On separating systems

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Abstract

We sharpen a result of Hansel on separating set systems. We also extend a theorem of Spencer on completely separating systems by proving an analogue of Hansel's result.

1 Introduction

A *weakly separating system* or, simply, a *separating system* on $[n] = \{1, \ldots, n\}$ is a collection $(S_1, T_1), \ldots, (S_N, T_N)$ of disjoint pairs of subsets of $[n]$ such that for every $i, j \in [n]$ with $i \neq j$ there is a k with $i \in S_k$ and $j \in T_k$, or $i \in T_k$ and $j \in S_k$. Equivalently, the complete bipartite graphs with vertex classes S_i and T_i cover the edges of the complete graph with vertex set $[n]$. Similarly, a *strongly separating system* on $[n]$ is a collection $(S_1, T_1), \ldots, (S_N, T_N)$ of disjoint pairs of subsets of $[n]$ such that for every $i, j \in [n]$ with $i \neq j$ there is a k with $i \in S_k$ and $j \in T_k$. The study of separating systems was started by Rényi [10] in 1961.

There are four basic extremal functions associated with separating systems. Write $s(n)$ for the minimal number of pairs (S_i, T_i) in a weakly separating system on $[n]$, and $t(n)$ for the corresponding minimum for a strongly

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separating system. Also, let

$$S(n) = \min\Big\{\sum_{i=1}^{N} |S_i \cup T_i| : \ (S_i, T_i)_{i=1}^{N} \text{ is a separating system on } [n]\Big\},$$

and let $T(n)$ be the corresponding minimum for a strongly separating system.

Let us recall some of the results concerning these functions. First, it is essentially trivial that $s(n) = \lceil \log_2 n \rceil$: this many bipartite graphs are necessary and sufficient to cover the edges of K_n. Hansel [3] (see also Katona and Szemerédi [5], Nilli [7], Radhakrishnan [8]) proved the following lower bound on $S(n)$.

Theorem 1. *$S(n) \geq n \log_2 n$ for every n.*

Note that this immediately implies the trivial bound $s(n) \geq \lceil \log_2 n \rceil$. However, the theorem gives a stronger bound on the minimal number of pairs in a weakly separating system (S_i, T_i) if we restrict the size of $S_i \cup T_i$.

The question of determining $t(n)$ was raised by Dickson [2], who proved that $t(n) = (1 + o(1)) \log_2 n$. (Note that, in this case, we may assume that $T_k = [n] \setminus S_k$.) The exact value of $t(n)$ was determined by Spencer [11].

Theorem 2. *Let t be the smallest positive integer with $\binom{t}{\lfloor t/2 \rfloor} \geq n$. Then $t(n) = t$.*

This implies that $t(n) = \log_2 n + \frac{1}{2} \log_2 \log_2 n + O(1)$. Thus $s(n)$ and $t(n)$ differ by about $\frac{1}{2} \log_2 \log_2 n$. Spencer's proof uses a correspondence between strongly separating systems of size k on $[n]$ and antichains on $[k]$.

Separating systems (S_i, T_i) with restrictions on the cardinalities $|S_i|$, $|T_i|$ have been studied by Katona [4], Wegener [13], Ramsay and Roberts [9], Kündgen, Mubayi and Tetali [6], among others.

Our aim in this brief note is to strengthen Hansel's theorem to a result that gives us the exact value of $S(n)$ for every n, and to prove a lower bound on $T(n)$ that extends Spencer's result and is analogous to Hansel's theorem.

2 Weakly Separating Systems

In this section we give a slight sharpening of Theorem 1. The main interest here is that the result is sharp for every n. Indeed, if $n = 2^k + l$, where

$0 \le l < 2^k$, then partition $[n]$ into $2^k - l$ sets of size 1 and l pairs. We can cover the edges between these 2^k sets with k complete bipartite graphs (with n vertices each); we can cover the l remaining edges with a single bipartite graph with $2l$ vertices. Then summing the orders of the graphs gives a total of $nk + 2l$, which equals the bound in the following result.

