

# $H$ -colouring $P_t$ -free graphs in subexponential time

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## Abstract

A graph is called  $P_t$ -free if it does not contain the path on  $t$  vertices as an induced subgraph. Let  $H$  be a multigraph with the property that any two distinct vertices share at most one common neighbour. We show that the generating function for (list) graph homomorphisms from  $G$  to  $H$  can be calculated in subexponential time  $2^{O(\sqrt{tn \log(n)})}$  for  $n = |V(G)|$  in the class of  $P_t$ -free graphs  $G$ . As a corollary, we show that the number of 3-colourings of a  $P_t$ -free graph  $G$  can be found in subexponential time. On the other hand, no subexponential time algorithm exists for 4-colourability of  $P_t$ -free graphs assuming the Exponential Time Hypothesis. Along the way, we prove that  $P_t$ -free graphs have pathwidth that is linear in their maximum degree.

## 1 Introduction

Throughout this paper, graphs do not have multiple edges or loops. When we need general graphs (with multiple edges and loops), we call them multigraphs. We use the notation  $vv'$  for the edge  $\{v, v'\}$ . For a multigraph  $G$ , the set  $N_G(v) = \{v' \in e : e \in E(G)\}$  contains  $v$  if and only if  $G$  has a loop at vertex  $v$ .

A  $k$ -colouring of a graph  $G$  is a function  $c : V(G) \rightarrow \{1, \dots, k\}$  such that  $c(v) \neq c(v')$  for all  $vv' \in E(G)$ . The decision problem  $k$ -COLOURABILITY asks whether a given graph  $G$  is  $k$ -colourable. This problem is NP-complete for  $k \geq 3$

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in general. In order to investigate what graph structure makes the decision problem hard, a natural question is whether the problem becomes easy if it is restricted to instances that do not contain a particular structure. Thus we restrict to the class of  $F$ -free graphs, that is, those graphs which do not contain  $F$  as an induced subgraph, for some fixed graph  $F$ . If  $k$  is part of the input, a full classification is given by Král et al. [18]. For fixed  $k \geq 3$ ,  $k$ -COLOURABILITY is shown to be NP-complete for  $F$ -free graphs if  $F$  is either a cycle  $C_\ell$  for  $\ell \geq 3$  [17] or the claw  $K_{1,3}$  [14, 19]. Any graph  $F$  which does not contain a cycle nor the claw is a disjoint union of paths.

Let  $P_t$  denote the path on  $t$  vertices. Polynomial-time algorithms for deciding  $k$ -COLOURABILITY for  $P_t$ -free graphs exist for  $t \leq 5$  [13],  $(k, t) = (4, 6)$  [6, 7] and  $(k, t) = (3, 7)$  [4]. On the other hand, Huang [15] showed 4-COLOURABILITY is NP-complete for  $P_7$ -free graphs and 5-COLOURABILITY is NP-complete for  $P_6$ -free graphs. It is an open problem to determine the complexity of 3-COLOURABILITY of  $P_t$ -free graphs for  $t \geq 8$ .

It is also open whether MAXIMUM INDEPENDENT SET is decidable in polynomial time for  $P_t$ -free graphs for  $t \geq 7$ . Brause [5] and Bascó, Marx and Tuza [2] independently showed a greedy exhaustive approach yields a subexponential-time algorithm for MAXIMUM INDEPENDENT SET on  $P_t$ -free graphs. In this paper we show that there are subexponential-time algorithms for a larger class of problems, including 3-COLOURABILITY and MAXIMUM INDEPENDENT SET, and also give counting results. Our algorithm builds on the following property of  $P_t$ -free graphs.

**Lemma 1.** *A  $P_t$ -free graph of maximum degree  $\Delta$  has pathwidth at most  $\Delta(t - 1) - 1$ . Moreover, a path-decomposition of this width can be found in polynomial time.*

This lemma is an improvement on the treewidth bound of Bascó et al. [1] for  $P_t$ -free graphs (which they used to improve the algorithm of Basco et al. [2]).

