose (for a contradiction) that each component of  $N_k$  is isomorphic to an ancestor of  $T$  different from  $T$ .

Let the components of  $N_k$  be  $H_1, \dots, H_k$ , where  $|H_1| \leq \dots \leq |H_k|$ , and for  $1 \leq i \leq k$  let  $h_i$  be the head of  $H_i$ . Now for  $1 \leq i \leq k$  there is an ancestor  $S_i$  of  $T$  such that  $H_i$  is the predecessor of  $S_i$ , since  $H_i$  is not isomorphic to  $T$ . We recall that  $S_i$  is obtained from  $H_i$  by adding a new leaf adjacent to  $h_i$ , and either keeping the same head, or making the new vertex the new head. Let  $I$  be the set of  $i \in \{1, \dots, k\}$  such that  $H_i, S_i$  have different heads. If  $I \neq \emptyset$ , choose  $i \in I$ , maximum, and otherwise let  $i = 1$ .

Let  $(X_v : v \in V(N_k))$  be a  $\mathcal{Y}$ -spread  $m_k$ -realization of  $N_k$  in  $G$ . Since  $k \geq 2$  and hence  $m_k \geq 3\epsilon$ , 5.1 implies that there exists  $Z \subseteq X_{h_i}$  with  $\mu(Z) > \mu(X_{h_i}) - \epsilon \geq m_k - \epsilon$ , such that  $Z$  is connected. Number the vertices of  $Z$  as  $z_1, \dots, z_n$  say, such that  $\{z_1, \dots, z_q\}$  is connected for  $1 \leq q \leq n$ . Since  $k \geq 2$ , there exists  $j \neq i$  with  $1 \leq j \leq k$ ; and since  $\mu(Z) \geq \epsilon$ , the set of vertices in  $X_{h_j}$  with a neighbour in  $Z$  has mass more than  $\mu(X_{h_j}) - \epsilon \geq m_{k-1}$ . Consequently we may choose  $q$  with  $0 \leq q \leq n$ , minimum such that for some  $j \in \{1, \dots, k\} \setminus \{i\}$ , the set of vertices in  $X_{h_j}$  with a neighbour in  $\{z_1, \dots, z_q\}$  has mass at least  $m_{k-1}$ . In particular,  $\{z_1, \dots, z_q\}$  is  $m_{k-1}$ -dominant, and  $q \geq 1$ .

- If  $j < i$ , it follows that  $i \in I$ . Let  $N_{k-1}$  be the graph obtained from  $N_k$  by adding the edge  $h_i h_j$ , and deleting all vertices in  $V(H_j) \setminus \{h_j\}$ . Let  $H'_i$  be the component of  $N_{k-1}$  that contains the edge  $h_i h_j$ , and let us assign its head to be  $h_j$ . Consequently  $H'_i$  is isomorphic to  $S_i$ , and so  $N_{k-1}$  is a nursery with  $k - 1$  components. Moreover,  $|H'_i| \geq |H_j|$  (because  $i > j$ ), and so  $\phi(N_{k-1}) \geq \phi(N_k)$ .
- If  $j > i$ , it follows that  $j \notin I$ . Let  $N_{k-1}$  be the graph obtained from  $N_k$  by adding the edge  $h_i h_j$ , and deleting all vertices in  $V(H_i) \setminus \{h_i\}$ . Let  $H'_j$  be the component of  $N_{k-1}$  that contains the edge  $h_i h_j$ , and let us assign its head to be  $h_j$ . Thus  $H'_j$  is isomorphic to  $S_j$ , and again  $N_{k-1}$  is a nursery with  $k - 1$  components and  $\phi(N_{k-1}) \geq \phi(N_k)$ .

For each  $v \in V(N_{k-1})$  define  $X'_v$  as follows:

- if  $v \neq \{h_1, \dots, h_k\}$  let  $X'_v = X_v$ ;
- let  $X'_{h_i} = \{z_1, \dots, z_t\}$ ;
- let  $X'_{h_j}$  be the set of vertices in  $X_{h_j}$  with a neighbour in  $\{z_1, \dots, z_q\}$ ;
- for  $1 \leq \ell \leq k$  with  $\ell \neq i, j$ , let  $X'_{h_\ell}$  be the set of vertices in  $X_{h_\ell}$  with no neighbour in  $\{z_1, \dots, z_q\}$ .

We see that  $X'_{h_i}$  covers  $X'_{h_j}$ , and has no edges to  $X'_{h_\ell}$  for  $1 \leq \ell \leq k$  with  $\ell \neq i, j$ ; and  $X'_{h_i}$  is connected and  $m_{k-1}$ -dominant. Moreover,  $\mu(X'_{h_j}) \geq m_{k-1}$ . Let  $1 \leq \ell \leq k$  with  $\ell \neq i, j$ ; then, since  $q \geq 1$  and from the choice of  $q$ , the mass of the set of vertices in  $X_{h_\ell}$  with a neighbour in  $\{z_1, \dots, z_{q-1}\}$  is less than  $m_{k-1}$ , and hence  $\mu(X_{h_\ell} \setminus X'_{h_\ell}) < m_{k-1} + \epsilon$ . Since  $\mu(X_{h_\ell}) \geq m_k$  and  $m_k = 2m_{k-1} + \epsilon$ , it follows that  $\mu(X'_{h_\ell}) \geq m_{k-1}$ . Thus  $(X'_v : v \in V(N_{k-1}))$  is a  $\mathcal{Y}$ -spread  $m_{k-1}$ -realization of  $N_{k-1}$  in  $G$ , contrary to the minimality of  $k$  since  $\phi(N_{k-1}) \geq \phi(N_k) \geq 2p$ .

Consequently some component of  $N_k$  is isomorphic to  $T$ ; but then the theorem holds (since  $m_k \geq \delta$ ). This proves 5.2. ■

## 6 If no small ball has large mass

In this section we prove 4.1 assuming that no ball with bounded radius has large mass. We need first:

**6.1** *Let  $k \geq 0$  be an integer, and let  $\epsilon, \kappa > 0$  such that  $\kappa + 4\epsilon \leq 1$  and  $(k-1)\kappa \leq 1$ . Let  $(G, \mu)$  be an  $\epsilon$ -coherent massed graph. Then there are  $2k+1$  subsets  $A_1, \dots, A_k, B_1, \dots, B_k, C$  of  $V(G)$ , pairwise disjoint, with the following properties:*

- for  $1 \leq i \leq k$ ,  $A_i$  is connected and covers  $B_i$ ;
- for  $1 \leq i \leq k$ ,  $A_i, C$  are anticomplete;
- for all distinct  $i, j \in \{1, \dots, k\}$ ,  $A_i$  is anticomplete to  $A_j \cup B_j$ ;
- $\mu(C) \geq 1 - 3k\epsilon$ ; and
- for  $1 \leq i \leq k$ , the set of vertices in  $C$  covered by  $B_i$  has mass at least  $\kappa - 3k\epsilon$ .

**Proof.** We proceed by induction on  $k$ ; the result is trivial for  $k = 0$ , taking  $C = V(G)$ , so we assume  $k \geq 1$ . Consequently we may assume that there are  $2k-1$  subsets  $A_1, \dots, A_{k-1}, B_1, \dots, B_{k-1}, C'$  of  $V(G)$ , pairwise disjoint, with the following properties:

- for  $1 \leq i \leq k-1$ ,  $A_i$  is connected and covers  $B_i$ ;
- for  $1 \leq i \leq k-1$ ,  $A_i, C'$  are anticomplete;
- for all distinct  $i, j \in \{1, \dots, k-1\}$ ,  $A_i$  is anticomplete to  $A_j \cup B_j$ ;

- $\mu(C') \geq 1 - 3(k-1)\epsilon$ ; and
- for  $1 \leq i \leq k-1$ , the set of vertices in  $C'$  covered by  $B_i$  has mass at least  $\kappa - 3(k-1)\epsilon$ .