Theorem 3. *Write n as $n = 2^k + l < 2^{k+1}$. Then $S(n) = nk + 2l$.*

Proof. Let G be the complete graph with vertex set $V = [n]$. For each i independently, we delete all vertices in either S_i or T_i, where S_i and T_i are chosen with equal probability. Since the pairs (S_i, T_i), $1 \le i \le N$, cover the edges of G, there is at most one vertex left after any sequence of deletions, and so the expected number of vertices left at the end is at most 1. If v is in $d(v)$ sets $S_i \cup T_i$, the probability that it survives is $2^{-d(v)}$. So

$$\sum_v 2^{-d(v)} \le 1. \tag{1}$$

Let $(e(v))_{v \in V}$ be a sequence of nonnegative integers that satisfies (1) and, subject to this, has $\sum_v e(v)$ minimal. Thus $\sum_v e(v) \le \sum_v d(v)$. If there are v, w with $e(v) \ge e(w) + 2$ then we can replace $e(v)$ by $e(v) - 1$ and $e(w)$ by $e(w) + 1$ without violating (1) or changing the sum. Thus we may assume that $e(v)$ takes at most two values, and these must be k and $k + 1$. If there are α vertices with $e(v) = k$, we have

$$\alpha 2^{-k} + (n - \alpha) 2^{-(k+1)} \le 1$$

and so

$$(n + \alpha) 2^{-(k+1)} \le 1.$$

It follows that $\alpha \le 2^k - l$, and so

$$\sum_{i=1}^{N} |S_i \cup T_i| = \sum_v d(v) \ge \sum_v e(v) = \alpha k + (n - \alpha)(k + 1) = nk + n - \alpha$$

which is at least $nk + n - 2^k + l = nk + 2l$. □

3 Strongly Separating Systems

The purpose of this section is to prove the following analogue of Hansel's result.

Theorem 4. *Let $n \geq 2$ and let t be the minimal integer such that $\binom{t+1}{\lfloor (t+1)/2 \rfloor} > n$. Then $T(n) \geq nt$, with equality if and only if $n = \binom{t}{\lfloor t/2 \rfloor}$.*

The role played by antichains in Spencer's proof of Theorem 2 is here played by cross-intersecting systems. Recall that a collection $\{(A_j, B_j) : 1 \leq j \leq n\}$ is *cross-intersecting* if $A_i \cap B_i = \emptyset$ for every i and $A_i \cap B_j \neq \emptyset$ for every $i \neq j$. Bollobás [1] proved the following inequality.

Lemma 5. *Suppose that $\{(A_j, B_j) : 1 \leq j \leq n\}$ is a cross-intersecting family. Then*

$$\sum_{i=1}^{n} \binom{|A_i| + |B_i|}{|A_i|}^{-1} \leq 1. \tag{2}$$

We use this inequality and the simple fact that if $1 \leq a \leq b - 2$ then

$$\binom{a}{\lfloor a/2 \rfloor}^{-1} + \binom{b}{\lfloor b/2 \rfloor}^{-1} \geq \binom{a+1}{\lfloor (a+1)/2 \rfloor}^{-1} + \binom{b-1}{\lfloor (b-1)/2 \rfloor}^{-1}. \tag{3}$$

We are now ready to prove the theorem.

Proof of Theorem 4. For $1 \leq j \leq n$, define

$$A_j = \{i : v_j \in S_i\}$$
$$B_j = \{i : v_j \in T_i\}.$$

Then $\{(A_j, B_j) : 1 \leq j \leq n\}$ is a cross-intersecting family if and only if $\big((S_i, T_i)\big)_{i=1}^{N}$ is a strongly separating system.

Now

$$\sum_{i=1}^{N} |S_i \cup T_i| = \sum_{i=1}^{n} |A_i \cup B_i|.$$

By (2) this is at least

$$\min\{\sum_{i=1}^{n} (a_i + b_i) : \sum_{i=1}^{n} \binom{a_i + b_i}{a_i}^{-1} \leq 1\},$$

which is at least

$$\min\{\sum_{i=1}^{n} c_i : \sum_{i=1}^{n} \binom{c_i}{\lfloor c_i/2 \rfloor}^{-1} \leq 1\},$$

where the minimum is taken over all sequences $c_1, \dots, c_n$ of positive integers.

Consider a sequence $c_1, \dots, c_n$ that achieves this minimum and (subject to this) has $\sum c_i^2$ minimal. It follows from (3), and the minimality of $\sum c_i^2$, that there are no i, j with $c_i \geq c_j + 2$, since we could then replace c_i by $c_i - 1$ and c_j by $c_j + 1$. Thus the c_i take at most two values, say t and $t+1$ (where $t = \min c_i$). We have

$$\binom{t}{\lfloor t/2 \rfloor} \leq n < \binom{t+1}{\lfloor (t+1)/2 \rfloor}$$

and so $\sum_{i=1}^{n} c_i \geq tn$, with equality only when $n = \binom{t}{\lfloor t/2 \rfloor}$. Note that, in this case, equality is achieved by starting with the cross-intersecting family $\{(A, [t] \setminus A) : A \in [t]^{\lfloor t/2 \rfloor}\}$. $\square$

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