We now introduce our framework. Let  $H$  be a multigraph and  $G$  a (simple) graph. A *graph homomorphism* from  $G$  to  $H$  is a map  $f : V(G) \rightarrow V(H)$  such that  $vv' \in E(G)$  implies  $f(v)f(v') \in E(H)$ . (Thus a 3-colouring is a graph homomorphism to  $K_3$ .) A *list  $H$ -colouring instance*  $I = (G, L)$  consists of a graph  $G$  together with a function  $L : V(G) \rightarrow \mathcal{P}(V(H))$  that assigns a subset  $L_v \subseteq V(H)$  to every  $v \in V(G)$ . A *list  $H$ -colouring* of such an instance is a graph homomorphism  $f$  from  $G$  to  $H$  such that  $f(v) \in L_v$  for all  $v \in V(G)$ . We denote the set of list  $H$ -colourings by  $\mathcal{LC}(H)$ .

A useful way to summarise information about  $H$ -colourings of a graph  $G$  is to use a multivariate generating function (see for example [20]). Given a multigraph  $H$  with  $\{0, 1\}$ -valued adjacency matrix  $A$ , we define the *partition function*  $p_{G \rightarrow H}(x)$  by

$$p_{G \rightarrow H}(x) := \sum_{f \in \mathcal{LC}(H)} \left( \prod_{vv' \in E(G)} A_{f(v)f(v')} \right) \left( \prod_{u \in V(G)} w_{u, f(u)} x_{f(u)} \right).$$

The weights  $w_{v,h}$  (for  $v \in V(G)$  and  $h \in V(H)$ ) are included to allow more general application and can be ignored by choosing them identically one. For  $H = (\{h, h'\}, \{hh', h'h'\})$ ,  $p_{G \rightarrow H}(x)$  gives the independent set polynomial when  $x_{h'}$  is set to one.

Summing appropriate coefficients of the polynomial, the partition function can be used to for example count the number of list  $H$ -colourings, or to count the number of “restrictive  $H$ -colourings” [8] in which a restriction is placed on the size of the preimages of the vertices of  $H$ .

The following theorem is our main result and will be proved in Section 3.

**Theorem 2.** *Let  $H$  be a multigraph so that  $|N_H(h) \cap N_H(h')| \leq 1$  for all distinct vertices  $h, h'$  of  $H$ . For  $t \geq 4$ , the polynomial  $p_{G \rightarrow H}(x)$  can be calculated for every  $P_t$ -free graph  $G$  in time  $2^{O(\sqrt{tn \log(n)})}$  where  $n = |V(G)|$ .*

For simple graphs  $H$ , the condition  $|N_H(h) \cap N_H(h')| \leq 1$  for all distinct  $h, h'$  is equivalent to  $H$  not having  $C_4$  as (not necessarily induced) subgraph.

**Corollary 3.** *The following problems can be solved for  $P_t$ -free graphs  $G$  in time  $2^{O(\sqrt{tn \log(n)})}$  where  $n = |V(G)|$ :*

- *Counting the number of  $H$ -colourings for any fixed simple graph  $H$  with no  $C_4$  subgraph.*
- *Computing the independent set polynomial.*

*In particular, it can be decided in subexponential time whether a  $P_t$ -free graph is 3-colourable (and the number of 3-colourings can be counted).*

For example, if  $G$  is  $P_t$ -free and  $H$  is an odd cycle, then we can count the number of  $H$ -colourings of  $G$  in subexponential time. This problem is  $\#P$ -complete if all graphs  $G$  are allowed [9] and the corresponding decision problem is NP-complete [12].

The Exponential Time Hypothesis (ETH) [16] states that there is an  $\epsilon > 0$  such that there is no  $O(2^{\epsilon n})$ -time algorithm for 3-SAT. In Section 4 we prove the following result that shows that it is in some sense unlikely that Corollary 3 will extend to  $k$ -colouring  $P_t$ -free graphs for  $k > 3$ .

**Proposition 4.** *If the Exponential Time Hypothesis is true, then there is no algorithm running in time subexponential in the number of vertices of the graph for*

- *4-COLOURABILITY on  $P_7$ -free graphs;*
- *3-COLOURABILITY for  $F$ -free graphs for any connected  $F$  which is not a path.*

This result might shed some light on why the complexity status of 3-COLOURABILITY of  $P_t$ -free graphs for  $t$  large has remained open whereas the complexity of  $k$ -COLOURABILITY of  $F$ -free graphs has been settled for other values of  $k$  or  $F$ .