Choose these subsets such that, in addition,  $|B_1| + \dots + |B_{k-1}|$  is minimum. For  $1 \leq i \leq k-1$ , let  $C_i$  be the set of vertices in  $C'$  covered by  $B_i$ . Thus  $\mu(C_i) \geq \kappa - 3(k-1)\epsilon$ , and from the minimality of  $|B_1| + \dots + |B_{k-1}|$ , it follows that  $\mu(C_i) \leq \kappa - (3k-4)\epsilon$ . Let  $D = C' \setminus (C_1 \cup \dots \cup C_{k-1})$ . Thus

$$\mu(D) \geq 1 - 3(k-1)\epsilon - (k-1)(\kappa - (3k-4)\epsilon) = (1 - (k-1)\kappa) + (k-1)(3k-7)\epsilon.$$

Note that  $D$  is anticomplete to  $A_i \cup B_i$  for  $1 \leq i < k$ ; and  $\mu(D) \geq 3\epsilon$ , from the hypotheses that  $\kappa + 4\epsilon \leq 1$  and  $(k-1)\kappa \leq 1$ .

For  $X \subseteq D$ , let  $B(X)$  denote the set of vertices in  $C' \setminus X$  with a neighbour in  $X$ . By 5.1, there exists a connected subset  $X \subseteq D$  with  $\mu(X) \geq \mu(D) - \epsilon$ ; and hence there is a connected subset  $X \subseteq D$  with  $\mu(X \cup B(X)) \geq \epsilon$ . Choose such a set  $X$  minimal, and let  $A_k = X$  and  $B_k = B(X)$ . Then  $A_k$  is anticomplete to  $A_i \cup B_i$  for  $1 \leq i < k$ , since  $A_k = X \subseteq D$ ; and  $B_k$  is anticomplete to  $A_i$  for  $1 \leq i < k$ , since  $B_k = B(X) \subseteq C'$ .

Choose  $x \in A_k$  such that  $A_k \setminus \{x\}$  is connected. Since  $\mu(x) < \epsilon$  and  $\mu(N(x)) < \epsilon$ , the minimality of  $X$  implies that  $\mu(A_k \cup B_k) \leq 3\epsilon$ . Let  $C = C' \setminus (A_k \cup B_k)$ . Since  $\mu(C') \geq 1 - 3(k-1)\epsilon$ , it follows that  $\mu(C) \geq 1 - 3k\epsilon$ , and since the set of vertices in  $C$  with no neighbour in  $A_k \cup B_k$  has mass less than  $\epsilon$ , it follows that the set  $C_k$  say of vertices in  $C$  with a neighbour in  $A_k \cup B_k$  satisfies  $\mu(C_k) \geq 1 - (3k+1)\epsilon \geq \kappa - 3k\epsilon$ . Also  $C_k$  is anticomplete to  $A_k$ , since  $B_k = B(X)$ , and so  $B_k$  covers  $C_k$ . For  $1 \leq i < k$ ,  $C_i \subseteq B_k \cup C$ , and  $\mu(C_i \cap B_k) \leq 3\epsilon$ , and so

$$\mu(C_i \cap C) \geq \mu(C_i) - 3\epsilon \geq \kappa - 3(k-1)\epsilon - 3\epsilon = \kappa - 3k\epsilon.$$

Since  $B_i$  covers  $C_i \cap C$ , this proves 6.1. ■

This is used to prove the following. Let us say a  $k$ -ladder in a graph  $G$  is a family of  $3k$  subsets

$$A_1, \dots, A_k, B_1, \dots, B_k, C_1, \dots, C_k$$

of  $V(G)$ , pairwise disjoint and such that

- for  $1 \leq i \leq k$ ,  $A_i$  is connected and covers  $B_i$ , and  $B_i$  covers  $C_i$ ;
- for  $1 \leq i \leq k$ ,  $A_i, C_i$  are anticomplete; and
- for all distinct  $i, j \in \{1, \dots, k\}$ ,  $A_i$  is anticomplete to  $A_j \cup B_j \cup C_j$ .

If in addition we have

- for  $1 \leq i < j \leq k$ ,  $B_i$  is anticomplete to  $C_j$

we say the ladder is *half-cleaned*. Let us say the *union* of the  $k$ -ladder is the triple  $(A, B, C)$  where  $A = \bigcup_{1 \leq i \leq k} A_i$ ,  $B = \bigcup_{1 \leq i \leq k} B_i$ , and  $C = \bigcup_{1 \leq i \leq k} C_i$ .

**6.2** *Let  $\epsilon, \kappa > 0$ , and let  $k \geq 0$  be an integer such that  $(k-1)k(\kappa + \epsilon) \leq 1$  and  $(k-1)(\kappa + \epsilon) + 4\epsilon \leq 1$ . Let  $(G, \mu)$  be an  $\epsilon$ -coherent massed graph. Then there is a half-cleaned  $k$ -ladder*

$$A_1, \dots, A_k, B_1, \dots, B_k, C_1, \dots, C_k$$

in  $G$  such that  $\mu(C_i) \geq \kappa$  for  $1 \leq i \leq k$ .

**Proof.** Let  $\kappa' = k(\kappa + \epsilon)$ . Since  $\kappa' + 4\epsilon \leq 1$  and  $(k-1)\kappa' \leq 1$ , it follows from 6.1 that there are  $2k+1$  subsets  $A_1, \dots, A_k, B'_1, \dots, B'_k, C$  of  $V(G)$ , all disjoint, with the following properties:

- for  $1 \leq i \leq k$ ,  $A_i$  is connected and covers  $B'_i$ ;
- for  $1 \leq i \leq k$ ,  $A_i, C$  are anticomplete;
- for all distinct  $i, j \in \{1, \dots, k\}$ ,  $A_i$  is anticomplete to  $A_j \cup B'_j$ ; and
- for  $1 \leq i \leq k$ , the set of vertices in  $C$  covered by  $B'_i$  has mass at least  $\kappa'$ .

Inductively, suppose that  $0 \leq j < k$  and we have defined  $B_1, \dots, B_j$  with  $B_i \subseteq B'_i$  for  $1 \leq i \leq j$ , and we have defined disjoint subsets  $C_1, \dots, C_j$  of  $C$  such that for  $1 \leq i \leq j$ ,  $B_i$  covers  $C_i$  and is anticomplete to  $C \setminus (C_1 \cup \dots \cup C_i)$ , with  $\kappa \leq \mu(C_i) \leq \kappa + \epsilon$ . Thus  $\mu(C_1 \cup \dots \cup C_j) \leq (k-1)(\kappa + \epsilon)$ , and since the set of vertices in  $C$  covered by  $B'_{j+1}$  has mass at least  $\kappa' = k(\kappa + \epsilon)$ , we may choose  $B_{j+1} \subseteq B'_{j+1}$  minimal such that the set,  $C_{j+1}$  say, of vertices in  $C \setminus (C_1 \cup \dots \cup C_j)$  covered by  $B_{j+1}$  has mass at least  $\kappa$ . From the minimality of  $B_{j+1}$ , it follows that  $\mu(C_{j+1}) \leq \kappa + \epsilon$ . This completes the inductive definition and so proves 6.2. ■

The distance in  $G$  between  $u, v$  is called the  $G$ -distance between  $u, v$ .

**6.3** Let  $\epsilon > 0$ , and let  $k \geq 0$  be an integer. Let  $(G, \mu)$  be an  $(\epsilon, 2)$ -coherent massed graph, and let

$$A_1, \dots, A_k, B_1, \dots, B_k, C_1, \dots, C_k$$

be a  $k$ -ladder in  $G$  such that  $\mu(C_i) \geq 3k\epsilon$  for  $1 \leq i \leq k$ . Let  $(A, B, C)$  be its union. For  $1 \leq i \leq k$  let  $b_i \in B_i$ , such that  $b_1, \dots, b_k$  are pairwise nonadjacent. Then every pairing of  $\{b_1, \dots, b_k\}$  is feasible in  $G[A \cup B \cup C]$ .

**Proof.** For each  $v \in B$ , let  $i(v) \in \{1, \dots, k\}$  such that  $v \in B_{i(v)}$ , and for  $1 \leq i \leq k$  let  $D_i = A_i \cup B_i \cup C_i$ . Let  $\Pi$  be a pairing of  $\{b_1, \dots, b_k\}$ , and let  $\{s_1, t_1\}, \dots, \{s_n, t_n\}$  be the members of  $\Pi$  with cardinality two. For  $1 \leq m \leq n$  we will construct inductively a path  $P_m$  between  $s_m, t_m$  with the following properties:

- $V(P_m) \subseteq D_{i(s_m)} \cup D_{i(t_m)}$ ;
- for  $1 \leq \ell < m$ , the sets  $V(P_\ell), V(P_m)$  are anticomplete;
- $V(P_m)$  is anticomplete to  $\{b_1, \dots, b_k\} \setminus \{s_m, t_m\}$ ; and
- at most two vertices of  $V(P_m)$  belong to  $C$ , and at most two to  $B \setminus \{s_m, t_m\}$ .

