

# BETTER BOUNDS FOR POSET DIMENSION AND BOXICITY

ALEX SCOTT AND DAVID R. WOOD

ABSTRACT. We prove that the dimension of every poset whose comparability graph has maximum degree  $\Delta$  is at most  $\Delta \log^{1+o(1)} \Delta$ . This result improves on a 30-year old bound of Füredi and Kahn, and is within a  $\log^{o(1)} \Delta$  factor of optimal. We prove this result via the notion of boxicity. The *boxicity* of a graph  $G$  is the minimum integer  $d$  such that  $G$  is the intersection graph of  $d$ -dimensional axis-aligned boxes. We prove that every graph with maximum degree  $\Delta$  has boxicity at most  $\Delta \log^{1+o(1)} \Delta$ , which is also within a  $\log^{o(1)} \Delta$  factor of optimal. We also show that the maximum boxicity of graphs with Euler genus  $g$  is  $\Theta(\sqrt{g \log g})$ , which solves an open problem of Esperet and Joret and is tight up to a  $O(1)$  factor.

## 1. INTRODUCTION

**1.1. Poset Dimension and Degree.** The *dimension* of a poset  $P$ , denoted by  $\dim(P)$ , is the minimum number of total orders whose intersection is  $P$ . Let  $\dim(\Delta)$  be the maximum dimension of a poset whose comparability graph has maximum degree at most  $\Delta$ . Several bounds on  $\dim(\Delta)$  have been proved in the literature. In unpublished work referenced in [24, 40], Rödl and Trotter proved the upper bound,  $\dim(\Delta) \leq 2\Delta^2 + 2$ . Füredi and Kahn [24] improved this result to

$$(1) \quad \dim(\Delta) \leq O(\Delta \log^2 \Delta).$$

On the other hand, Erdős, Kierstead, and Trotter [16] proved the lower bound,

$$(2) \quad \dim(\Delta) \geq \Omega(\Delta \log \Delta).$$

Both these proofs use probabilistic methods. The problem of narrowing the gap between (1) and (2) was described as “an important topic for further research” by Erdős et al. [16]; Trotter [40] speculated that the lower bound

---

(A. Scott) MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, OXFORD OX2 6GG, UNITED KINGDOM

(D. R. Wood) SCHOOL OF MATHEMATICAL SCIENCES, MONASH UNIVERSITY, MELBOURNE, AUSTRALIA

*E-mail addresses:* `scott@maths.ox.ac.uk`, `david.wood@monash.edu`.

*Date:* April 10, 2018.

2010 *Mathematics Subject Classification.* Primary 05C62, 06A07.

Alex Scott is supported by a Leverhulme Research Fellowship.

could be improved and wrote “a really new idea will be necessary to improve the upper bound—if this is at all possible”; and Wang [43, page 52] described the problem as “one of the most challenging (and probably quite difficult) problems in dimension theory.”

Our first contribution is the following result, which is the first improvement to the Füredi–Kahn upper bound in 30 years, and shows that (2) is sharp to within a  $(\log \Delta)^{o(1)}$  factor:

$$(3) \quad \dim(\Delta) = \Delta \log^{1+o(1)} \Delta.$$

A more precise result is given below (see Theorem 14).

**1.2. Boxicity and Degree.** We prove (3) via the notion of boxicity. The *boxicity* of a (finite undirected) graph  $G$ , denoted by  $\text{box}(G)$ , is the minimum integer  $d$ , such that  $G$  is the intersection graph of boxes in  $\mathbb{R}^d$ . Here a *box* is a Cartesian product  $I_1 \times I_2 \times \cdots \times I_d$ , where  $I_i \subseteq \mathbb{R}$  is an interval for each  $i \in [d]$ . So a graph  $G$  has boxicity at most  $d$  if and only if there is a set of  $d$ -dimensional boxes  $\{B_v : v \in V(G)\}$  such that  $B_v \cap B_w \neq \emptyset$  if and only if  $vw \in E(G)$ . Note that a graph has boxicity 1 if and only if it is an interval graph. It is easily seen that every graph has finite boxicity.

Let  $\text{box}(\Delta)$  be the maximum boxicity of a graph with maximum degree  $\Delta$ . It is easily seen that  $\text{box}(2) = 2$ , and Adiga and Chandran [2] proved that  $\text{box}(3) = 3$ . Chandran, Francis, and Sivadasan [10] proved the first general upper bound of  $\text{box}(\Delta) \leq 2\Delta^2 + 2$ , which was improved to  $\Delta^2 + 2$  by Esperet [18]. A breakthrough was made by Adiga, Bhowmick, and Chandran [1] via the following connection to poset dimension.

Given a graph  $G$ , let  $P$  be the poset on  $V(G) \times \{0, 1\}$  where  $(u, i) \prec (v, j)$  if and only if  $i = 0$  and  $j = 1$ , and  $u = v$  or  $uv \in E(G)$ . Adiga et al. [1] proved that  $\frac{1}{2} \dim(P) - 2 \leq \text{box}(G) \leq 2 \dim(P)$ . The comparability graph of  $P$  has maximum degree  $\Delta(G) + 1$ . Thus  $\text{box}(\Delta) \leq 2 \dim(\Delta + 1)$ . Conversely, Adiga et al. [1] proved that if  $G$  is the comparability graph of a poset  $P$ , then  $\dim(P) \leq 2 \text{box}(G)$ , implying  $\dim(\Delta) \leq 2 \text{box}(\Delta)$ . This says that  $\dim(\Delta) = \Theta(\text{box}(\Delta))$ . Thus Adiga et al. [1] concluded from (1) and (2) that

$$(4) \quad \Omega(\Delta \log \Delta) \leq \text{box}(\Delta) \leq O(\Delta \log^2 \Delta).$$

We improve the upper bound, giving the following result, which is equivalent to (3):

$$(5) \quad \text{box}(\Delta) = \Delta \log^{1+o(1)} \Delta.$$

Again, a more precise result is given below (see Theorem 13).

**1.3. Boxicity and Genus.** Now consider the boxicity of graphs embeddable in a given surface. Scheinerman [36] proved that every outerplanar graph has boxicity at most 2. Thomassen [39] proved that every planar graph has boxicity at most 3 (generalised to ‘cubicity’ by Felsner and Francis [23]). Esperet and Joret [21] proved that every toroidal graph has boxicity at most

7. The *Euler genus* of an orientable surface with  $h$  handles is  $2h$ . The *Euler genus* of a non-orientable surface with  $c$  cross-caps is  $c$ . The *Euler genus* of a graph  $G$  is the minimum Euler genus of a surface in which  $G$  embeds (with no crossings). Esperet and Joret [21] proved that every graph with Euler genus  $g$  has boxicity at most  $5g + 3$ . Esperet [19] improved this upper bound to  $O(\sqrt{g} \log g)$  and also noted that there are graphs of Euler genus  $g$  with boxicity  $\Omega(\sqrt{g \log g})$ , which follows from the result of Erdős et al. [16] mentioned above (see [20] for more on the boxicity of graphs embedded in surfaces).

The second contribution of this paper is to improve the upper bound to match the lower bound up to a constant factor (see Theorem 15). We conclude that the maximum boxicity of a graph with Euler genus  $g$  is

$$(6) \quad \Theta(\sqrt{g \log g}).$$

Furthermore, the implicit constant in (6) is not large: the upper bound in Theorem 15 is  $(12 + o(1))\sqrt{g \log g}$ .

**1.4. Boxicity and Layered Treewidth.** The third contribution of the paper is to prove a new upper bound on boxicity in terms of layered treewidth, which is a graph parameter recently introduced by Dujmović et al. [14] (see Section 5). This generalises the known bound in terms of treewidth, and leads to generalisations of known results for graphs embedded in surfaces where each edge is in a bounded number of crossings.

**1.5. Related Work.** The present paper can be considered to be part of a body of research connecting poset dimension and graph structure theory. Several recent papers [26–31, 33, 38, 42] show that structural properties of the comparability graph of a poset lead to bounds on its dimension. Finally, we mention the following relationships between boxicity and chromatic number. Graphs with boxicity 1 (interval graphs) are perfect. Asplund and Grünbaum [5] proved that graphs with boxicity 2 are  $\chi$ -bounded. But Burling [8] constructed triangle-free graphs with boxicity 3 and unbounded chromatic number.

## 2. TOOLS

Roberts [35] introduced boxicity and proved the following two fundamental results.