## 2 Pathwidth of $P_t$ -free graphs and dynamic programming

A *path-decomposition* of a graph  $G$  is a sequence of subsets  $X_i$  of vertices of  $G$  with three properties:

- The vertex set of  $G$  equals  $\cup_i X_i$ ,
- For each edge of  $G$ , there exists an  $i$  such that both endpoints of the edge belong to subset  $X_i$ , and
- $X_\ell \cap X_j \subseteq X_i$  for every three indices  $1 \leq \ell \leq i \leq j$ .

The *pathwidth* of  $G$  is defined as the minimum of  $\max_i |X_i| - 1$  over the path-decompositions of  $G$ .

Let  $T$  be a rooted tree with root  $r$ . For a vertex  $v$ , let  $T_v$  denote the path of  $T$  between  $r$  and  $v$ . For each vertex  $w$ , fix a linear order of the set of children of  $w$ . We call this a *plane tree*. If  $u, v$  are both children of  $w$  and  $u$  is earlier than  $v$  in the corresponding ordering, we say  $u$  is an *elder sibling* of  $v$ .

Let  $T$  be a spanning tree of a graph  $G$ , equipped with orders to make it a plane tree. We call it an *uncle tree* of  $G$  if for every edge  $uv$  of  $G$  that is not an edge of  $T$ , one of  $u, v$  has an elder sibling that is an ancestor of the other. We use the following result of Seymour [21].

**Theorem 5.** *For every connected graph we can compute an uncle tree with any specified vertex as root in polynomial time.*

(The proof is easy; grow a depth-first tree, subject to the condition that the path of the tree between each vertex and the root is induced.)

*Proof of Lemma 1.* We may assume that  $G$  is connected. Take an uncle tree  $T$  of  $G$ , and order its leaves as  $p_1, \dots, p_k$  say, in the natural order of the leaves of a plane tree. For  $1 \leq i \leq k$ , let  $X_i$  be the set of vertices of  $T$  that either belong to  $T_{p_i}$ , or have an elder sibling (and hence also a parent) in this set. We claim that the sequence  $(X_1, \dots, X_k)$  is a path-decomposition of  $G$ . We check that:

- Every vertex belongs to some  $X_i$  (this is clear).
- For all  $u, v$  adjacent in  $G$ , there exists  $i$  with  $u, v \in X_i$ . To see this, since  $T$  is an uncle tree we may assume that  $u$  has an elder sibling  $u'$  that is an ancestor of  $v$ . Choose a leaf  $p_i$  of  $T$  such that  $T_{p_i}$  contains  $v$ ; then the common parent of  $u, u'$  belongs to  $T_{p_i}$ , and hence  $u, v \in X_i$ .
- $X_\ell \cap X_j \subseteq X_i$  for  $1 \leq \ell \leq i \leq j \leq k$ . To see this, let  $v \in X_\ell \cap X_j$ ; consequently either  $v$  or an elder sibling of  $v$  belongs to  $T_{p_\ell}$ , and either  $v$  or an elder sibling belongs to  $T_{p_j}$ . It follows that either  $v$  or an elder sibling of  $v$  belongs to  $T_{p_i}$ , from the ordering of the leaves of  $T$ .

This proves the claim.

If  $T$  is an uncle tree of  $G$ , then each path of  $T$  starting from  $r$  is induced; and so if  $G$  does not contain  $P_t$ , then  $|T_{p_i}| \leq t - 1$  for each  $i$ , and so  $|X_i| \leq \Delta(t - 1)$ , where  $\Delta$  denotes the maximum degree of  $G$ .  $\square$

We give a short outline of how the standard dynamic programming approach can be applied to compute  $p_{G \rightarrow H}(x)$  in time  $2^{O(p)}n$  given a path-decomposition of width  $p$  of a graph  $G$  on  $n$  vertices.

Let  $(X_1, \dots, X_k)$  be the given path-decomposition of  $G$ . We define  $X(i) = \cup_{j \leq i} X_j$ . For each  $i \in V(T)$  and list  $H$ -colouring  $g : X_i \rightarrow V(H)$ , we compute the polynomial  $p_{G[X(i)] \rightarrow H}$  with the vertices in  $X_i$  precoloured: we define  $p_i(g)$  as the sum, over the list  $H$ -colourings  $f : X(i) \rightarrow V(H)$  such that  $f|_{X_i} = g$ , of

$$\left( \prod_{vv' \in E(G[X(i)])} A_{f(v)f(v')} \right) \left( \prod_{u \in X(i)} w_{u,f(u)} x_{f(u)} \right).$$