Let  $1 \leq m \leq n$ , and suppose we have constructed  $P_1, \dots, P_{m-1}$ ; we construct  $P_m$  as follows. Let

$$Z = \{b_1, \dots, b_k\} \cup ((V(P_1) \cup \dots \cup V(P_{m-1})) \cap (B \cup C)).$$

Thus  $|Z| \leq k + 4(m-1) \leq 3k - 4$ , since  $m \leq n \leq k/2$ . Let  $X$  be the set of vertices in  $C_{i(s_m)}$  that have  $G$ -distance at least three from every vertex of  $Z$ . Since  $(G, \mu)$  is  $(\epsilon, 2)$ -coherent, it follows that  $\mu(X) \geq \mu(C_{i(s_m)}) - (3k-4)\epsilon \geq \epsilon$ . Let  $Y$  be the set of vertices in  $C_{i(t_m)}$  that have  $G$ -distance at least three from every vertex of  $Z$ ; then similarly  $\mu(Y) \geq \epsilon$ . Since  $X \cap Y = \emptyset$  and  $(G, \mu)$  is  $\epsilon$ -coherent,

there exist  $x \in X$  and  $y \in Y$ , adjacent. Since  $B_{i(s_m)}$  covers  $C_{i(s_m)}$ , there exists  $x' \in B_{i(s_m)}$  adjacent to  $x$ ; and since the distance between  $x$  and  $Z$  is at least three, it follows that  $x'$  has no neighbour in  $Z$ . Similarly there exists  $y' \in B_{i(t_m)}$  adjacent to  $y$ . Since  $A_{i(s_m)}$  is connected and covers  $B_{i(s_m)}$ , and similarly for  $t_m$ , it follows that the subgraph of  $G$  induced on  $A_{i(s_m)} \cup A_{i(t_m)} \cup \{s_m, t_m, x, y, x', y'\}$  is connected. Choose an induced path  $P_m$  joining  $s_m, t_m$  in this subgraph. Then  $P_m$  satisfies the first and fourth bullets above.

We claim that for  $1 \leq \ell < m$ ,  $V(P_\ell)$  and  $V(P_m)$  are anticomplete. Suppose not, and let  $u \in V(P_\ell)$  and  $v \in V(P_m)$  be adjacent or equal. Now either  $u$  is one of  $s_\ell, t_\ell$ , or  $u$  belongs to one of  $A_{i(s_\ell)}, A_{i(t_\ell)}$ , or  $u \in Z$ ; and either  $v$  is one of  $s_m, t_m$ , or  $v$  belongs to one of  $A_{i(s_m)}, A_{i(t_m)}$ , or  $v \in \{x, y, x', y'\}$ . If  $u \in A_{i(s_\ell)}$ , then all its neighbours in  $A \cup B \cup C$  belong to  $A_{i(s_\ell)} \cup B_{i(s_\ell)}$  from the definition of a  $k$ -ladder; and since  $v$  is not in the latter set, it follows that  $u \notin A_{i(s_\ell)}$ . Similarly  $u \notin A_{i(t_\ell)}$ , and  $v \notin A_{i(s_m)} \cup A_{i(t_m)}$ . Consequently  $u, v \in B \cup C$ . Thus  $u \in Z$ , and so  $v \notin \{x, y, x', y'\}$  from the choice of  $x, y, x', y'$ . Hence  $v$  is one of  $s_m, t_m$ . But from the choice of  $P_\ell$ ,  $V(P_\ell)$  is anticomplete to  $\{b_1, \dots, b_k\} \setminus \{s_\ell, t_\ell\}$ , a contradiction. Thus  $P_m$  satisfies the second bullet.

For the third bullet, suppose that  $u \in \{b_1, \dots, b_k\} \setminus \{s_m, t_m\}$  is adjacent to  $v \in V(P_m)$ . Since  $u \in Z$ , it follows that  $v \neq x, y, x', y'$ ; and as before  $v \notin A_{i(s_m)} \cup A_{i(t_m)}$ , and so  $v$  is one of  $s_m, t_m$ , a contradiction since  $b_1, \dots, b_k$  are pairwise nonadjacent. Thus  $P_m$  satisfies the third bullet.

This completes the inductive definition, and hence proves 6.3. ■

In turn, 6.3 is used to prove the following, the main result of this section.

**6.4** *For every caterpillar  $T$ , there exist  $\epsilon > 0$  and an integer  $r \geq 1$ , such that for every  $(\epsilon, r)$ -coherent massed graph  $(G, \mu)$ , there is a versatile copy of  $T$  in  $G$ .*

**Proof.** We may assume that  $|T| \geq 3$ . Assign a head to  $T$  to make it rooted, not one of the leaves. Let  $k = 2^{|T|}$ . Let  $r = 5k$ , and choose  $\epsilon > 0$  such that  $(k-1)k(2^k(3k+2)+1)\epsilon \leq 1$ . Let  $(G, \mu)$  be an  $(\epsilon, r)$ -coherent massed graph. By 6.2, there is a half-cleaned  $k$ -ladder

$$A_1, \dots, A_k, B_1, \dots, B_k, C_1, \dots, C_k$$

in  $G$  such that  $\mu(C_i) \geq 2^k(3k+2)\epsilon$  for  $1 \leq i \leq k$ . Then 5.2 (taking  $\delta = (3k+1)\epsilon$ ) implies that there is a  $\{C_1, \dots, C_k\}$ -spread  $(3k+1)\epsilon$ -realization  $(X_v : v \in V(T))$  of  $T$  in  $G$ .

Let  $t_1, \dots, t_q$  be the vertices of  $T$  that are not leaves, where  $t_1$  is the head of  $T$  and  $t_i t_{i+1}$  is an edge of  $T$  for  $1 \leq i < q$ . Choose  $x_1 \in X_{t_1}$ . For  $2 \leq i \leq q$  in turn, since  $X_{t_i}$  covers  $X_{t_{i-1}}$  by definition of an realization, we may choose  $x_i \in X_{t_i}$  adjacent to  $x_{i-1}$ . Since each  $x_i$  belongs to one of  $C_1, \dots, C_k$ , it follows that  $\{x_1, \dots, x_q\}$  is anticomplete to  $A_1 \cup \dots \cup A_k$ . Also, since there are no edges between  $X_u, X_v$  for nonadjacent  $u, v \in V(T)$ , it follows that  $x_1, \dots, x_q$  are the vertices in order of an induced path of  $G$ . For the same reason we have the following two statements:

(1) *For each leaf  $v$  of  $T$  with neighbour  $t_j$  say,  $x_j$  has a neighbour in  $X_v$ , and  $X_v$  is anticomplete to  $\{x_1, \dots, x_q\} \setminus \{x_j\}$ .*

(2) *For all distinct leaves  $u, v$  of  $T$ ,  $X_u$  is anticomplete to  $X_v$ .*

We recall that  $C_1, \dots, C_k$  are pairwise disjoint, and  $X_v$  ( $v \in V(T)$ ) is  $\{C_1, \dots, C_k\}$ -spread; let  $I$  be the set of  $i \in \{1, \dots, k\}$  such that  $X_v \subseteq C_i$  for some leaf  $v$  of  $T$ . For each  $i \in I$ , let  $v_i$  be the

(unique) leaf of  $T$  with  $X_{v_i} \subseteq C_i$ . Let  $i \in I$ , and let  $j \in \{1, \dots, q\}$  such that  $v_i$  is adjacent to  $t_j$  in  $T$ . We define  $x^i = x_j$ . Thus there may be distinct values  $i, i' \in I$  with  $x^i = x^{i'}$ .