**Lemma 1** ([35]). *For all graphs  $G, G_1, \dots, G_r$  such that  $G = G_1 \cap \dots \cap G_r$ ,*

$$\text{box}(G) \leq \sum_{i=1}^r \text{box}(G_i).$$

Note that Lemma 1 is proved trivially via the product construction.

**Lemma 2** ([35]). *Every  $n$ -vertex graph has boxicity at most  $\lfloor \frac{n}{2} \rfloor$ .*

Note that Trotter [41] characterised those graphs for which equality holds in Lemma 2.

A vertex-colouring of a graph is *acyclic* if adjacent vertices receive distinct colours and every pair of colour classes induces a forest. Esperet and Joret [21] made the following connection to boxicity.

**Lemma 3** ([21]). *For  $k \geq 2$ , every acyclically  $k$ -colourable graph has boxicity at most  $k(k-1)$ .*

A graph is  *$k$ -degenerate* if every subgraph has a vertex of degree at most  $k$ . Note that 1-degenerate graphs (that is, forests) have boxicity at most 2, but 2-degenerate graphs have unbounded boxicity, since the 1-subdivision of  $K_n$  is 2-degenerate and has boxicity  $\Theta(\log \log n)$  [21]. Adiga et al. [3] proved the following bound. Throughout this paper, all logarithms are natural unless otherwise indicated.

**Lemma 4** ([3]). *Every  $k$ -degenerate graph on  $n$  vertices has boxicity at most  $(k+2)\lceil 2e \log n \rceil$ .*

The following lemma, due to Kawarabayashi and Thomassen [32], is the starting point for our work on embedded graphs.

**Lemma 5** ([32]). *Every graph  $G$  with Euler genus  $g$  has a set  $X$  of at most  $1000g$  vertices such that  $G - X$  is acyclically 7-colorable.*

Let  $[n] := \{1, 2, \dots, n\}$ . For our purposes, a *permutation* of a set  $X$  is a bijection from  $X$  to  $[|X|]$ . A set  $\{\pi_1, \dots, \pi_p\}$  of permutations of a set  $X$  is  *$r$ -suitable* if for every  $r$ -subset  $S$  of  $X$  and for every element  $x \in S$ , there is permutation  $\pi_i$  such that  $\pi_i(x) < \pi_i(y)$  for all  $y \in S \setminus \{x\}$ . This definition was introduced by Dushnik [15]; see [9, 12, 37] for further results on suitable sets. Spencer [37] attributes the following result to Hajnal. We include the proof for completeness, and so that the dependence on  $k$  is absolutely clear (since Spencer assumed that  $k$  is fixed).

**Lemma 6** ([37]). *For every  $k \geq 2$  and  $n \geq 10^4$  there is a  $k$ -suitable collection of permutations of size at most  $k2^k \log \log n$ .*

*Proof.* A sequence  $S_1, \dots, S_r$  of subsets of  $[s]$  is  *$t$ -scrambling* if for every set  $I \subseteq [r]$  with  $|I| \leq t$  and every  $A \subseteq I$ , we have

$$(7) \quad \bigcap_{i \in A} S_i \cap \bigcap_{j \in I \setminus A} ([s] \setminus S_j) \neq \emptyset.$$

For  $s \geq t \geq 1$ , let  $M(s, t)$  be the maximum cardinality of a  $t$ -scrambling family of subsets of  $[s]$ . Note that  $M(s, t)$  is monotone increasing in  $s$  (and trivially  $M(s, t) \leq 2^s$ ).

**Claim.** *Let  $s \geq t \geq 1$ . If  $m$  is a positive integer such that*

$$(8) \quad 2^t \binom{m}{t} (1 - 2^{-t})^s < 1,$$

*then  $M(s, t) \geq m$ .*

*Proof of Claim.* Choose subsets  $S_1, \dots, S_m$  of  $[s]$  independently and uniformly at random. For any  $t$ -set  $I \subseteq [m]$  and any  $A \subseteq I$ , the probability that (7) is not satisfied is  $(1 - 2^{-t})^s$ . There are  $\binom{m}{t}$  choices for  $I$  and then  $2^t$  choices for  $A$ , so taking a union bound, the probability that there is some pair  $(I, A)$  such that (7) is not satisfied is at most the left hand side of (8) (note that it is enough to consider sets  $I$  of size exactly  $t$ ). Since this is smaller than 1, we are done.  $\square$

Given  $n$  and  $k$ , choose  $s$  minimal so that  $2^{M(s, k-1)} \geq n$ . Let  $M := M(s, k-1)$ , let  $S_1, \dots, S_M$  be a  $(k-1)$ -scrambling set of subsets of  $[s]$ , and let  $Q_1, \dots, Q_n$  be distinct subsets of  $[M]$ . We define orders  $<_1, \dots, <_s$  on  $[n]$  as follows: for  $a, b \in [n]$ , let  $j(a, b) = \min(Q_a \Delta Q_b)$ ; then  $a <_i b$  if either

- $i \in S_{j(a,b)}$  and  $j(a, b) \in Q_a$ ; or
- $i \notin S_{j(a,b)}$  and  $j(a, b) \in Q_b$ .

(Note that if  $S_i = [M]$  then this gives the lex order on the  $Q_s$ , and if  $S_i = \emptyset$  it is reverse lex.)

Now  $<_1, \dots, <_s$  is a  $k$ -suitable collection of orders on  $[n]$ . This is straightforward, but a little tricky: given a set  $B$  of  $k$  elements of  $[n]$  and  $b \in B$ , let  $I := \{\min(Q_h \Delta Q_b) : h \in B \setminus \{b\}\}$ , let  $A = I \cap Q_b$ , and choose an element  $i$  of the intersection on the left hand side of (7). Consider the order  $<_i$ . It is enough to show that  $b <_i h$  for each  $h \in B \setminus \{b\}$ . Given  $h \in B \setminus \{b\}$ , let  $q = j(b, h) = \min Q_b \Delta Q_h$ . If  $q \in Q_b$  then  $q \in A$  and so  $i \in S_q$ , and therefore  $b <_i h$ ; if  $q \notin Q_b$  then  $q \notin A$  and so  $i \notin S_q$ , and again  $b <_i h$ .

How big is  $s$ ? By the choice of  $s$  and monotonicity,  $M(s-1, k-1) < \log_2 n$ . The left hand side of (8) is less than

$$(9) \quad (2em/t)^t \exp(-s2^{-t}) = \left( \frac{2em \exp(-s/t2^t)}{t} \right)^t,$$

and so

$$M(s, t) \geq \frac{t}{2e} e^{s/t2^t},$$

as setting  $m$  equal to the right hand side of this expression leaves (9) less than 1. Thus, bounding  $M(s-1, k-1)$ , we have

$$\frac{k-1}{2e} \exp\left(\frac{s-1}{(k-1)2^{k-1}}\right) - 1 \leq M(s-1, k-1) \leq \log_2 n$$

and so

$$s \leq 1 + (k-1)2^{k-1} \log\left(\frac{2e}{k-1} \log_2(2n)\right),$$

which is at most  $k2^k \log \log n$  for  $k \geq 2$  and  $n \geq 10^4$ .  $\square$

We use the Lovász Local Lemma:

**Lemma 7** ([17]). *Let  $E_1, \dots, E_n$  be events in a probability space, each with probability at most  $p$  and mutually independent of all but at most  $D$  other events. If  $4pD \leq 1$  then with positive probability, none of  $E_1, \dots, E_n$  occur.*

For a graph  $G$  and set  $X \subseteq V(G)$ , the graph  $G[X]$  with vertex set  $X$  and edge set  $\{vw \in E(G) : v, w \in X\}$  is called the subgraph of  $G$  induced by  $X$ . Let  $G\langle X \rangle$  be the graph obtained from  $G$  by adding an edge between every pair of non-adjacent vertices at least one of which is not in  $X$ .