For each list  $H$ -colouring  $g : X_1 \rightarrow V(H)$ , we set  $p_1(g) = \prod_{v \in X_1} w_{v,g(v)} x_{g(v)}$ . Having computed all  $p_i(g)$  for some  $i$ , for each list  $H$ -colouring  $g : X_{i+1} \rightarrow V(H)$  we select the list  $H$ -colourings  $g_1, \dots, g_\ell$  of  $X(i)$  that are compatible with  $g$ , that is,  $g_i(v) = g(v)$  for  $v \in X_i \cap X_{i+1}$  and  $g(u)g(v) \in E(H)$  if  $uv \in E(G)$  for all  $u \in X_i$  and  $v \in X_{i+1}$ . Since  $N_G[v] \cap X(i) \subseteq X_i$  for all  $v \in X_{i+1} \setminus X_i$ , we can then compute

$$p_{i+1}(g) = \sum_{j=1}^{\ell} p_i(g_j) \prod_{v \in X_{i+1} \setminus X_i} w_{v,g(v)} x_{g(v)}.$$

Finally, we calculate the desired  $p_{G \rightarrow H}$ , which is the sum, over all list  $H$ -colourings  $g$ , of  $p_k(g)$ .

### 3 Algorithm and time analysis

Throughout this section,  $H$  is a fixed multigraph such that  $|N_H(h) \cap N_H(h')| \leq 1$  for all distinct  $h, h'$  in  $H$ . We allow loops in  $H$  but no multiple edges.<sup>1</sup> The  $P_t$ -free graphs  $G$  are assumed to be simple.

We shall say a list colouring instance  $I = (G, L)$  has *weight*  $w(I) = \sum_{v \in V(G)} |L_v|$  and is *reduced* if  $|L_v| \geq 2$  for all  $v \in V(G)$ . The key observation we need is the following.

**Lemma 6.** *Let  $I = (G, L)$  be a reduced list  $H$ -colouring instance and let  $v \in V(G)$  with degree  $d(v)$ . For  $h \in V(H)$ , let*

$$C_h = \{v' \in N_G(v) : L_{v'} \subseteq N_H(h)\}.$$

*Then there is at most one  $h \in L_v$  for which  $|C_h| > \frac{1}{2}d(v)$ .*

<sup>1</sup>It is possible to extend the algorithm so that entries unequal to 0 or 1 are allowed in the adjacency matrix of  $H$  (which can be interpreted as ‘‘edge weights’’), but in this case the weights  $w_{v',h'}$  have to be updated in Line 3 and 5 of Algorithm HCol to  $w_{v',h'} A_{h,h'}$  for all  $v' \in N_v$ .

*Proof.* Suppose  $h \neq h'$  in  $L_v$  both satisfy  $|C_h|, |C_{h'}| > \frac{1}{2}d(v)$ . Then there exists  $v' \in N_G(v)$  such that  $v' \in C_h \cap C_{h'}$ . Hence  $L_{v'} \subseteq N_H(h) \cap N_H(h')$ , so that by our assumption on  $H$  we find  $|L_{v'}| \leq 1$ , contradicting the assumption that  $I$  is reduced.  $\square$

This lemma tells us that “colouring” a vertex  $v$  of degree  $d(v) = \Delta$  decreases the weight of a reduced instance by at least  $\frac{1}{2}\Delta$  for all but one “colour” in  $L_v$ . Either there is a vertex of high degree and we can reduce the weight significantly by colouring this vertex, or  $\Delta$  is “small” and we can apply the results from the previous section to compute  $p_{G \rightarrow H}(x)$  in time  $2^{O(t\Delta)}$ .

Our algorithm “HCol” for computing the list  $H$ -colouring function of a graph  $G$  is given below. This algorithm either terminates or recurses on instances of strictly smaller weight. Therefore, it always terminates in finite time. We can represent the recursions by a tree: the root is the first call of the algorithm and each recursive call creates a child. For  $P_t$ -free graphs, we can bound the number of nodes in this recursion tree.