Let  $i \in I$ . Since  $X_{v_i}$  is  $(2k+1)\epsilon$ -dominant, and  $(G, \mu)$  is  $(\epsilon, r)$ -coherent, there exists  $u \in X_{v_i}$  with a neighbour  $v$  such that the  $G$ -distance between  $v$  and  $x_1$  is at least  $r+1$ . Consequently the  $G$ -distance between  $u$  and  $x_1$  is at least  $r$ . By (1),  $X_{v_i} \cup \{x^i\}$  is connected, and so there is a path of  $G[X_{v_i} \cup \{x^i\}]$  between  $x^i$  and  $u$ ; and hence there is a minimal path  $P_i$  of  $G[X_{v_i} \cup \{x^i\}]$  with one end  $x^i$  and the other  $u_i$  say, such that the  $G$ -distance between  $x_1$  and  $u_i$  is at least  $q+4i$ . It follows that the  $G$ -distance between  $x_1$  and  $u_i$  is exactly  $q+4i$ . Choose a vertex  $b_i \in B_i$  adjacent to  $u_i$ . Let  $c_i$  be the second vertex of  $P_i$ , that is, the vertex adjacent to  $x^i$ , and let  $Q_i = P_i \setminus \{x^i\}$ . Thus  $Q_i$  is a path of  $G[X_{v_i}]$ . The subgraph  $T'$  of  $G$  induced on  $\{x_1, \dots, x_q\} \cup \{c_i : i \in I\}$  is a copy of  $T$ , by (1) and (2), and we will show that it is versatile.

(3) For all distinct  $i, j \in I$ , the sets  $V(Q_i) \cup \{b_i\}$  and  $V(Q_j) \cup \{b_j\}$  are anticomplete.

We may assume that  $i < j$ . Certainly  $V(Q_i)$  and  $V(Q_j)$  are anticomplete, by (2). Also  $b_i$  has no neighbour in  $V(Q_j)$  since the  $k$ -ladder is half-cleaned; so it remains to check that  $b_j$  has no neighbour in  $V(Q_i) \cup \{b_i\}$ . Let  $v \in V(Q_i)$ ; then from the minimality of  $Q_i$ , the  $G$ -distance between  $v, x_1$  is at most  $q+4i$ . But the  $G$ -distance between  $u_j$  and  $x_1$  is  $q+4j$ , and so the  $G$ -distance between  $v, u_j$  is at least four. Consequently the  $G$ -distance between  $v, b_j$  is at least three, and in particular  $b_j$  has no neighbour in  $V(Q_i)$ ; and setting  $v = u_i$ , since the  $G$ -distance between  $v, b_j$  is at least three it follows that  $b_i, b_j$  are nonadjacent. This proves (3).

(4) For all  $i \in I$ , the sets  $V(Q_i) \cup \{b_i\}$  and  $\{x_1, \dots, x_q\} \setminus \{x^i\}$  are anticomplete.

Let  $j \in \{1, \dots, q\}$  and suppose that  $x_j$  is adjacent to some  $v \in V(Q_i) \cup \{b_i\}$  and  $x_j \neq x^i$ . Since the  $G$ -distance between  $x_1, b_i$  is  $q+4i$ , and the  $G$ -distance between  $x_1, x_j$  is at most  $q-1$ , it follows that  $b_i, x_j$  are nonadjacent, and so  $v \in Q_i$ , contrary to (1).

Let  $Z$  be the set of vertices in  $G$  with  $G$ -distance from  $x_1$  at most  $4k+q+4$ , and for  $1 \leq i \leq k$  let  $C'_i = C_i \setminus Z$ . Since  $(G, \mu)$  is  $(\epsilon, 4k+q+4)$ -coherent, it follows that  $\mu(Z) \leq \epsilon$ . Thus for  $1 \leq i \leq k$ ,  $\mu(C'_i) \geq \mu(C_i) - \epsilon \geq 3k\epsilon$ . Let  $Q = \bigcup_{i \in I} V(Q_i) \cup \{b_i\}$ . Since every vertex in  $Q$  has  $G$ -distance at most  $4k+q+1$  from  $x_1$ , it follows that every vertex in  $C'_i$  has  $G$ -distance at least three from  $Q$ . Let  $B'_i$  be the set of vertices in  $B_i$  with no neighbours in  $V(Q)$ . Since every vertex in  $C'_i$  has a neighbour in  $B_i$  and has  $G$ -distance at least three from  $Q$ , it follows that  $B'_i$  covers  $C'_i$ . Hence the sets

$$A_i \ (i \in I), B'_i \cup \{b_i\} \ (i \in I), C'_i \ (i \in I)$$

form an  $|I|$ -ladder, with union  $(A', B', C')$  say. Since  $\mu(C'_i) \geq 3k\epsilon$  and  $r \geq 2$ , every pairing of  $\{b_i : i \in I\}$  is feasible in  $G[A' \cup B' \cup C']$  by 6.3.

Since  $A' \cup B' \cup C' \setminus \{b_i : i \in I\}$  is anticomplete to

$$(Q \setminus \{b_i : i \in I\}) \cup \{x_1, \dots, x_q\},$$

it follows from (3) and (4) that every pairing of  $\{c_i : i \in I\}$  is feasible in the subgraph induced on  $A' \cup B' \cup C' \cup \bigcup_{i \in I} V(Q_i)$ . But  $\{c_i : i \in I\}$  is the set of leaves of  $T'$ , and since

$$A' \cup B' \cup C' \cup \bigcup_{i \in I} (V(Q_i) \setminus \{c_i\})$$

is anticomplete to  $\{x_1, \dots, x_q\}$ , it follows that  $T'$  is versatile. This proves 6.4. ▀

## 7 The general proof

Now we turn to the proof of 4.1 in general. Fix the caterpillar  $T$ , and choose  $\epsilon_r$  and  $r$  to satisfy 6.4 with  $\epsilon$  replaced by  $\epsilon_r$ . Now we will choose  $\epsilon$  much smaller than  $\epsilon_0$ , and try to prove that in every  $\epsilon$ -coherent massed graph  $(G, \mu)$ , some copy of  $T$  is versatile. We can therefore assume that for every  $Z \subseteq V(G)$ , there is no mass  $\mu'$  on  $G[Z]$  that is  $(\epsilon_r, r)$ -coherent; and in particular (assuming  $\mu(Z) > 0$ ), the mass  $\mu'$  on  $G[Z]$  defined by  $\mu'(X) = \mu(X)/\mu(Z)$  for each  $X \subseteq Z$  is not  $(\epsilon_r, r)$ -coherent. Consequently, either there is a vertex  $v \in Z$  with  $\mu(N_{G[Z]}^r[v]) \geq \epsilon_0\mu(Z)$ , or there are two anticomplete sets  $A, B \subseteq Z$  with  $\mu(A), \mu(B) \geq \epsilon_r\mu(Z)$ . The latter is only helpful if  $\epsilon_r\mu(Z) \geq \epsilon$ , but in that case we can assume the latter never occurs. Thus, for every  $Z \subseteq V(G)$  with  $\mu(Z) \geq \epsilon/\epsilon_r$ , there is a vertex  $v \in Z$  with  $\mu(N_{G[Z]}^r[v]) \geq \epsilon_r\mu(Z)$ . Since  $N_{G[Z]}^r[v]$  is anticomplete to  $Z \setminus N_{G[Z]}^{r+1}[v]$ , and  $\mu'(N_{G[Z]}^r[v]) \geq \epsilon_r$  (and because of the ‘latter never occurs’ assumption above), it follows that  $\mu'(Z \setminus N_{G[Z]}^{r+1}[v]) < \epsilon_r$ , and so  $\mu'(N_{G[Z]}^{r+1}[v]) \geq 1 - \epsilon_r$ , that is,  $\mu(N_{G[Z]}^{r+1}[v]) \geq (1 - \epsilon_r)\mu(Z)$ . Initially we could have chosen  $\epsilon_r$  as small as we want, and in particular we may assume that  $\epsilon_r \leq 1/2$ ; and so  $\mu(N_{G[Z]}^{r+1}[v]) \geq \mu(Z)/2$ .

That motivates the following: let us say a massed graph  $(G, \mu)$  is  $(\delta, r)$ -focussed if for every  $Z \subseteq V(G)$  with  $\mu(Z) \geq \delta$ , there is a vertex  $v \in Z$  with  $\mu(N_{G[Z]}^r[v]) \geq \mu(Z)/2$ . For such a graph we can find a useful variant of 5.2. For  $X \subseteq V(G)$  and  $v \in X$ , let us say  $v$  is an  $r$ -centre of  $X$  if every vertex in  $X$  has  $G[X]$ -distance at most  $r$  from  $v$  (and consequently  $X$  is connected).

