On graph decompositions modulo k

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Abstract. We prove that, for every integer $k \geq 2$, every graph has an edge-partition into $5k^2 \log k$ sets, each of which is the edge-set of a graph with all degrees congruent to 1 mod k. This answers a question of Pyber.

Pyber [8] proved that every graph G has an edge-partition into four sets, each of which is the edge set of a graph with all degrees odd; if every component of G has even order then three sets will do. This is best possible, as can be seen by considering K_5 with two independent edges removed, which cannot be partitioned into fewer than four subgraphs with all degrees odd; and K_4 with one edge removed, which requires three. Motivated by this result, Pyber [8] asked what happens when we consider residues mod k, rather than mod 2. In particular he asked whether for every integer k there is an integer $c(k)$ such that every graph has an edge-partition into at most $c(k)$ sets, each of which is the edge-set of a graph with all degrees congruent to 1 mod k. The following theorem answers this question.

Theorem 1. *For every integer $k \geq 2$ and every graph G there is a partition of $E(G)$ into at most $5k^2 \log k$ sets, each of which is the edge set of a graph with all degrees congruent to 1 modulo k.*

In order to prove this, we will need to show that if a graph has sufficiently large average degree then it must contain a nonempty subgraph with all degrees divisible by k. The simplest such subgraph would be a k-regular graph, but Pyber, Rödl and Szemerédi [9] have shown that a graph can have as many as $n \log \log n$ edges without containing a k-regular subgraph. We will instead make use of a standard result about subgraphs with all degrees divisible by k (see [1]), which we prove for the sake of completeness. We use the following theorem of van Emde Boas and Kruyswijk ([4]; see also [6], [2], [1]) and Meshulam [5].

Theorem A. *Let G be a finite abelian group and let $m = m(G)$ be the maximal order of elements of G. Then for every sequence $a_1, \ldots, a_s$ with*

$$s \geq m \left(1 + \log \frac{|G|}{m}\right)$$

there is a nonempty set $T \subset \{1, \ldots, s\}$ such that

$$\sum_{i \in T} a_i = 0.$$

□

For a graph G and $S \subset V(G)$, we write $e(S)$ for the number of edges in the induced subgraph $G[S]$. The following lemma is an easy consequence of Theorem A.

Lemma 2. *Let $k \geq 3$ be an integer and let G be a graph that has no nonempty subgraph with all degrees divisible by k. Then, for every $S \subset V(G)$,*

$$e(S) \leq |S| k \log k.$$

Proof. Suppose $S \subset V(G)$ and $t := e(S) > |S| k \log k$. Let

$$A = \sum_{v \in S} \mathbb{Z}_k^{(v)}$$

be the direct sum of $|S|$ copies of $\mathbb{Z}_k$, and let $x_1 y_1, \ldots, x_t y_t$ be the edges in $G[S]$. Now, for $i = 1, \ldots, t$, let $v_i = 1^{(x_i)} + 1^{(y_i)}$, where $1^{(v)}$ denotes the vector in A which is 1 in $\mathbb{Z}_k^{(v)}$ and 0 in the other coordinates. Clearly $m(G) = k$, so

$$\begin{aligned} m\left(1 + \log \frac{|A|}{m}\right) &= k(1 + \log k^{|S|-1}) \\ &< |S| k \log k. \end{aligned}$$

Therefore for some subset $T \subset \{1, \ldots, t\}$ we have $\sum_{t \in T} v_i = 0$, and so $\{e_i : i \in T\}$ is the edge set of a graph with all degrees divisible by k. □

Armed with this result, we can prove Theorem 1.

Proof of Theorem 1. We already know that the theorem holds for $k = 2$; thus we assume $k \geq 3$. We begin by noting that any star, and thus any star-forest (that is, a forest consisting only of stars) has an edge-partition into k or fewer subgraphs with all degrees congruent to 1 modulo k. Now let G be a graph, and let H be a subgraph of maximal size with all degrees congruent to 1 modulo k. It is clearly enough to show that $E(G) \setminus E(H)$ can be covered by fewer than $5k \log k$ star-forests.

Let $W = V(H)$, let $E_1 = E(G[W]) \setminus E(H)$ and let $E_2 = E(W, V(G) \setminus W)$. Thus $E(G) = E(H) \cup E_1 \cup E_2$, since $V(G) \setminus W$ is an independent set by maximality of H. Now every $w \in W$ has at most $k - 1$ neighbours in $V(G) \setminus W$, or else we could have added to H any k edges from w to $V(G) \setminus W$. Therefore E_2 can be partitioned into $k-1$ sets $F_1, \ldots, F_{k-1}$ such that each vertex in W belongs to at most one edge in each F_i. In other words, each F_i is a star-forest.

Now consider the graph $G_0 = (W, E_1)$. By maximality of H, G_0 contains no nontrivial subgraph with all degrees divisible by k. We claim that $E(G_0) = E_1$ can be partitioned into at most $2k \log k$ star-forests. Indeed, it follows from Lemma 2 that every subgraph of G_0 has a vertex of degree at most $2k \log k$, and thus G_0 is $(2k \log k)$-degenerate. (A graph G is *h-degenerate* if there is some ordering $v_1, \ldots, v_n$ of $V(G)$ such that $|\Gamma(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq h$ for $i > 1$.) It follows immediately that G can be covered by $2k \log k$ forests, since an h-degenerate graph can be decomposed into h 1-degenerate graphs and a 1-degenerate graph is just a forest. Now any forest is the edge-disjoint union of two star-forests (by the greedy algorithm), so E_1 can be partitioned into $4k \log k$ star-forests.

We have partitioned $E(G) \setminus E(W)$ into $4k \log k + k - 1 < 5k \log k$ star-forests, so G has an edge-partition into $5k^2 \log k$ subgraphs with all degrees congruent to 1 modulo k. □

Pyber [8] asked further for the best possible value of $c(k)$, if well-defined. We have shown that we may take $c(k) = O(k^2 \log k)$; it would be interesting to know whether $c(k)$ can be taken linear in k. Pyber also asked whether every graph can be *covered* by at most three subgraphs with all degrees odd. This question can also be asked more generally: for $k \geq 2$, what is the smallest integer $d(k)$ such that the edges of any graph can be covered by at most $d(k)$ graphs with all degrees congruent to 1 mod k? Related results and conjectures involving packing and covering with *induced* subgraphs satisfying degree restrictions mod k can be found in [10]and [3].

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