**Proposition 7.** *Let  $t \geq 4$ ,  $c > 4\sqrt{t|V(H)|/\log(2)}$  and  $f(w) = 2^{c\sqrt{w \log(w)}}$ . Then there exists an  $n_0$  for Algorithm HCol such that if it is applied to an instance  $I = (G, L)$  of weight  $w(I)$  with  $G$  a  $P_t$ -free graph, then the number of nodes in the corresponding recursion tree is bounded by  $f(w(I))$ .*

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**Algorithm HCol:** Outputs the list  $H$ -colouring function.

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Input: a list  $H$ -colouring instance  $I = (G, L)$  for  $G = (V, E)$ .

1. If  $|V| \leq n_0$ , compute the list  $H$ -colouring function exhaustively.
  2. If there exists  $v \in V$  such that  $|L_v| = 0$ , return 0.
  3. If there exists  $v \in V$  such that  $|L_v| = 1$ , say  $L_v = \{h\}$ , then set  $L_{v'}^h = L_{v'} \cap N_H(h)$  for  $v' \in N_G(v)$  and  $L_{v'}^h = L_{v'}$  for  $v' \notin N_G(v)$ . Return  $w_{v,h}x_h \text{HCol}(G - v, L^h)$ .
  4. If  $G$  is not connected, let  $G_1, \dots, G_k$  be the connected components. Return  $\prod_{i=1}^k \text{HCol}(G_i, L|_{V(G_i)})$ .
  5. If the maximum degree of  $G$  is at most  $\sqrt{n \log(n)}/t$ , compute a path-decomposition of  $G$  of width  $O(\sqrt{n \log(n)}/t)$  and compute the result using dynamic programming.
  6. Otherwise take  $v \in V$  of maximal degree. For  $h \in L_v$ , set  $L_{v'}^h = L_v \cap N_H(h)$  if  $v' \in N_G(v)$  and  $L_{v'}^h = L_{v'}$  if  $v' \notin N_G(v)$ . Return  $\sum_{h \in L_v} w_{v,h}x_h \text{HCol}(G - v, L^h)$ .
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Algorithm HCol gives the correct answer for any graph  $G$ : in line 2 we note that if some vertex has an empty list, then  $p_{G \rightarrow H} = 0$ ; in line 3 we note that if the list of a vertex  $v \in V(G)$  has a single element  $h \in V(H)$ , then  $v$  has to be mapped to  $h$ , i.e.  $p_{G \rightarrow H}(x) = w_{v,h}x_h p_{G-v \rightarrow H}(x)$ ; in line 4 we use the algebraic identity  $p_{G_1 \sqcup G_2 \rightarrow H}(x) = p_{G_1 \rightarrow H}(x)p_{G_2 \rightarrow H}(x)$ ; in line 6, we use  $p_{G \rightarrow H}(x) = \sum_{h \in L_v} w_{v,h}x_h p_{G-v \rightarrow H}(x)$ .<sup>2</sup>

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<sup>2</sup>We left the lists of the vertices implicit in the notation; the precise way in which the lists need to be updated is given in the algorithm.

Inspecting and updating the lists of vertices, finding a vertex of maximal degree and finding the connected components of a graph on  $n$  vertices can all be done in time  $Cn^2$ , where the constant  $C$  may depend on  $H$  and  $n_0$ . Line 5 is applied at most once and takes  $2^{O(\sqrt{tn \log(n)})}$ . Since  $|L_v| \leq |V(H)|$  for all  $v \in V(G)$ , it follows that  $w(I) \leq |V(H)|n = O(n)$ . Theorem 2 follows from Proposition 7 by observing that  $Cn^2 2^{O(\sqrt{tn \log(n)})} 2^{O(\sqrt{tn \log(n)})} = 2^{O(\sqrt{tn \log(n)})}$ .

We require the following simple estimate.

**Lemma 8.** *Let  $m, y > 0$  and  $c > y^{-1}$ . There exists an  $n_0 \in \mathbb{N}$  such that  $f(w) = e^{c\sqrt{w \log(w)}}$  satisfies*

$$f(n-2) + mf\left(n - y\sqrt{n \log(n)}\right) \leq f(n)$$

for all  $n \geq n_0$ .