**7.1** *Let  $T$  be a rooted caterpillar  $T$ , let  $\delta, \epsilon > 0$  with  $\epsilon \leq \delta/2$ , let  $(G, \mu)$  be an  $\epsilon$ -coherent  $(\delta, r)$ -focussed massed graph, and let  $\mathcal{Y}$  be a set of disjoint subsets of  $V(G)$ , such that  $|\mathcal{Y}| = 2^{|T|}$  and  $\mu(Y) \geq 2^{2^{|T|}}(\delta + \epsilon)$  for each  $Y \in \mathcal{Y}$ . Then there is a  $\mathcal{Y}$ -spread  $\delta$ -realization  $(X_v : v \in V(T))$  of  $T$  in  $G$  such that  $X_v$  has an  $r$ -centre for each  $v \in V(T)$  except the head.*

The proof of 7.1 is the same as that of 5.2 except at one step. In the fourth paragraph of the proof of 5.2 we applied 5.1 to deduce that there exists  $Z \subseteq X_{h_i}$  with  $\mu(Z) > \mu(X_{h_i}) - \epsilon$ , such that  $Z$  is connected. Instead, let us apply the fact that  $(G, \mu)$  is  $(\delta, r)$ -focussed to deduce that there exists  $Z \subseteq X_{h_i}$  with  $\mu(Z) > \mu(X_{h_i})/2$  and with an  $r$ -centre. This is justified, since  $\mu(X_{h_i}) \geq m_k \geq \delta$ . The next step, to deduce that  $\mu(Z) \geq \epsilon$ , is also justified since  $\mu(Z) \geq \mu(X_{h_i})/2 \geq m_k/2 \geq \delta/2 \geq \epsilon$ , by hypothesis. Let  $z_1$  be an  $r$ -centre of  $Z$ , and choose the ordering  $z_1, \dots, z_n$  of the vertices of  $Z$  in increasing order of  $G[Z]$ -distance from  $z_1$ ; and then the remainder of the proof is unchanged.

For  $X \subseteq V(G)$  and  $v \in V(G)$ , we say  $v$  touches  $X$  if either  $v \in X$  or  $v$  has a neighbour in  $X$ ; and otherwise  $v$  is anticomplete to  $X$ . We need the following.

**7.2** *Let  $t, r \geq 1$  be integers and  $\epsilon > 0$ , and let  $(G, \mu)$  be an  $\epsilon$ -coherent massed graph. Let  $T$  be a caterpillar in  $G$  with  $t$  vertices, and let  $x_1, \dots, x_q$  be the vertices of  $T$  with degree more than one. For each leaf  $v$  of  $T$  let  $x^v$  be its neighbour in  $\{x_1, \dots, x_q\}$ , and let  $X_v \subseteq V(G)$ , such that*

- for each leaf  $v$ ,  $v \in X_v$ , and  $X_v \cap \{x_1, \dots, x_q\} = \emptyset$ , and  $X_v$  is anticomplete to  $\{x_1, \dots, x_q\} \setminus \{x^v\}$ ;
- for all distinct leaves  $u, v$ ,  $X_u$  is anticomplete to  $X_v$ ;

- for each leaf  $v$ ,  $x^v$  is an  $r$ -centre for  $X_v \cup \{x^v\}$ ;
- for each leaf  $v$ ,  $v$  is the unique neighbour of  $x^v$  in  $X_v$ ; and
- for each leaf  $v$ ,  $X_v \cup \{x^v\}$  is  $(r+2)t^{t+1}\epsilon$ -dominant.

Then  $T$  is versatile.

**Proof.** Define  $\kappa_i = (r+2)t^{t-i+1}\epsilon$  for  $0 \leq i \leq t$ . Let  $\Pi$  be a pairing of the set of leaves  $L$  of  $T$ . Let  $L = \{v_1, \dots, v_\ell\}$ , where  $\Pi$  consists of the sets  $\{v_{2i-1}, v_{2i}\}$  for  $1 \leq i \leq k$  for some  $k \leq \ell/2$ , together with the singleton sets  $\{v_i\}$  for  $2k+1 \leq i \leq \ell$ . Let  $X_v^0 = X_v \cup \{x^v\}$  for each  $v \in L$ . For  $1 \leq i \leq k$ , we define  $X_v^i$  ( $v \in \{v_{2i+1}, \dots, v_\ell\}$ ) and  $P_i$  inductively as follows. We assume  $P_1, \dots, P_{i-1}$  and  $X_v^{i-1}$  ( $v \in \{v_{2i-1}, \dots, v_\ell\}$ ) have been defined, such that

- for  $1 \leq h \leq i-1$ ,  $P_h$  is an induced path between  $v_{2h-1}, v_{2h}$ , of length at most  $2r+1$ ;
- for  $1 \leq h \leq i-1$ ,  $V(P_h) \setminus \{v_{2h-1}, v_{2h}\}$  is anticomplete to  $V(T) \setminus \{v_{2h-1}, v_{2h}\}$ ;
- for  $1 \leq h < h' \leq i-1$ ,  $V(P_h)$  is anticomplete to  $V(P_{h'})$ ;
- for  $1 \leq h \leq i-1$  and  $v \in \{v_{2i-1}, \dots, v_\ell\}$ ,  $V(P_h)$  is anticomplete to  $X_v^{i-1}$ ;
- for  $v \in \{v_{2i-1}, \dots, v_\ell\}$ ,  $X_v^{i-1}$  is  $\kappa_{i-1}$ -dominant; and
- for  $v \in \{v_{2i-1}, \dots, v_\ell\}$ ,  $\{x^v, v\} \subseteq X_v^{i-1}$  and  $x^v$  is an  $r$ -centre for  $X_v^{i-1}$ .

For each  $w \in \{v_{2i+1}, \dots, v_\ell\}$ , choose  $X_w^i \subseteq X_w^{i-1}$ , minimal such that  $X_w^i$  is  $\kappa_i$ -dominant and  $x^w$  is a  $r$ -centre for  $X_w^i$ . By deleting a vertex in  $X_w^i$  with maximum  $G[X_w^i]$ -distance from  $x^w$ , the minimality of  $X_w^i$  implies that the set of vertices that touch  $X_w^i$  has mass at most  $\kappa_i + \epsilon$ . Also the set of vertices that touch  $\{x_1, \dots, x_q\} \cup \bigcup_{h < i} V(P_h)$  has mass at most  $(q + (k-1)(2r+2))\epsilon$ . Let  $u = v_{2i-1}$  and  $v = v_{2i}$ . Since  $X_u^{i-1}$  is  $\kappa_{i-1}$ -dominant, the set of vertices that touch  $X_u^{i-1}$  and do not touch  $\{x_1, \dots, x_q\} \cup \bigcup_{h < i} V(P_h)$  and do not touch  $X_w^i$  for  $w \in \{v_{2i+1}, \dots, v_\ell\}$  has mass at least

$$\kappa_{i-1} - (q + (k-1)(2r+2))\epsilon - \ell(\kappa_i + \epsilon) \geq \epsilon$$

since  $q + \ell = t$  and  $\ell \leq t-1$  and  $k \leq \ell/2$  and  $r \geq 1$  (we leave checking this to the reader). The same holds for  $X_v^{i-1}$ ; and so these two sets are not anticomplete. Consequently there are vertices  $a, b$ , adjacent or equal, such that  $a$  touches  $X_u^{i-1}$  and  $b$  touches  $X_v^{i-1}$ , and  $a, b$  are anticomplete to  $\{x_1, \dots, x_q\} \cup \bigcup_{h < i} V(P_h)$  and to  $X_w^i$  for  $w \in \{v_{2i+1}, \dots, v_\ell\}$ . Since  $x^u$  is an  $r$ -centre for  $X_u^{i-1}$ , and  $u$  is the unique neighbour of  $x^u$  in  $X_u^{i-1}$ , there is a path of  $G[X_u^{i-1} \cup \{a\}]$  between  $u$  and  $a$  of length at most  $r$ , and the same for  $v$ ; and therefore there is an induced path  $P_i$  between  $u, v$  of length at most  $2r+1$ , and all its vertices belong to  $X_u^{i-1} \cup X_v^{i-1} \cup \{a, b\}$ . In particular,  $V(P_i)$  is anticomplete to  $V(P_h)$  for  $h < i$ , since  $X_u^{i-1} \cup X_v^{i-1} \cup \{a, b\}$  is anticomplete to  $V(P_h)$ ; and  $V(P_i)$  is anticomplete to  $X_w^i$  for  $w \in \{v_{2i+1}, \dots, v_\ell\}$ , since  $X_u^{i-1} \cup X_v^{i-1} \cup \{a, b\}$  is complete to  $X_w^i$ . This completes the inductive definition.