**Lemma 8.**  $\text{box}(G\langle X \rangle) = \text{box}(G[X])$ .

*Proof.* Given a  $d$ -dimensional box-representation of  $G\langle X \rangle$ , delete the boxes representing the vertices in  $V(G) \setminus X$  to obtain a  $d$ -dimensional box-representation of  $G[X]$ . Thus  $\text{box}(G[X]) \leq \text{box}(G\langle X \rangle)$ . Given a  $d$ -dimensional box-representation of  $G[X]$ , for every vertex  $x$  in  $V(G) \setminus X$ , add a box with interval  $\mathbb{R}$  in every dimension (so that it meets all other boxes). We obtain a  $d$ -dimensional box-representation of  $G\langle X \rangle$ . Thus  $\text{box}(G\langle X \rangle) \leq \text{box}(G[X])$ .  $\square$

For a graph  $G$  and disjoint sets  $X, Y \subseteq V(G)$ , the graph  $G[X, Y]$  with vertex set  $X \cup Y$  and edge set  $\{vw \in E(G) : v \in X, w \in Y\}$  is called the bipartite subgraph of  $G$  induced by  $X, Y$ . For non-adjacent vertices  $v \in X$  and  $w \in Y$ , we say  $vw$  is a *non-edge* of  $G[X, Y]$ . Let  $G\langle X, Y \rangle$  be the graph obtained from  $G$  by adding an edge between distinct vertices  $v$  and  $w$  whenever  $v, w \in V(G) \setminus X$  or  $v, w \in V(G) \setminus Y$ .

### 3. BOUNDED DEGREE

The first ingredient in our proof is the following colouring result that bounds the number of monochromatic neighbours of each vertex.

**Lemma 9.** *For every graph  $G$  with maximum degree  $\Delta > 0$  and for all integers  $d \geq 1$  and  $k \geq \frac{(4d+4)^{1/d}e}{d} \Delta^{1+1/d}$ , there is a partition  $V_1, \dots, V_k$  of  $V(G)$ , such that  $|N_G(v) \cap V_i| \leq d$  for each  $v \in V(G)$  and  $i \in [k]$ .*

*Proof.* Colour each vertex of  $G$  independently and randomly with one of  $k$  colours. Let  $V_1, \dots, V_k$  be the corresponding colour classes. For each set  $S$  of exactly  $d+1$  vertices in  $G$ , such that  $S \subseteq N_G(v)$  for some vertex  $v \in V(G)$ , introduce an event which holds if only if  $S \subseteq V_i$  for some  $i \in [k]$ . Each such event has probability  $p := k^{-d}$ . The colour on one vertex affects at most  $\Delta \binom{\Delta-1}{d}$  events. Thus each event is mutually independent of all but at most  $D$  other events, where

$$D := (d+1)\Delta \binom{\Delta-1}{d} \leq (d+1)\Delta \left(\frac{e\Delta}{d}\right)^d = (d+1) \left(\frac{e}{d}\right)^d \Delta^{d+1}.$$

It follows that  $4pD \leq 1$ . By Lemma 7, with positive probability, no event occurs. Thus the desired partition exists.  $\square$

As an aside, we now show that for fixed  $d$ , the number of colours in Lemma 9 is within a constant factor of optimal. Let  $G$  be the graph with vertex set  $[n]^{d+1}$ , where  $vw \in E(G)$  if and only if  $v$  and  $w$  have at least one coordinate in common. Thus  $\Delta(G) \leq (d+1)n^d$ . Consider a  $k$ -colouring of  $G$  in which each colour is assigned to at most  $d$  neighbours of each vertex.

Suppose that distinct vertices  $v_1, \dots, v_{d+1}$  are assigned the same colour. Let  $w$  be the vertex whose  $i$ -th coordinate equals the  $i$ -th coordinate of  $v_i$ . Thus  $v_1, \dots, v_{d+1} \in N_G(w)$ , which is a contradiction. Thus at most  $d$  vertices are assigned the same colour, and the number of colours is at least  $n^{d+1}/d \geq (\Delta/(d+1))^{(d+1)/d}/d$ .

Lemma 9 leads to the following (see [4, 24] for similar results).

**Corollary 10.** *For every graph  $G$  with maximum degree  $\Delta \geq 2$  and for all integers  $d \geq 100 \log \Delta$  and  $k \geq \frac{3\Delta}{d}$ , there is a partition  $V_1, \dots, V_k$  of  $V(G)$ , such that  $|N_G(v) \cap V_i| \leq d$  for each  $v \in V(G)$  and  $i \in [k]$ .*

*Proof.* Since  $d \geq 100 \log \Delta \geq 69$ , we have  $(4d+4)^{1/d}e \leq 2.95$  and  $d \geq \log^{-1}(\frac{3}{2.95}) \log \Delta$  and  $\Delta^{1/d} \leq \frac{3}{2.95}$ . Thus  $\frac{(4d+4)^{1/d}e}{d} \Delta^{1+1/d} \leq \frac{2.95}{d} \Delta^{1+1/d} \leq \frac{3\Delta}{d} \leq k$ . The result follows from Lemma 9.  $\square$

**Lemma 11.** *Let  $G$  be a bipartite graph with bipartition  $\{A, B\}$ , where vertices in  $A$  have degree at most  $d$  and vertices in  $B$  have degree at most  $\Delta$ . Let  $r, t, \ell$  be positive integers such that*

$$\ell \geq e \left( \frac{ed}{r+1} \right)^{1+1/r} \quad \text{and } t \geq \log(4d\Delta).$$

*Then there exist  $t$  colourings  $c_1, \dots, c_t$  of  $B$ , each with  $\ell$  colours, such that for each vertex  $v \in A$ , for some colouring  $c_i$ , each colour is assigned to at most  $r$  neighbours of  $v$  under  $c_i$ .*

*Proof.* For  $i \in [t]$  and for each vertex  $w \in B$ , let  $c_i(w)$  be a random colour in  $[\ell]$ . Let  $X_v$  be the event that for each  $i \in [t]$ , some set of  $r+1$  neighbours of  $v$  are monochromatic under  $c_i$ . The probability that there is a monochromatic set of at least  $r+1$  neighbours of  $v$  under  $c_i$  is

$$\binom{\deg(v)}{r+1} \ell^{-r} \leq \binom{d}{r+1} \ell^{-r} \leq \left( \frac{ed}{r+1} \right)^{r+1} \ell^{-r} \leq e^{-1}.$$

Thus  $\mathbb{P}(X_v) \leq e^{-t}$ . Observe that  $X_v$  is mutually independent of all but at most  $d\Delta$  other events. By assumption,  $4e^{-t}d\Delta \leq 1$ . By Lemma 7, with positive probability no event  $X_v$  occurs. Therefore, the desired colourings exists.  $\square$

**Lemma 12.** *Let  $G$  be a bipartite graph with bipartition  $\{A, B\}$ , where vertices in  $A$  have degree at most  $d$  and vertices in  $B$  have degree at most  $\Delta$ , for some  $\Delta \geq d \geq 2$ . Let  $G' = G\langle A, B \rangle$  be the graph obtained from  $G$  by adding a clique on  $A$  and a clique on  $B$ . Then, as  $d \rightarrow \infty$ ,*

$$\text{box}(G') \leq (60 + o(1)) d \log(d\Delta) \log \log(\Delta) (2e)^{\sqrt{\log d}}.$$

*Proof.* Let  $r := \lceil \sqrt{\log d} \rceil$  and  $\ell := \lceil e \left( \frac{ed}{r+1} \right)^{1+1/r} \rceil$  and  $t := \lceil \log(4d\Delta) \rceil$ . As  $d \rightarrow \infty$ , we may assume that  $d$  is large.

By Lemma 11, there exist  $t$  colourings  $c_1, \dots, c_t$  of  $B$ , each with  $\ell$  colours, such that for each vertex  $v \in A$ , for some colouring  $c_j$ , each colour is assigned

to at most  $r$  neighbours of  $v$ . Let  $\{A_j : j \in [t]\}$  be a partition of  $A$  such that for each  $v \in A_j$ , at most  $r$  neighbours of  $v$  are assigned the same colour under  $c_j$ . Assume that  $[\ell]$  is the set of colours used by each  $c_j$ .

Our aim is to construct a box representation for each of the graphs  $G\langle A_i, B \rangle$ , and then take their intersection using Lemma 1. In light of Lemma 8, it is enough to concentrate on the subgraphs  $G[A_i \cup B]$ . To handle  $G[A_i \cup B]$ , we further decompose  $B$  according to  $c_1, \dots, c_t$  as follows. For each  $j \in [t]$  and each colour  $\alpha \in [\ell]$ , let  $B_{j,\alpha} := \{w \in B : c_j(w) = \alpha\}$ . Let  $G_{j,\alpha} := G\langle A_j, B_{j,\alpha} \rangle$ . Note that  $G' = \bigcap_{j,\alpha} G_{j,\alpha}$ .