*Proof.* Let  $\epsilon > 0$  be given such that  $\sqrt{1-x} \leq 1 - \frac{1}{2}x$  and  $e^{-x} \leq 1 - x/2$  for all  $0 \leq x \leq \epsilon$ . Choose  $n_0$  sufficiently large such that  $2/n, y\sqrt{\log(n)/n}, c \log(n)/\sqrt{n} \leq \epsilon$  and  $mn^{-cy/2} < \frac{\epsilon}{2}\sqrt{\log(n)/n}$  for all  $n \geq n_0$  (by assumption,  $cy > 1$ ). We calculate

$$\begin{aligned} f(n-2) + mf\left(n - y\sqrt{n \log(n)}\right) &\leq e^{c\sqrt{(n-2)\log(n)}} + me^{c\sqrt{\log(n)}\left(n - y\sqrt{n \log(n)}\right)^{\frac{1}{2}}} \\ &= f(n)\sqrt{1-2/n} + mf(n)\left(1 - y\sqrt{\log(n)/n}\right)^{\frac{1}{2}} \\ &\leq f(n)^{1-1/n} + mf(n)^{1-\frac{1}{2}y\sqrt{\log(n)/n}} \\ &= f(n)\left[e^{-c\sqrt{\log(n)/n}} + me^{-\frac{1}{2}cy \log(n)}\right] \end{aligned}$$

But

$$e^{-c\sqrt{\log(n)/n}} + me^{-\frac{1}{2}cy \log(n)} \leq 1 - \frac{1}{2}c\sqrt{\log(n)/n} + mn^{-\frac{1}{2}cy} < 1. \quad \square$$

*Proof of Proposition 7.* Let  $n_0$  be given from Lemma 8 applied with  $m = |V(H)|$ ,  $y = \frac{1}{4\sqrt{t|V(H)|}}$  and  $c = c \log(2)$ . Enlarging  $n_0$  if necessary for the last three properties, we may now assume that

$$\begin{aligned} f(w-2) + |V(H)|f\left(w - y\sqrt{w \log(w)}\right) &\leq f(w) && \text{for all } w \geq n_0, \\ f(k) + f(\ell) + 1 &\leq f(k + \ell) && \text{for all } k, \ell \geq n_0, \\ f(k) + (w - k) + 1 &\leq f(w) && \text{for all } w \geq n_0, k < w, \\ w + 1 &\leq f(w) && \text{for all } w \geq n_0. \end{aligned}$$

The proposition is proved by induction on  $w = w(I)$ . If the algorithm terminates in line 1, 2 or 5, then there is only one iteration and  $f(n) \geq 1$  for all  $n \in \mathbb{Z}_{\geq 0}$ .

If the algorithm reaches line 3, then  $G$  has at least  $n_0$  vertices and  $1 \leq |L_v|$  for all  $v \in V$ , so that  $n_0 \leq w(I)$ . Therefore, the statement holds for all  $w < n_0$ .

Suppose the statement has been shown for instances with  $w(I) < w$  for some  $w \geq n_0$ . If the algorithm recurses on line 3, then the removed vertex contributed at least 1 to the weight, and so by induction at most  $f(w-1) + 1 \leq f(w)$  iterations are taken.

If the algorithm reaches line 4, we may assume the instance is reduced, and so  $2|V(G)| \leq w(I)$ . Suppose the graph  $G$  is disconnected with connected components  $G_1, \dots, G_k$ . Let  $I_i = (G_i, L|_{V(G_i)})$  and note that  $I_i$  is also reduced. Hence if  $|V(G_i)| \leq \frac{1}{2}w(I_i) \leq n_0$ , then the recursive call on  $G_i$  will take a single iteration. Renumber so that  $I_1, \dots, I_\ell$  have weight at most  $2n_0$  and  $I_{\ell+1}, \dots, I_k$  have weight at least  $2n_0$ . By induction the algorithm takes at most

$$\sum_{i=1}^{\ell} 1 + \sum_{i=\ell+1}^k f(w(I_i)) + 1 \leq f(w)$$

iterations, where the inequality follows from the assumptions we placed on  $n_0$ , considering  $k - \ell = 0$ ,  $k - \ell = 1$  and  $k - \ell > 1$  separately.