But then the paths  $P_1, \dots, P_k$  together with the singletons  $\{c_i\}$  ( $2k+1 \leq i \leq \ell$ ) show that  $\Pi$  is feasible relative to  $T$ . Consequently  $T$  is versatile. This proves 7.2.  $\blacksquare$



This is used to prove:

**7.3** *Let  $T$  be a caterpillar with  $t$  vertices, let  $r \geq 1$  be an integer, and let  $\epsilon, \delta > 0$  with  $\epsilon \leq \delta/2$ , such that  $\delta \leq 2^{-(t+2^t)}t^{-t}$  and  $\epsilon \leq 2^{-(t+2^t)}t^{-2t}(3r+5)^{-1}$ . Let  $(G, \mu)$  be a  $(\delta, r)$ -focussed  $\epsilon$ -coherent massed graph. Then there is a versatile copy of  $T$  in  $G$ .*

**Proof.** We may assume that  $|T| \geq 3$ ,  $T$  is rooted, and its head is an internal vertex. Let  $t = |T|$  and  $k = 2^t$ . Let  $\lambda = 1/k - \epsilon$ .

(1) *There exist pairwise disjoint subsets  $Y_1, \dots, Y_k$  of  $V(G)$  with  $\mu(Y_i) \geq \lambda$  for  $1 \leq i \leq k$ .*

We define  $Y_1, \dots, Y_k \subseteq V(G)$  inductively as follows. Let  $1 \leq i < k$ , and assume we have chosen  $Y_1, \dots, Y_i \subseteq V(G)$ , pairwise disjoint and each with  $\lambda \leq \mu(Y_i) \leq \lambda + \epsilon$ . Thus

$$\mu(Y_1 \cup \dots \cup Y_i) \leq (k-1)(\lambda + \epsilon),$$

and so

$$\mu(V(G) \setminus (Y_1 \cup \dots \cup Y_i)) \geq 1 - (k-1)(\lambda + \epsilon) \geq \lambda.$$

Consequently we may choose  $Y_{i+1}$  disjoint from  $Y_1 \cup \dots \cup Y_i$  with  $\mu(Y_i) \geq \lambda$ . Choose  $Y_{i+1}$  minimal with this property; then  $\mu(Y_i) \leq \lambda + \epsilon$ . This completes the inductive definition of  $Y_1, \dots, Y_k$  and so proves (1).

Let  $\kappa_0 = 2^{-k}\lambda - \epsilon$ , and for  $1 \leq i \leq t$  let  $\kappa_i = \kappa_0 t^{-i}$ . Since  $G$  is  $(\delta, r)$ -focussed, it is also  $(\kappa_0, r)$ -focussed, since  $\kappa_0 \geq \delta$ . From 7.1, there is a  $\{Y_1, \dots, Y_k\}$ -spread  $\kappa_0$ -realization  $(X_v : v \in V(T))$  of  $T$  in  $G$  such that  $X_v$  has an  $r$ -centre for each  $v \in V(T)$  except the head. Let the vertices of degree more than one in  $T$  be  $t_1, \dots, t_q$ , where  $t_1$  is the head and  $t_i t_{i+1}$  is an edge of  $T$  for  $1 \leq i < q$ . Choose  $x_1 \in X_{t_1}$ , and inductively for  $i = 2, \dots, q$ , choose  $x_i \in X_{t_i}$  adjacent to  $x_{i-1}$ . As in the proof of 6.4,  $x_1, \dots, x_t$  are the vertices in order of an induced path of  $G$ . We need to arrange that for each leaf  $v$  of  $T$  adjacent to  $t_i$  say,  $x_i$  has a unique neighbour in  $X_v$ .

Let  $L = \{v_1, \dots, v_\ell\}$  be the set of leaves of  $T$ , and for each  $v \in L$  let  $x^v$  be the vertex  $x_j$  such that  $v$  is adjacent to  $t_j$  in  $T$ . Thus  $x^v$  is the unique vertex in  $\{x_1, \dots, x_q\}$  covered by  $X_v$ . Since  $X_v$  has an  $r$ -centre, and  $x^v$  has a neighbour in  $X_v$ , it follows that  $x^v$  is a  $(2r+1)$ -centre of  $X_v \cup \{x^v\}$ .

Let  $X_v^0 = X_v \cup \{x^v\}$  for  $v \in L$ . For  $0 \leq i \leq \ell$  we will inductively define  $X_v^i$  ( $v \in L$ ) satisfying the following: for  $0 \leq i \leq \ell$ ,

- for each  $v \in L$ ,  $x^v \in X_v^i$  and  $X_v^i \setminus \{x^v\}$  is anticomplete to  $\{x_1, \dots, x_q\} \setminus \{x^v\}$ ;
- for all distinct  $u, v \in L$ ,  $X_u^i \setminus \{x^u\}$  is anticomplete to  $X_v^i \setminus \{x^v\}$ ;
- for  $1 \leq j \leq i$ ,  $x^{v_j}$  is a  $(3r+2)$ -centre of  $X_{v_j}^i$ , and  $x^{v_j}$  has a unique neighbour in  $X_{v_j}^i$ ;
- for  $i < j \leq \ell$ ,  $x^{v_j}$  is a  $(2r+1)$ -centre of  $X_{v_j}^i$ ; and
- for each  $v \in L$ ,  $X_v^i$  is  $\kappa_i$ -dominant.

Suppose that  $1 \leq i \leq \ell$ , and we have defined  $X_v^{i-1}$  ( $v \in L$ ) as above. For each  $u \in L \setminus \{v_i\}$ , let  $u = v_j$  say. Let  $r_j = 3r + 2$  if  $j < i$ , and  $r_j = 2r + 1$  if  $j > i$ , so in either case  $x^u$  is an  $r_j$ -centre of  $X_u^{i-1}$ . Choose  $X_u^i \subseteq X_u^{i-1}$ , minimal such that  $X_u^i$  is  $\kappa_i$ -dominant and  $x^u$  is an  $r_j$ -centre for  $X_u^i$ . It follows from the minimality of each  $X_u^i$  that the set of vertices that touch  $X_u^i$  has mass at most  $\kappa_i + \epsilon$ .

Let  $v = v_i$ , and let  $Y''$  be the set of all vertices that touch  $X_v^{i-1}$ ; then  $\mu(Y'') \geq \kappa_{i-1}$  since  $X_v^{i-1}$  is  $\kappa_{i-1}$ -dominant. Let  $Y'$  be the set of vertices in  $Y''$  that are anticomplete to

$$\{x_1, \dots, x_q\} \cup \bigcup_{u \in L \setminus \{v\}} X_u^i.$$

Thus  $\mu(Y') \geq \mu(Y'') - q\epsilon - (\ell - 1)(\kappa_i + \epsilon)$ . Since  $\mu(Y'') \geq \kappa_{i-1}$  and  $\delta \leq 2^{-(t+2^t)}t^{-t}$  by hypothesis, it follows that  $\mu(Y') \geq \delta$  (we leave it to the reader to check this arithmetic). Since  $G$  is  $(\delta, r)$ -focussed, there is a subset  $Y \subseteq Y'$  with  $\mu(Y) \geq \mu(Y')/2$ , such that  $Y$  has an  $r$ -centre  $y$  say. Since  $x^v$  is a  $(2r + 1)$ -centre for  $X_v^{i-1}$ , and  $y$  touches this set, there is an induced path  $P$  of  $G[X_v^{i-1} \cup \{y\}]$  between  $x^v$  and  $y$ , of length at most  $2r + 2$ . Let  $X_v^i = Y \cup V(P)$ . Then  $x^v$  is a  $(3r + 2)$ -centre for  $X_v^i$ . Since  $x^v$  is anticomplete to  $Y$  and has only one neighbour in  $P$ , it follows that  $x^v$  has only one neighbour in  $Y \cup V(P)$ . Moreover,  $Y$  is  $\kappa_i$ -dominant since  $\mu(Y) \geq \epsilon$ . This completes the inductive definition.