We now bound the boxicity of  $G_{j,\alpha}$ . Let  $H$  be the graph with vertex set  $B_{j,\alpha}$ , where distinct vertices  $x, y \in B_{j,\alpha}$  are adjacent in  $H$  whenever  $x$  and  $y$  have a common neighbour in  $A_j$ . Since each vertex in  $A_j$  has at most  $r$  neighbours in  $B_{j,\alpha}$ , the graph  $H$  has maximum degree at most  $r\Delta$ . Thus  $\chi(H) \leq h := r\Delta + 1$ . Let  $X_1, \dots, X_h$  be the colour classes in a proper colouring of  $H$ . For  $q \in [h]$ , let  $\overrightarrow{X}_q$  denote an arbitrary linear ordering of  $X_q$ . Let  $\overleftarrow{X}_q$  be the reverse of  $\overrightarrow{X}_q$ . Since we may assume that  $d$  is large, Lemma 6 shows that there exists a set of  $(r+1)$ -suitable permutations  $\pi_1, \dots, \pi_p$  of  $[h]$  for some  $p \leq (r+1)2^{r+1} \log \log(h)$ .

For each  $a \in [p]$ , we introduce two 2-dimensional representations of  $G_{j,\alpha}$ . Let  $\sigma_a$  be the ordering  $\overrightarrow{X_{\pi_a(1)}}, \overrightarrow{X_{\pi_a(2)}}, \dots, \overrightarrow{X_{\pi_a(h)}}$  of  $B_{j,\alpha}$ . Similarly, let  $\sigma'_a$  be the ordering  $\overleftarrow{X_{\pi_a(1)}}, \overleftarrow{X_{\pi_a(2)}}, \dots, \overleftarrow{X_{\pi_a(h)}}$  of  $B_{j,\alpha}$ . For each vertex  $x$  in  $B$ , say  $x$  is the  $b_x$ -th vertex in  $\sigma_a$  and  $x$  is the  $b'_x$ -th vertex in  $\sigma'_a$ . Then represent  $x$  by the box with corners  $(-\infty, +\infty)$  and  $(2b_x, 2b_x)$ . For each vertex  $v \in A_j$ , if  $v$  has no neighbours in  $B$ , then represent  $v$  by the point  $(2|B|, -2|B|)$ ; otherwise, if  $x$  is the leftmost neighbour of  $v$  in  $\sigma_a$  and  $y$  is the rightmost neighbour of  $v$  in  $\sigma_a$ , then represent  $v$  by the box with corners  $(\infty, -\infty)$  and  $(2b_x - 1, 2b_y + 1)$ , as illustrated in Figure 1. Now, in two new dimensions introduce the following representation. Represent each  $x$  in  $B_{j,\alpha}$  by the box with corners  $(-\infty, +\infty)$  and  $(2b'_x, 2b'_x)$ . For each vertex  $v \in A_j$ , if  $v$  has no neighbours in  $B$ , then represent  $v$  by the point  $(2|B|, -2|B|)$ ; otherwise, if  $x$  is the leftmost neighbour of  $v$  in  $\sigma'_a$  and  $y$  is the rightmost neighbour of  $v$  in  $\sigma'_a$ , then represent  $v$  by the box with corners  $(\infty, -\infty)$  and  $(2b'_x - 1, 2b'_y + 1)$ . In each of these four dimensions, add every vertex in  $V(G) \setminus (B_{j,\alpha} \cup A_j)$  with interval  $\mathbb{R}$ . Observe that  $A$  and  $B$  are both cliques in this representation.

By construction, for every edge  $vw$  of  $G_{j,\alpha}$  the boxes of  $v$  and  $w$  intersect in every dimension. Now consider a non-edge  $zv$  of  $G_{j,\alpha}$  with  $z \in B_{j,\alpha}$  and  $v \in A_j$ . Let  $C$  be the set of integers  $q \in [h]$  such that some neighbour of  $v$  is in  $X_q$ . Thus  $|C| \leq r$ . Say  $z$  is in  $X_{q'}$ . First suppose that  $q' \notin C$ . Since  $|C \cup \{q'\}| \leq r+1$ , for some permutation  $\pi_a$ , we have  $\pi_a(q') < \pi_a(q)$  for each  $q \in C$ . Let  $x$  be the leftmost neighbour of  $v$  in  $\sigma_a$ . Thus  $b_z < b_x$ , and in the first 2-dimensional representation corresponding to  $\pi_a$ , the right-hand-side of the box representing  $z$  is to the left of the left-hand-side of the box representing  $v$ , as illustrated in Figure 2(a). Thus the boxes representing  $v$  and  $z$  do not intersect. Now assume that  $q' \in C$ . By construction, there is



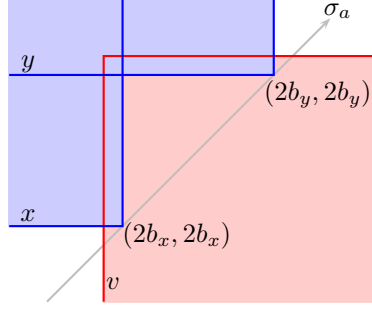


FIGURE 1. Representation of  $G_{j,\alpha}$  with respect to  $\sigma_\alpha$ .

exactly one neighbour  $x$  of  $v$  in  $X_{q'}$ . Since  $|C| \leq r$ , for some permutation  $\pi_\alpha$ , we have  $\pi_\alpha(q') \leq \pi_\alpha(q)$  for each  $q \in C$ . If  $z < x$  in  $\overrightarrow{X_q}$ , then  $b_z < b_x$ , and as argued above and illustrated in Figure 2(b), the boxes representing  $v$  and  $z$  do not intersect. Otherwise,  $z < x$  in  $\overleftarrow{X_q}$ . Then  $b'_z < b'_x$ , and in the second 2-dimensional representation corresponding to  $\pi_\alpha$ , the right-hand-side of the box representing  $z$  is to the left of the left-hand-side of the box representing  $v$ . Hence the boxes representing  $v$  and  $z$  do not intersect. Therefore  $\text{box}(G_{j,\alpha}) \leq 4p$ .

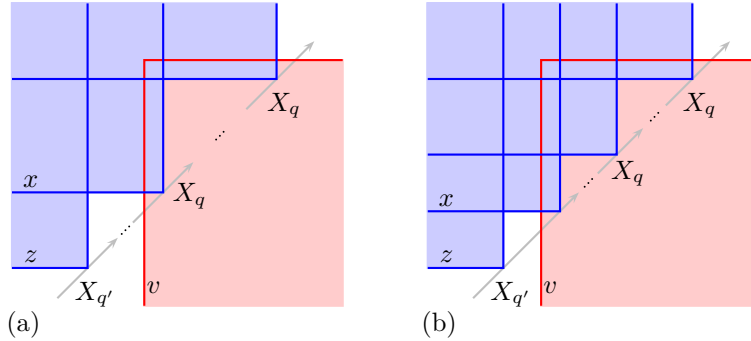


FIGURE 2. Proof for the representation of  $G_{j,\alpha}$ .

By Lemma 1,

$$\begin{aligned} \text{box}(G') &\leq 4t\ell p \\ &\leq 4\lceil \log(4d\Delta) \rceil \left[ e \left( \frac{ed}{r+1} \right)^{1+1/r} \right] (r+1)2^{r+1} \log \log(r\Delta + 1). \end{aligned}$$

Since  $\log(4d\Delta) \leq (1 + o(1)) \log(d\Delta)$  and  $e^{1+1/r} \leq (1 + o(1))e$  and  $\log(r\Delta + 1) \leq (1 + o(1)) \log(\Delta)$ ,

$$\begin{aligned} \text{box}(G') &\leq (8e^2 + o(1)) \log(d\Delta) \left( \frac{d}{r+1} \right)^{1+1/r} (r+1)2^r \log \log(\Delta) \\ &\leq (60 + o(1)) \log(d\Delta) \log \log(\Delta) d^{1+1/r} \frac{2^r}{(r+1)^{1/r}} \\ &\leq (60 + o(1)) d \log(d\Delta) \log \log(\Delta) \left( d^{1/r} 2^r \right) \\ &\leq (60 + o(1)) d \log(d\Delta) \log \log(\Delta) (2e)^{\sqrt{\log d}}. \quad \square \end{aligned}$$

We now prove our first main result.