At line 6, the hypothesis of Lemma 6 is satisfied. All but one of the instances have their weight reduced by at least

$$\frac{1}{2} \sqrt{n \log(n)/t} \geq \frac{1}{2\sqrt{t}|V(H)|} \sqrt{w(\log(w) - \log(|V(H)|))} \geq y \sqrt{w \log(w)}$$

(using that  $w = \sum_{v \in V(G)} |L_v| \leq |V(H)|n$  and assuming  $\log(w) - \log(|V(H)|) \geq \frac{1}{4} \log(w)$  by enlarging  $n_0$  if necessary). The other instance has its weight reduced by at least 2, since the vertex  $v$  with  $|L_v| \geq 2$  is removed. By induction, the number of nodes in the recursion tree is at most  $f(w-2) + (|V(H)|-1)f(w - y\sqrt{w \log(w)}) + 1 \leq f(w)$ .  $\square$

## 4 Extensions

Our algorithm can easily be adapted to find the  $H$ -colouring  $f$  of  $G$  which minimises the cost  $\sum_{v \in V(G)} w_{v,f(v)}$ ; in particular, MINIMUM COST HOMOMORPHISM [11] can be solved in subexponential time for  $G$  and  $H$  as above.

Note that Lemma 6 is the bottleneck for extending the time complexity to, for example, 4-colouring ( $H = K_4$ ): it is possible that the neighbourhood of a vertex  $v$  with  $L_v = \{1, 2, 3, 4\}$  has mostly neighbours with list  $\{1, 2\}$ , so that both  $C_3$  and  $C_4$  are large. Assuming the Exponential Time Hypothesis (ETH), this is to be expected in view of Proposition 4: under ETH, no such subexponential time algorithm can exist for 4-COLOURABILITY on  $P_7$ -free graphs.

*Proof of Proposition 4.* Huang [15] gives a reduction of an instance of 3-SAT with  $m$  formulas and  $n$  variables into an instance of 4-COLOURABILITY for a  $P_7$ -free graph on  $O(n + m)$  vertices. Therefore, any algorithm for 4-COLOURABILITY on



$P_7$ -free graphs yields an algorithm for 3-SAT with the same time dependence on the input size.

Let  $F$  be a connected graph which is not a path. Then  $F$  contains either the claw  $K_{1,3}$  or a cycle. We prove ETH implies that there is no subexponential time algorithm for 3-COLOURABILITY of  $F$ -free graphs. The standard reduction (for example [10, Prop. 2.26]) from 3-SAT to 3-COLOURABILITY creates a graph on  $O(n + m)$  vertices. Kamiński and Lozin [17] reduce 3-COLOURABILITY to 3-COLOURABILITY on graphs of girth  $g$  (for every  $g \geq 3$ ) by (in the worst case) replacing each vertex by a constant-sized gadget. This handles the case when  $F$  contains a cycle.

For  $F$  containing the claw  $K_{1,3}$ , note that claw-free graphs are a superset of line graphs. Holyer [14] reduces 3-SAT to 3-EDGE COLOURABILITY on 3-regular graphs. Given  $n$  variables and  $m$  clauses, constant-sized gadgets are created for each variable and clause; additional components are added to the variable gadgets for each time it occurs in a clause. Since there are at most  $3m$  such occurrences, this creates a graph on  $O(n + m)$  vertices. Since the graph is 3-regular, the number of edges is  $O(n + m)$  as well. Hence the line graph of such a graph will have  $O(n + m)$  vertices. Since 3-colourings of the vertices of the line graph are in one-to-one correspondence with 3-colourings of the edges of the original graph, we can hence reduce 3-SAT to 3-COLOURABILITY of line graphs on  $O(n + m)$  vertices.  $\square$

If ETH holds, then any polynomial-time reduction from 3-SAT to a problem with a subexponential algorithm must “blow up” the instance size. Our result therefore suggests that, if one attempts to prove NP-completeness of 3-COLOURABILITY for  $P_t$ -free graphs (for  $t$  large) by designing gadgets, it may be necessary either to start from a problem whose instance size has already been “blown up” or to use gadgets which are not of bounded size.

Our algorithm only uses the property of  $P_t$ -free graphs that every induced subgraph has pathwidth  $O(t\Delta)$ . Seymour [21] proves that a tree-decomposition of width  $O(\ell\Delta)$  can be computed efficiently for graphs that do not contain cycles of length at least  $\ell$  as induced subgraph; therefore, our algorithm extends to this class of graphs (after adjusting the standard dynamic programming approach of for example [3] to our setting in a similar fashion as done in Section 2). This motivates the following question.

**Problem.** *For fixed  $t$ , are 3-COLOURABILITY and MAXIMUM INDEPENDENT SET solvable in polynomial time on graphs that have no induced cycles of length greater than  $t$ ?*

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