For each  $v \in L$ , let  $c_v$  be the unique neighbour of  $x^v$  in  $X_v^\ell$ . The subgraph  $T'$  induced on  $\{x_1, \dots, x_q\} \cup \{c_v : v \in L\}$  is a copy of  $T$ . Since  $\epsilon \leq 2^{-(t+2^t)}t^{-2t}(3r + 5)^{-1}$  by hypothesis, it follows that  $\kappa_\ell \geq (3r + 4)t^{t+1}\epsilon$ , and so 7.2 (with  $r$  replaced by  $3r + 2$ ) implies that  $T'$  is versatile. This proves 7.3.  $\blacksquare$

Now let us put these pieces together, to prove 4.1, which we restate:

**7.4** *For every caterpillar  $T$ , there exists  $\epsilon > 0$ , such that for every  $\epsilon$ -coherent massed graph  $(G, \mu)$ , there is a versatile copy of  $T$  in  $G$ .*

**Proof.** By 6.4 there exist  $\epsilon_r > 0$  and an integer  $r \geq 1$ , such that for every  $(\epsilon_r, r)$ -coherent massed graph  $(G, \mu)$ , there is a versatile copy of  $T$  in  $G$ . We may assume  $\epsilon_r \leq t^{-t}(3r + 5)^{-1}$ . Choose  $\epsilon$  such that  $\epsilon \leq 2^{-(t+2^t)}t^{-t}\epsilon_r$ . Let  $\delta = \epsilon/\epsilon_r$ . Then  $\epsilon, \delta$  satisfy the hypotheses of 7.3.

Let  $(G, \mu)$  be an  $\epsilon$ -coherent massed graph; we will prove there is a versatile copy of  $T$  in  $G$ . If  $(G, \mu)$  is  $(\delta, r + 1)$ -focussed, the result follows from 7.3, so we assume not. Hence there exists  $Z \subseteq V(G)$  with  $\mu(Z) \geq \delta$ , such that  $\mu(N_{G[Z]}^{r+1}[v]) < \mu(Z)/2$  for each  $v \in Z$ . Let  $\mu'(X) = \mu(X)/\mu(Z)$  for all  $X \subseteq Z$ ; then  $(G[Z], \mu')$  is a massed graph. If it is  $(\epsilon_r, r)$ -coherent then the result follows from 6.4, so we assume not, for a contradiction. If there exist anticomplete subsets  $A, B$  of  $Z$  with  $\mu'(A), \mu'(B) > \epsilon_r$ , then  $\mu(A), \mu(B) \geq \epsilon$ , which is impossible. Thus there exists  $v \in Z$  such that  $\mu'(N_{G[Z]}^r[v]) \geq \epsilon_r$ , and hence such that  $\mu(N_{G[Z]}^r[v]) \geq \epsilon_r\mu(Z) \geq \epsilon$ . But  $\mu(N_{G[Z]}^{r+1}[v]) < \mu(Z)/2$  from the choice of  $Z$ , and so  $\mu(Z \setminus N_{G[Z]}^{r+1}[v]) > \mu(Z)/2 \geq \epsilon$ , a contradiction since the two sets  $N_{G[Z]}^r[v]$  and  $Z \setminus N_{G[Z]}^{r+1}[v]$  are anticomplete. This proves 7.4.  $\blacksquare$

## 8 Parallels with $\chi$ -boundedness

Here are some consequences of 2.3.

- By setting  $H = K_t$  and  $|P| = 1$ , it follows that the ideal of graphs with no induced subgraph a subdivision of  $K_t$  is incoherent.
- For a graph  $H$ , let  $\mathcal{I}$  be the ideal of all graphs  $G$  such that no induced subgraph of  $G$  or its complement is a subdivision of  $H$ . Then  $\mathcal{I}$  satisfies the Erdős-Hajnal conjecture 2.1, by 3.3. Consequently 1.4 holds.
- Fix  $H, P$  as before, let  $k \geq 1$  be an integer, and let  $H_k$  be obtained from  $H$  by subdividing every edge not in  $E(P)$  exactly  $k$  times. Since every coherent ideal contains a  $P$ -filleting of  $H_k$ , it follows that every coherent ideal contains a  $P$ -filleting of  $H$  where every edge not in  $P$  is subdivided at least  $k$  times.
- If  $T$  is a tree such that some path  $P$  of  $T$  contains all vertices of degree at least three (such a tree is called a *caterpillar subdivision*), it follows that every coherent ideal contains  $T$  (this is the main theorem of [18]). To see this, observe that since  $T$  is a caterpillar subdivision, every  $P$ -filleting of  $T$  contains a copy of  $T$  as an induced subgraph.

An ideal is  $\chi$ -bounded if there is a function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for each graph  $G$  in the ideal. Such ideals have been intensively studied, and it turns out that incoherent ideals and  $\chi$ -bounded ideals are in some ways very similar. Here are two instances:

- Take a graph  $H$ , and let  $\mathcal{I}$  be the ideal of all  $H$ -free graphs. The Gyárfás-Sumner conjecture [17, 25] asserts that  $\mathcal{I}$  is  $\chi$ -bounded if and only if  $H$  is a forest, and a conjecture of [18] asserts that  $\mathcal{I}$  is incoherent if and only if  $H$  is a forest. The second is now (very recently) a theorem [8], and the ‘only if’ part of the first is true, and the ‘if’ part has been shown for some forests.
- A *hole* in  $G$  means an induced cycle of length at least four. Let  $\mathcal{I}$  be the class of all graphs with no hole of length at least  $k$ , for some fixed integer  $k$ . A theorem of [6] says that  $\mathcal{I}$  is  $\chi$ -bounded, and a theorem of [2] says that  $\mathcal{I}$  is incoherent.

But the parallel does not always work, and in fact neither property implies the other. Here are two examples showing this, one for each direction:

- The ideal of all perfect graphs is  $\chi$ -bounded, and indeed so is the ideal of all graphs with no odd hole [21], but an example of Fox [15] shows that these ideals are coherent.
- Fix a graph  $H$ , and let  $\mathcal{I}$  be the ideal of all graphs such that no induced subgraph is isomorphic to a subdivision of  $H$ . Then  $\mathcal{I}$  is not necessarily  $\chi$ -bounded [19], but our main result proves that it is incoherent.

There are a number of hard results and longstanding open conjectures about ideals that are not  $\chi$ -bounded, and it is entertaining to try their parallels for coherent ideals. (The proofs below are just sketched.)

- It is proved in [24] that every ideal that is not  $\chi$ -bounded contains cycles of all lengths modulo  $k$ , for any integer  $k \geq 1$ . The same is not true for coherent ideals, as the example of Fox [15] shows; a coherent ideal need not contain a cycle of odd length more than three, and in particular need not contain a cycle of length 1 modulo 6. But it follows from 2.3 that for all integers  $\ell$ , every coherent ideal contains a cycle of length  $2\ell$  modulo  $k$ , and hence contains one of every

length modulo  $k$  if  $k$  is odd. To see this, choose  $\epsilon > 0$  very small, and choose an  $\epsilon$ -coherent graph  $G$  from the ideal. By 2.3,  $G$  contains a  $P$ -filleting of complete graph  $H$  of some large (constant) size, where  $P$  is a Hamilton path of  $H$ . Choose many disjoint subpaths of  $P$ , each of length  $2k$ , with no edges joining them. Let these paths be  $P_1, \dots, P_n$  say, and let the  $i$ th vertex of  $P_j$  be  $v_j^i$ . For each  $i$ , and  $1 \leq h < k \leq n$ , there is a path of the  $P$ -filleting that joins  $v_h^i, v_j^i$ , say  $Q_i^{h,j}$ ; and by Ramsey's theorem, we may choose many of the paths  $P_j$  such that all the paths  $Q_i^{h,j}$  have the same length modulo  $k$  depending on  $i$ . (Redefine  $n$ , and renumber  $P_1, \dots, P_n$  so that this holds.) Let  $R_i$  be the union of

$$Q_i^{1,2}, Q_i^{2,3}, \dots, Q_i^{k,k+1},$$

then  $R_i$  has length divisible by  $k$ . But then for any  $t$  modulo  $k$ , the union of  $R_1, R_{t+1}$  and subpaths of  $P_1$  and  $P_{k+1}$  (both of length  $t$ ) makes a hole of length  $2t$  modulo  $k$ , and this cycle belongs to the ideal.