**Theorem 13.** *For every graph  $G$  with maximum degree  $\Delta$ , as  $\Delta \rightarrow \infty$ ,*

$$\text{box}(G) \leq (180 + o(1)) \Delta \log(\Delta) (2e)^{\sqrt{\log \log \Delta}} \log \log \Delta.$$

*Proof.* Let  $d := \lceil 100 \log \Delta \rceil$  and  $k := \lceil \frac{3\Delta}{d} \rceil$ . By Corollary 10, there is a partition  $V_1, \dots, V_k$  of  $V(G)$ , such that  $|N_G(v) \cap V_i| \leq d$  for each  $v \in V(G)$  and  $i \in [k]$ . Note that

$$\text{box}(G) = \bigcap_i G[V_i] \cap G[V_i, V(G) \setminus V_i].$$

Since  $G[V_i]$  has maximum degree at most  $d$ , by the result of Esperet [18], the graph  $G[V_i]$  has boxicity at most  $d^2 + 2$ . By Lemma 8,

$$\text{box}(G[V_i]) \leq d^2 + 2.$$

Let  $G_i := G[V_i, V(G) \setminus V_i]$ . Every vertex in  $V(G) \setminus V_i$  has degree at most  $d$  in  $G_i$ . Let  $G'_i$  be obtained from  $G_i$  by adding a clique on  $V_i$  and a clique on  $V(G) \setminus V_i$ . By Lemmas 8 and 12 and since  $\log(d\Delta) \leq (1 + o(1)) \log \Delta$ ,

$$\text{box}(G[V_i, V(G) \setminus V_i]) \leq \text{box}(G'_i) \leq (60 + o(1)) d \log(\Delta) \log \log(\Delta) (2e)^{\sqrt{\log d}}.$$

Applying Lemma 1 again,

$$\begin{aligned} \text{box}(G) &\leq k(d^2 + 2) + (60 + o(1)) k d \log(\Delta) \log \log(\Delta) (2e)^{\sqrt{\log d}} \\ &\leq (9 + o(1)) (\Delta \log \Delta) + (180 + o(1)) \Delta \log(\Delta) \log \log(\Delta) (2e)^{\sqrt{\log \log \Delta}} \\ &\leq (180 + o(1)) \Delta \log(\Delta) \log \log(\Delta) (2e)^{\sqrt{\log \log \Delta}}. \quad \square \end{aligned}$$

Since  $(2e)^{\sqrt{\log \log \Delta}} \log \log \Delta \leq \log^{o(1)} \Delta$ , Theorem 13 implies (5). More precisely,

$$\text{box}(\Delta) = \Delta(\log \Delta) e^{O(\sqrt{\log \log \Delta})}.$$

Theorem 13 and the result of Adiga et al. [1] mentioned in Section 1 imply the following quantitative version of (3).

**Theorem 14.** *For every poset  $P$  whose comparability graph has maximum degree  $\Delta$ , as  $\Delta \rightarrow \infty$ ,*

$$\dim(P) \leq (360 + o(1)) \Delta \log(\Delta) (2e)^{\sqrt{\log \log \Delta}} \log \log \Delta.$$

Again, with (2), this gives

$$\dim(\Delta) = \Delta(\log \Delta)e^{O(\sqrt{\log \log \Delta})}.$$

#### 4. EULER GENUS

We now prove our second main result.

**Theorem 15.** *For every graph  $G$  with Euler genus  $g$ , as  $g \rightarrow \infty$ .*

$$\text{box}(G) \leq (12 + o(1))\sqrt{g \log g}.$$

*Proof.* By Lemma 5,  $G$  contains a set  $X$  of at most  $1000g$  vertices such that  $G - X$  is acyclically 7-colorable. By Lemma 3,  $\text{box}(G - X) \leq 42$ . First suppose that  $|X| < 10^4$ . Deleting one vertex reduces boxicity by at most 1. Thus  $\text{box}(G) \leq \text{box}(G - X) + |X| \leq 10042$ , and we are done since  $g \rightarrow \infty$ . Now assume that  $|X| \geq 10^4$ .

Let  $G_1 := G \setminus X$ . Let  $Y$  be the set of vertices in  $G - X$  with exactly one or exactly two neighbours in  $X$ . Let  $G_2 := \langle X, Y \rangle$ . By Lemma 6, there is a 3-suitable set of permutations  $\pi_1, \dots, \pi_p$  of  $X$  for some  $p \leq 24 \log \log |X|$ . For each  $\pi_i$  we introduce two dimensions, as illustrated in Figure 3. Represent each vertex  $w \in X$  by the box with corners  $(-\infty, +\infty)$  and  $(2\pi_i(w), 2\pi_i(w))$ . For each vertex  $v \in Y$ , if  $x$  and  $y$  are respectively the leftmost and rightmost neighbours of  $v$  in  $\pi_i$ , then represent  $v$  by the box with corners  $(2\pi_i(x) - 1, 2\pi_i(y) + 1)$  and  $(+\infty, -\infty)$ . Observe that  $X$  and  $Y$  are both cliques in this representation. If  $vw \in E(G)$  and  $v \in Y$  and  $w \in X$ , then the box representing  $v$  intersects the box representing  $w$ . Consider a non-edge  $vz$  with  $v \in Y$  and  $z \in X$ . Since  $\pi_1, \dots, \pi_p$  is 3-suitable and  $\deg_X(v) \leq 2$ , for some  $i$ , we have  $\pi_i(z) < \pi_i(x)$  for each  $x \in N_G(v)$ . Thus, for the 2-dimensional representation defined with respect to  $\pi_i$ , the boxes representing  $v$  and  $z$  do not intersect.

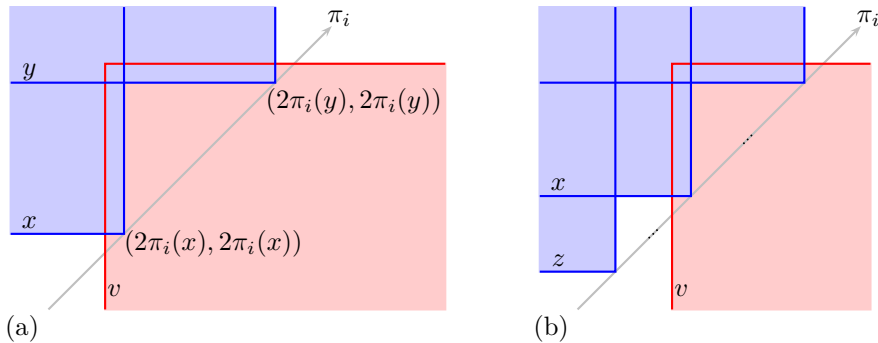


FIGURE 3. Representation of  $G_2$  with respect to  $\pi_i$ .

Add each vertex in  $G - (X \cup Y)$  to every dimension with interval  $\mathbb{R}$ . We obtain a box representation of  $G_2$ . Thus  $\text{box}(G_2) \leq 48 \log \log |X| \leq 48 \log \log(1000g)$ .

Let  $Z$  be the set of vertices in  $G - X$  with at least three neighbours in  $X$ . Let  $G_3 := G\langle X \cup Z \rangle$ . Observe that  $G = G_1 \cap G_2 \cap G_3$ .

To bound  $\text{box}(G_3)$ , we first bound  $\text{box}(H)$ , where  $H := G[X \cup Z]$ . The number of edges in  $G[X, Z]$  is least  $3|Z|$  and at most  $2(|X| + |Z| + g - 2)$  by Euler's formula. Thus  $|Z| < 2(|X| + g)$ , implying  $|X \cup Z| < 3002g$ . Let  $n := |V(H)| < 3002g$ . Let  $v_1, \dots, v_n$  be an ordering of  $V(H)$ , where  $v_i$  has minimum degree in  $H[\{v_i, \dots, v_n\}]$ . Define  $k := 7 + \lceil \sqrt{g/\log g} \rceil$ . Let  $i$  be minimum such that  $v_i$  has degree at least  $k$  in  $H[\{v_i, \dots, v_n\}]$ . If  $i$  is defined, then let  $A := \{v_1, \dots, v_{i-1}\}$  and  $B := \{v_i, \dots, v_n\}$ , otherwise let  $A := V(H)$  and  $B := \emptyset$ .

Observe that  $H = H\langle A \rangle \cap H\langle B \rangle \cap H\langle A, B \rangle$ .

By construction,  $H[A]$  is  $k$ -degenerate and has at most  $n$  vertices. By Lemma 4,  $\text{box}(H[A]) \leq (k+2)\lceil 2e \log n \rceil$ . By Lemma 8,  $H\langle A \rangle \leq (k+2)\lceil 2e \log n \rceil$ .

By construction,  $H[B]$  has minimum degree at least  $k$ . The number of edges in  $H[B]$  is at least  $\frac{1}{2}k|B|$  and at most  $3(|B| + g - 2)$ , implying  $(\frac{k}{2} - 3)|B| < 3g$ . By Lemma 2,  $\text{box}(H[B]) \leq \frac{|B|}{2} < \frac{3g}{k-6}$ . By Lemma 8,  $H\langle B \rangle < \frac{3g}{k-6}$ .

Now consider  $H[A, B]$ . By construction, every vertex in  $A$  has degree at most  $k$  in  $H[A, B]$ . A permutation  $\sigma$  of  $B$  catches a non-edge  $vw$  of  $H[A, B]$  with  $v \in A$  and  $w \in B$  if there are edges  $vx, vy$  in  $H[A, B]$ , such that  $w$  is between  $x$  and  $y$  in  $\sigma$ . Let  $t := \lceil \frac{3}{2}(k+1) \log n \rceil$ . Let  $\sigma_1, \dots, \sigma_t$  be random permutations of  $B$ . For each non-edge  $vw$  of  $H[A, B]$ , the probability that  $\sigma_i$  catches  $vw$  equals  $1 - \frac{2}{\deg(v)+1} \leq e^{-2/(k+1)}$ . Thus, the probability that every  $\sigma_i$  catches  $vw$  is at most  $e^{-2t/(k+1)} \leq n^{-3}$ . Since the number of non-edges is at most  $n^2$ , by the union bound, the probability that for some non-edge  $vw$ , every  $\sigma_i$  catches  $vw$  is at most  $n^{-1} < 1$ . Hence, with positive probability, for every non-edge  $vw$ , some  $\sigma_i$  does not catch  $vw$ . Therefore, there exists permutations  $\sigma_1, \dots, \sigma_t$  of  $B$ , such that for every non-edge  $vw$ , some  $\sigma_i$  does not catch  $vw$ .

For each permutation  $\sigma_i$  we introduce two dimensions. Represent each vertex  $w \in B$  by the box with corners  $(-\infty, +\infty)$  and  $(2\sigma_i(w), 2\sigma_i(w))$ . For each vertex  $v \in A$ , if  $v$  has no neighbours in  $B$  then represent  $v$  by the point  $(2|B|, -2|B|)$ ; otherwise, if  $x$  and  $y$  are respectively the leftmost and rightmost neighbours of  $v$  in  $\sigma_i$ , then represent  $v$  by the box with corners  $(2\sigma_i(x) - 1, 2\sigma_i(y) + 1)$  and  $(+\infty, -\infty)$ . Observe that  $A$  and  $B$  are both cliques in this representation. If  $vw \in E(G)$  and  $v \in A$  and  $w \in B$ , then the box representing  $v$  intersects the box representing  $w$ . For a non-edge  $vw$  with  $v \in A$  and  $w \in B$ , the box representing  $v$  intersects the box representing  $w$  if and only if  $\sigma_i$  catches  $vw$ . Since for every non-edge  $vw$ , some  $\sigma_i$  does not catch  $vw$ , the boxes representing  $v$  and  $w$  do not intersect. Thus  $\text{box}(H\langle A, B \rangle) \leq 2t \leq 2 + 3(k+1) \log n$ .

By Lemma 1,

$$\begin{aligned} \text{box}(H) &\leq \text{box}(H\langle A \rangle) + \text{box}(H\langle B \rangle) + \text{box}(H\langle A, B \rangle) \\ &\leq (k+2)\lceil 2e \log n \rceil + \frac{3g}{k-6} + 2 + 3(k+1) \log n \\ &\leq (9k+15) \log(3002g) + 3\sqrt{g \log g} \\ &\leq 12\sqrt{g \log g} + O(\sqrt{g/\log g}). \end{aligned}$$

Applying Lemma 1 again,

$$\begin{aligned} \text{box}(G) &\leq \text{box}(G_1) + \text{box}(G_2) + \text{box}(G_3) \\ &\leq 42 + 48 \log \log(1000g) + 12\sqrt{g \log g} + O(\sqrt{g/\log g}) \\ &\leq 12\sqrt{g \log g} + O(\sqrt{g/\log g}). \quad \square \end{aligned}$$

As noted in Section 1, when combined with the lower bound proved by Esperet [19], this shows that the maximum possible boxicity of a graph with Euler genus  $g$  is  $\Theta(\sqrt{g \log g})$ .

## 5. LAYERED TREewidth

A *tree decomposition* of a graph  $G$  is a set  $(B_x : x \in V(T))$  of non-empty sets  $B_x \subseteq V(G)$  (called *bags*) indexed by the nodes of a tree  $T$ , such that for each vertex  $v \in V(G)$ , the set  $\{x \in V(T) : v \in B_x\}$  induces a non-empty (connected) subtree of  $T$ , and for each edge  $vw \in E(G)$  there is a node  $x \in V(T)$  such that  $v, w \in B_x$ . The *width* of a tree decomposition  $(B_x : x \in V(T))$  is  $\max\{|B_x| - 1 : x \in V(T)\}$ . The *treewidth* of a graph  $G$  is the minimum width of a tree decomposition of  $G$ . Treewidth is a key parameter in algorithmic and structural graph theory (see [7, 25, 34] for surveys). Chandran and Sivadasan [11] proved:

**Theorem 16** ([11]). *For every graph  $G$ ,*

$$\text{box}(G) \leq \text{tw}(G) + 2.$$

A *layering* of a graph  $G$  is a partition  $(V_1, V_2, \dots, V_n)$  of  $V(G)$  such that for every edge  $vw \in E(G)$ , for some  $i \in [n-1]$ , both  $v$  and  $w$  are in  $V_i \cup V_{i+1}$ . For example, if  $r$  is a vertex of a connected graph  $G$  and  $V_i := \{v \in V(G) : \text{dist}(r, v) = i\}$  for  $i \geq 0$ , then  $(V_0, V_1, \dots)$  is a layering of  $G$ . The *layered tree-width* of a graph  $G$  is the minimum integer  $k$  such that there is a tree decomposition  $(B_x : x \in V(T))$  and a layering  $(V_1, V_2, \dots, V_n)$  of  $G$ , such that  $|B_x \cap V_i| \leq k$  for each node  $x \in V(T)$  and for each layer  $V_i$ . Of course,  $\text{ltw}(G) \leq \text{tw}(G) + 1$  and often  $\text{ltw}(G)$  is much less than  $\text{tw}(G)$ . For example, Dujmović et al. [14] proved that every planar graph has layered treewidth at most 3, whereas the  $n \times n$  planar grid has treewidth  $n$ . Thus the following result provides a qualitative generalisation of Theorem 16.

**Theorem 17.** *For every graph  $G$ ,*

$$\text{box}(G) \leq 6 \text{ltw}(G) + 4.$$

*Proof.* Consider a tree decomposition  $(B_x : x \in V(T))$  and a layering  $(V_1, V_2, \dots, V_n)$  of  $G$ , such that  $|B_x \cap V_i| \leq \text{ltw}(G)$  for each node  $x \in V(T)$  and for each layer  $V_i$ . Note that  $(B_x \cap (V_i \cup V_{i+1}) : x \in V(T))$  is a tree-decomposition of  $G[V_i \cup V_{i+1}]$  with bags of size at most  $2\text{ltw}(G)$ . Thus  $\text{tw}(G[V_i \cup V_{i+1}]) \leq 2\text{ltw}(G) - 1$ . For  $i \in \{0, 1, 2\}$ , let

$$G_i := \bigcup_{j \equiv i \pmod{3}} G[V_j \cup V_{j+1}].$$

Each component of  $G_i$  is contained in  $V_j \cup V_{j+1}$  for some  $j \equiv i \pmod{3}$ . The treewidth of a graph equals the maximum treewidth of its connected components. Thus  $\text{tw}(G_i) \leq 2\text{ltw}(G) - 1$ , and  $\text{box}(G_i) \leq 2\text{ltw}(G) + 1$  by Theorem 16. Use three disjoint sets of  $2\text{ltw}(G) + 1$  dimensions for each  $G_i$ , and add each vertex not in  $G_i$  to the dimensions used by  $G_i$  with interval  $\mathbb{R}$ . Finally, add one more dimension, where the interval for each vertex  $v \in V_i$  is  $[i, i + 1]$ . For adjacent vertices in  $G$ , the corresponding boxes intersect in every dimension. Consider non-adjacent vertices  $v$  and  $w$  in  $G$ . Say  $v \in V_a$  and  $w \in V_b$ . If  $|a - b| \geq 2$  then in the final dimension, the intervals for  $v$  and  $w$  are disjoint, as desired. If  $|a - b| = 1$ , then  $vw$  is a non-edge in some  $G_i$ , and thus the intervals for  $v$  and  $w$  are disjoint in some dimension corresponding to  $G_i$ . Hence we have a  $3(2\text{ltw}(G) + 1) + 1$ -dimensional box representation of  $G$ .  $\square$