- It is conjectured in [22] that in every graph with huge chromatic number and bounded clique number, there are  $k$  holes with consecutive lengths. The same is not true in  $\epsilon$ -coherent graphs  $G$  with  $\epsilon$  very small, because they need not have odd holes; but perhaps there must always be  $k$  even holes with successive lengths differing by two?
- It is proved in [23] that, in any colouring of a graph with huge chromatic number and bounded clique number, some induced  $k$ -vertex path is *rainbow* (that is, all its vertices have different colours), and no other types of connected subgraph have this property. What if we colour a graph which is  $\epsilon$ -coherent for  $\epsilon$  very small? Then we can do better than just paths; the results of this paper show that we can get a rainbow copy of any caterpillar. Each colour class has cardinality at most  $2\epsilon|G|$ , so by grouping the colour classes, we can partition the vertex set into many disjoint sets each of about the same size (differing by at most  $2\epsilon|G|$ ), and each a union of colour classes. Then 5.2 gives a copy  $T'$  of  $T$  with at most one vertex from each block of the partition; and in particular,  $T'$  is rainbow. Actually, we can do even better; results of [8] show we can get a rainbow copy of any forest.
- If we direct the edges of a graph with huge chromatic number and bounded clique number, some digraphs must be present as induced subdigraphs. For instance, it is proved in [7] that every oriented star has this property, and so does a three-edge path where both ends point outwards. What if we direct the edges of a graph which is  $\epsilon$ -coherent for  $\epsilon$  very small? Now much less is true. We need not get a directed two-edge path, because of Fox's example from [15] (this is a comparability graph, and so can be directed so that there is no induced directed two-edge path). We also need not get an outdirected 3-star (a tree with four vertices, three of them adjacent from the fourth). To see this, fix  $\epsilon$ , choose  $k$  with  $k\epsilon \geq 2$ , and take  $k$  disjoint sets  $A_1, \dots, A_k$  each of the same size  $n/k$  say, with  $n$  large. Now take a random graph on  $A_1 \cup \dots \cup A_k$  with average degree  $\log n$ ; with high probability the outcome is  $\epsilon$ -coherent, and its maximum degree is  $O(\log(n))$ . For  $1 \leq i \leq k$  in turn, and for every pair of vertices  $u, v \in A_{i+1} \cup \dots \cup A_k$  with a common neighbour in  $A_i$ , add an edge  $uv$ . Let the result be  $G'$ . Since this process is repeated only  $k$  times and the maximum degree at most squares at each step, the maximum degree of  $G'$  is still less than  $\epsilon n$ . Now add more edges so that each  $A_i$  is a clique, forming  $G''$ . Thus  $G''$  is  $\epsilon$ -coherent. Orient every edge  $uv$  of  $G'$ , from  $u$  to  $v$  if  $u \in A_i$  and  $v \in A_j$  where  $i < j$ ,

and arbitrarily if  $u, v$  belong to the same  $A_i$ . In the resulting digraph, there is no induced outdirected 3-star. (Incidentally, because of our main theorem 2.3 we always get a subdivision of  $K_{2,3}$  as an induced subgraph, and however this is oriented it contains an outdirected 2-star; so we always get an outdirected 2-star.)

And to finish, here is another conjecture, again similar to the Gyárfás-Sumner conjecture and the conjecture of [18] in that the conclusion is false for all graphs  $T$  that are not forests:

**8.1 Conjecture:** *For every forest  $T$  there exists  $\epsilon > 0$  with the following property. Let  $G$  be a  $T$ -free bipartite graph with bipartition  $(A, B)$ , where  $|A| = |B| = n \geq 1$ . Then either some vertex has degree at least  $en$ , or  $\min(|A'|, |B'|) \geq en$  for some anticomplete pair of subsets  $A' \subseteq A$  and  $B' \subseteq B$ .*

We can currently prove this for all trees of radius two and for all subdivided caterpillars (we omit the proofs).

## References

- [1] N. Alon, J. Pach, R. Pinchasi, R. Radoičić and M. Sharir, "Crossing patterns of semi-algebraic sets", *J. Combin. Theory Ser. A*, **111** (2005), 31–326.
- [2] M. Bonamy, N. Bousquet, and S. Thomassé, "The Erdős-Hajnal conjecture for long holes and antiholes", *SIAM J. Discrete Math.*, **30** (2016), 1159–1164.
- [3] N. Bousquet, A. Lagoutte, and S. Thomassé, "The Erdős-Hajnal conjecture for paths and antipaths", *J. Combinatorial Theory, Ser. B*, **113** (2015), 261–264.
- [4] K. Choromanski, D. Falik, A. Liebenau, V. Patel, M. Pilipczuk, "Excluding hooks and their complements", [arXiv:1508.00634](https://arxiv.org/abs/1508.00634).
- [5] M. Chudnovsky, "The Erdős-Hajnal conjecture – a survey", *J. Graph Theory* **75** (2014), 178–190.
- [6] M. Chudnovsky, A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. III. Long holes", *Combinatorica* **37** (2017), 1057–72.
- [7] M. Chudnovsky, A. Scott and P. Seymour, "Induced subgraphs of graphs with large chromatic number. XI. Orientations", (manuscript February 2017).
- [8] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, "Trees and linear anticomplete pairs" (manuscript March 2018).
- [9] M. Chudnovsky and P. Seymour, "Excluding paths and antipaths", *Combinatorica* **35** (2015), 389–412.
- [10] M. Chudnovsky and Y. Zwols, "Large cliques or stable sets in graphs with no four-edge path and no five-edge path in the complement", *J. Graph Theory*, **70** (2012), 449–472.
- [11] P. Erdős, "Some remarks on the theory of graphs", *Bull. Amer. Math. Soc.* **53** (1947), 292–294.

- [12] P. Erdős and A. Hajnal, “On spanned subgraphs of graphs”, *Graphentheorie und Ihre Anwendungen* (Oberhof, 1977), [www.renyi.hu/~p\\_erdos/1977-19.pdf](http://www.renyi.hu/~p_erdos/1977-19.pdf).
- [13] P. Erdős and A. Hajnal, “Ramsey-type theorems”, *Discrete Applied Mathematics* **25** (1989), 37–52.
- [14] P. Erdős and G. Szekeres, “A combinatorial problem in geometry”, *Compositio Mathematica* **2** (1935), 463–470.
- [15] J. Fox, “A bipartite analogue of Dilworth’s theorem”, *Order* **23** (2006), 197–209.
- [16] J. Fox and J. Pach, “Erdős-Hajnal-type results on intersection patterns of geometric objects”, in *Horizon of Combinatorics* (G.O.H. Katona et al., eds.), Bolyai Society Studies in Mathematics, Springer, 79–103, 2008.
- [17] A. Gyárfás, “On Ramsey covering-numbers”, *Coll. Math. Soc. János Bolyai*, in *Infinite and Finite Sets*, North Holland/American Elsevier, New York (1975), 10.
- [18] A. Liebenau, M. Pilipczuk, P. Seymour and S. Spirkl, “Caterpillars in Erdős-Hajnal”, submitted for publication (manuscript October 2017).
- [19] A. Pawlik, J. Kozik, T. Krawczyk, M. Lasoń, P. Micek, W. T. Trotter and B. Walczak, “Triangle-free intersection graphs of line segments with large chromatic number”, *J. Combinatorial Theory, Ser. B*, **105** (2014), 6–10.
- [20] V. Rödl, “On universality of graphs with uniformly distributed edges”, *Discrete Math.* **59** (1986), 125–134.
- [21] A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. I. Odd holes”, *J. Combinatorial Theory, Ser. B*, **121** (2016), 68–84.
- [22] A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. IV. Consecutive holes”, submitted for publication, [arXiv:1509.06563](https://arxiv.org/abs/1509.06563).
- [23] A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. IX. Rainbow paths”, *Electronic J. Combinatorics*, 24.2:#P2.53, 2017.
- [24] A. Scott and P. Seymour, “Induced subgraphs of graphs with large chromatic number. X. Holes with specific residue”, submitted for publication, [arXiv:1705.04609](https://arxiv.org/abs/1705.04609).
- [25] D.P. Sumner, “Subtrees of a graph and chromatic number”, in *The Theory and Applications of Graphs*, (G. Chartrand, ed.), John Wiley & Sons, New York (1981), 557–576.