The following two examples illustrate the generality of Theorem 17. A graph is  $(g, k)$ -planar if it has a drawing in a surface of Euler genus at most  $g$  with at most  $k$  crossings per edge. Dujmović et al. [13] proved that every  $(g, k)$ -planar graph has layered treewidth at most  $(4g + 6)(k + 1)$ . Theorem 17 then implies that every  $(g, k)$ -planar graph has boxicity at most  $6(4g + 6)(k + 1) + 4$ . Map graphs provide a second example. Start with a graph  $G_0$  embedded in a surface of Euler genus  $g$ , with each face labelled a nation or a lake, where each vertex of  $G_0$  is incident with at most  $d$  nations. Let  $G$  be the graph whose vertices are the nations of  $G_0$ , where two vertices are adjacent in  $G$  if the corresponding faces in  $G_0$  share a vertex. Then  $G$  is called a  $(g, d)$ -map graph. Dujmović et al. [13] proved that every  $(g, d)$ -map graph has layered treewidth at most  $(2g + 3)(2d + 1)$ . Theorem 17 then implies that every  $(g, d)$ -map graph has boxicity at most  $6(2g + 3)(2d + 1) + 4$ . By definition, a graph is  $(g, 0)$ -planar if and only if it has Euler genus at most  $g$ . Similarly, it is easily seen that a graph is a  $(g, 3)$ -map graph if and only if it has Euler genus at most  $g$  (see [13]). Thus these results provide qualitative generalisations of the fact that graphs with Euler genus  $g$  have boxicity  $O(g)$ , as proved by Esperet and Joret [21]. As discussed above, Esperet [19] improved this upper bound to  $O(\sqrt{g} \log g)$  and Theorem 15 improves it further to  $O(\sqrt{g \log g})$ . On the other hand, Theorem 17 is within a constant factor of optimal, since Chandran and Sivadasan [11] constructed a family of graphs  $G$  with  $\text{box}(G) \geq (1 - o(1))\text{tw}(G)$ . See [6] for more examples

of graph classes with bounded layered treewidth, for which Theorem 17 is applicable.

## 6. OPEN PROBLEMS

We conclude with several open problems.

- What is the maximum boxicity of graphs with maximum degree 4?
- What is the maximum boxicity of  $k$ -degenerate graphs with maximum degree  $\Delta$ ?
- What is the maximum boxicity of graphs with treewidth  $k$ ? Chandran and Sivadasan [11] proved lower and upper bounds of  $k - 2\sqrt{k}$  and  $k + 2$  respectively.
- What is the maximum boxicity of graphs with no  $K_t$  minor? The best known upper bound is  $O(t^2 \log t)$  due to Esperet and Wiechert [22]. A lower bound of  $\Omega(t\sqrt{\log t})$  follows from results of Esperet [19].

## REFERENCES

- [1] ABHIJIN ADIGA, DIPTENDU BHOWMICK, AND L. SUNIL CHANDRAN. Boxicity and poset dimension. *SIAM J. Discrete Math.*, 25(4):1687–1698, 2011. doi: [10.1137/100786290](https://doi.org/10.1137/100786290). MR: [2873210](https://www.ams.org/mathscinet-getitem?mr=2873210).
- [2] ABHIJIN ADIGA AND L. SUNIL CHANDRAN. Representing a cubic graph as the intersection graph of axis-parallel boxes in three dimensions. *SIAM J. Discrete Math.*, 28(3):1515–1539, 2014. doi: [10.1137/120861795](https://doi.org/10.1137/120861795).
- [3] ABHIJIN ADIGA, L. SUNIL CHANDRAN, AND ROGERS MATHEW. Cubicity, degeneracy, and crossing number. *European J. Combin.*, 35:2–12, 2014. doi: [10.1016/j.ejc.2013.06.021](https://doi.org/10.1016/j.ejc.2013.06.021). MR: [3090481](https://www.ams.org/mathscinet-getitem?mr=3090481).
- [4] NOGA ALON, MANU BASAVARAJU, L. SUNIL CHANDRAN, ROGERS MATHEW, AND DEEPAK RAJENDRAPRASAD. Separation dimension of bounded degree graphs. *SIAM J. Discrete Math.*, 29(1):59–64, 2015. doi: [10.1137/140973013](https://doi.org/10.1137/140973013). MR: [3295685](https://www.ams.org/mathscinet-getitem?mr=3295685).
- [5] EDGAR ASPLUND AND BRANKO GRÜNBAUM. On a colouring problem. *Math. Scand.*, 8:181–188, 1960.
- [6] MICHAEL J. BANNISTER, WILLIAM E. DEVANNY, VIDA DUJMOVIĆ, DAVID EPPSTEIN, AND DAVID R. WOOD. Track layouts, layered path decompositions, and leveled planarity, 2015. arXiv: [1506.09145](https://arxiv.org/abs/1506.09145).
- [7] HANS L. BODLAENDER. A partial  $k$ -arboretum of graphs with bounded treewidth. *Theoret. Comput. Sci.*, 209(1-2):1–45, 1998. doi: [10.1016/S0304-3975\(97\)00228-4](https://doi.org/10.1016/S0304-3975(97)00228-4).
- [8] JAMES PERKINS BURLING. *On coloring problems of families of prototypes*. Ph.D. thesis, University of Colorado, 1965.
- [9] JUSTIN H. C. CHAN AND JONATHAN JEDWAB. Constructions and nonexistence results for suitable sets of permutations. *J. Combin.*

- Theory Ser. A*, 148:183–196, 2017. doi:[10.1016/j.jcta.2016.12.006](https://doi.org/10.1016/j.jcta.2016.12.006). MR: [3603319](https://doi.org/10.1016/j.jcta.2016.12.006).
- [10] L. SUNIL CHANDRAN, MATHEW C. FRANCIS, AND NAVEEN SIVADASAN. Boxicity and maximum degree. *J. Combin. Theory Ser. B*, 98(2):443–445, 2008. doi:[10.1016/j.jctb.2007.08.002](https://doi.org/10.1016/j.jctb.2007.08.002). MR: [2389609](https://doi.org/10.1016/j.jctb.2007.08.002).
- [11] L. SUNIL CHANDRAN AND NAVEEN SIVADASAN. Boxicity and treewidth. *J. Combin. Theory Ser. B*, 97(5):733–744, 2007. doi:[10.1016/j.jctb.2006.12.004](https://doi.org/10.1016/j.jctb.2006.12.004). MR: [2344136](https://doi.org/10.1016/j.jctb.2006.12.004).
- [12] CHARLES J. COLBOURN. Suitable permutations, binary covering arrays, and Paley matrices. In *Algebraic design theory and Hadamard matrices*, vol. 133 of *Springer Proc. Math. Stat.*, pp. 29–42. 2015. doi:[10.1007/978-3-319-17729-8\\_3](https://doi.org/10.1007/978-3-319-17729-8_3). MR: [3440525](https://doi.org/10.1007/978-3-319-17729-8_3).
- [13] VIDA DUJMOVIĆ, DAVID EPPSTEIN, AND DAVID R. WOOD. Structure of graphs with locally restricted crossings. *SIAM J. Disc. Math.*, 31(2):805–824, 2017. doi:[10.1137/16M1062879](https://doi.org/10.1137/16M1062879). MR: [3639571](https://doi.org/10.1137/16M1062879).
- [14] VIDA DUJMOVIĆ, PAT MORIN, AND DAVID R. WOOD. Layered separators in minor-closed graph classes with applications. *J. Combin. Theory Ser. B*, 127:111–147, 2017. doi:[10.1016/j.jctb.2017.05.006](https://doi.org/10.1016/j.jctb.2017.05.006). MR: [3704658](https://doi.org/10.1016/j.jctb.2017.05.006).
- [15] BEN DUSHNIK. Concerning a certain set of arrangements. *Proc. Amer. Math. Soc.*, 1:788–796, 1950. doi:[10.2307/2031986](https://doi.org/10.2307/2031986). MR: [0038922](https://doi.org/10.2307/2031986).
- [16] PAUL ERDŐS, HAL A. KIERSTEAD, AND WILLIAM T. TROTTER, JR. The dimension of random ordered sets. *Random Structures Algorithms*, 2(3):253–275, 1991. doi:[10.1002/rsa.3240020302](https://doi.org/10.1002/rsa.3240020302). MR: [1109694](https://doi.org/10.1002/rsa.3240020302).
- [17] PAUL ERDŐS AND LÁSZLÓ LOVÁSZ. Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and Finite Sets*, vol. 10 of *Colloq. Math. Soc. János Bolyai*, pp. 609–627. North-Holland, 1975. [https://www.renyi.hu/~p\\_erdos/1975-34.pdf](https://www.renyi.hu/~p_erdos/1975-34.pdf). MR: [0382050](https://doi.org/10.1016/j.jctb.2017.05.006).
- [18] LOUIS ESPERET. Boxicity of graphs with bounded degree. *European J. Combin.*, 30(5):1277–1280, 2009. doi:[10.1016/j.ejc.2008.10.003](https://doi.org/10.1016/j.ejc.2008.10.003). MR: [2514650](https://doi.org/10.1016/j.ejc.2008.10.003).
- [19] LOUIS ESPERET. Boxicity and topological invariants. *European J. Combin.*, 51:495–499, 2016. doi:[10.1016/j.ejc.2015.07.020](https://doi.org/10.1016/j.ejc.2015.07.020). MR: [3398874](https://doi.org/10.1016/j.ejc.2015.07.020).
- [20] LOUIS ESPERET. Box representations of embedded graphs. *Discrete Comput. Geom.*, 57(3):590–606, 2017. doi:[10.1007/s00454-016-9837-8](https://doi.org/10.1007/s00454-016-9837-8). MR: [3614774](https://doi.org/10.1007/s00454-016-9837-8).
- [21] LOUIS ESPERET AND GWENAËL JORET. Boxicity of graphs on surfaces. *Graphs Combin.*, 29(3):417–427, 2013. doi:[10.1007/s00373-012-1130-x](https://doi.org/10.1007/s00373-012-1130-x). MR: [3053591](https://doi.org/10.1007/s00373-012-1130-x).
- [22] LOUIS ESPERET AND VEIT WIECHERT. Boxicity, poset dimension, and excluded minors. 2018. arXiv: [1804.00850](https://arxiv.org/abs/1804.00850).
- [23] STEFAN FELSNER AND MATHEW C. FRANCIS. Contact representations of planar graphs with cubes. In *Proc. 17th Symposium on Computational Geometry (SoCG'11)*, pp. 315–320. ACM, 2011.



- doi: [10.1145/1998196.1998250](https://doi.org/10.1145/1998196.1998250). MR: [2919622](https://doi.org/10.1007/BF00403406).
- [24] ZOLTÁN FÜREDI AND JEFF KAHN. On the dimensions of ordered sets of bounded degree. *Order*, 3(1):15–20, 1986. doi: [10.1007/BF00403406](https://doi.org/10.1007/BF00403406). MR: [850394](https://doi.org/10.1002/jgt.22030).
- [25] DANIEL J. HARVEY AND DAVID R. WOOD. Parameters tied to treewidth. *J. Graph Theory*, 84(4):364–385, 2017. doi: [10.1002/jgt.22030](https://doi.org/10.1002/jgt.22030). MR: [3623383](https://doi.org/10.1007/s00493-014-3081-8).
- [26] GWENAËL JORET, PIOTR MICEK, PATRICE OSSONA DE MENDEZ, AND VEIT WIECHERT. Nowhere dense graph classes and dimension. 2017. arXiv: [1708.05424](https://arxiv.org/abs/1708.05424).
- [27] GWENAËL JORET, PIOTR MICEK, KEVIN G. MILANS, WILLIAM T. TROTTER, BARTOSZ WALCZAK, AND RUIDONG WANG. Tree-width and dimension. *Combinatorica*, 36(4):431–450, 2016. doi: [10.1007/s00493-014-3081-8](https://doi.org/10.1007/s00493-014-3081-8). MR: [3537035](https://doi.org/10.1137/1.9781611974331.ch125).
- [28] GWENAËL JORET, PIOTR MICEK, WILLIAM T. TROTTER, RUIDONG WANG, AND VEIT WIECHERT. On the dimension of posets with cover graphs of treewidth 2. *Order*, 34(2):185–234, 2017. doi: [10.1007/s11083-016-9395-y](https://doi.org/10.1007/s11083-016-9395-y). MR: [3669250](https://doi.org/10.1137/17M111300X).
- [29] GWENAËL JORET, PIOTR MICEK, AND VEIT WIECHERT. Sparsity and dimension. In *Proc. 27th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '16)*, pp. 1804–1813. 2016. doi: [10.1137/1.9781611974331.ch125](https://doi.org/10.1137/1.9781611974331.ch125).
- [30] GWENAËL JORET, PIOTR MICEK, AND VEIT WIECHERT. Planar posets have dimension at most linear in their height. *SIAM J. Discrete Math.*, 31(4):2754–2790, 2017. doi: [10.1137/17M111300X](https://doi.org/10.1137/17M111300X). MR: [3738844](https://doi.org/10.1016/j.jctb.2012.03.001).
- [31] GWENAËL JORET, PIOTR MICEK, AND VEIT WIECHERT. Sparsity and dimension. *Combinatorica*, 2017. doi: [10.1007/s00493-017-3638-4](https://doi.org/10.1007/s00493-017-3638-4).
- [32] KEN-ICHI KAWARABAYASHI AND CARSTEN THOMASSEN. From the plane to higher surfaces. *J. Comb. Theory, Ser. B*, 102(4):852–868, 2012. doi: [10.1016/j.jctb.2012.03.001](https://doi.org/10.1016/j.jctb.2012.03.001). MR: [2927410](https://doi.org/10.1017/CBO9780511662119.006).
- [33] PIOTR MICEK AND VEIT WIECHERT. Topological minors of cover graphs and dimension. *J. Graph Theory*, 86(3):295–314, 2017. doi: [10.1002/jgt.22127](https://doi.org/10.1002/jgt.22127). MR: [3697925](https://doi.org/10.1017/CBO9780511662119.006).
- [34] BRUCE A. REED. Tree width and tangles: a new connectivity measure and some applications. In *Surveys in combinatorics*, vol. 241 of *London Math. Soc. Lecture Note Ser.*, pp. 87–162. Cambridge Univ. Press, 1997. doi: [10.1017/CBO9780511662119.006](https://doi.org/10.1017/CBO9780511662119.006).
- [35] FRED S. ROBERTS. On the boxicity and cubicity of a graph. In *Recent Progress in Combinatorics (Proc. Third Waterloo Conf. on Combinatorics, 1968)*, pp. 301–310. Academic Press, 1969. MR: [0252268](https://doi.org/10.1017/CBO9780511662119.006).
- [36] EDWARD R. SCHEINERMAN. *Intersection classes and multiple intersection parameters*. Ph.D. thesis, Princeton University, 1984.
- [37] JOEL SPENCER. Minimal scrambling sets of simple orders. *Acta Math. Acad. Sci. Hungar.*, 22:349–353, 1971/72. doi: [10.1007/BF01896428](https://doi.org/10.1007/BF01896428).

- [38] NOAH STREIB AND WILLIAM T. TROTTER. Dimension and height for posets with planar cover graphs. *European J. Combin.*, 35:474–489, 2014. doi: [10.1016/j.ejc.2013.06.017](https://doi.org/10.1016/j.ejc.2013.06.017). MR: [3090518](#).
- [39] CARSTEN THOMASSEN. Interval representations of planar graphs. *J. Comb. Theory, Ser. B*, 40(1):9–20, 1986. doi: [10.1016/0095-8956\(86\)90061-4](https://doi.org/10.1016/0095-8956(86)90061-4).
- [40] WILLIAM T. TROTTER. Graphs and partially ordered sets: recent results and new directions. *Congr. Numer.*, 116:253–278, 1996. MR: [1411256](#).
- [41] WILLIAM T. TROTTER, JR. A characterization of Roberts’ inequality for boxicity. *Discrete Math.*, 28(3):303–313, 1979. doi: [10.1016/0012-365X\(79\)90137-7](https://doi.org/10.1016/0012-365X(79)90137-7). MR: [548629](#).
- [42] BARTOSZ WALCZAK. Minors and dimension. *J. Combin. Theory Ser. B*, 122:668–689, 2017. doi: [10.1016/j.jctb.2016.09.001](https://doi.org/10.1016/j.jctb.2016.09.001). MR: [3575223](#).
- [43] RUIDONG WANG. *Combinatorial problems for graphs and partially ordered sets*. Ph.D. thesis, Georgia Tech, 2015. <http://hdl.handle.net/1853/54483>.