Powers of paths in tournaments

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Abstract

In this short note we prove that every tournament contains the $k$-th power of a directed path of linear length. This improves upon recent results of Yuster and of Girão. We also give a complete solution for this problem when $k = 2$, showing that there is always a square of a directed path of length $\lceil 2n/3 \rceil - 1$, which is best possible.

1 Introduction

One of the main themes in extremal graph theory is the study of embedding long paths and cycles in graphs. Some of the classical examples include the Erdős–Gallai theorem [3] that every $n$-vertex graph with average degree $d$ contains a path of length $d$, and Dirac’s theorem [2] that every graph with minimum degree $n/2$ contains a Hamilton cycle. A famous generalization of this, conjectured by Pósa and Seymour, and proved for large $n$ by Komlós, Sárközy and Szemerédi [5], asserts that if the minimum degree is at least $kn/(k + 1)$, then the graph contains the $k$-th power of a Hamilton cycle.

In this note, we are interested in embedding directed graphs in a tournament. A tournament is an oriented complete graph. The $k$-th power of the directed path $\overrightarrow{P}_\ell = v_0 \ldots v_\ell$ of length $\ell$ is the graph $\overrightarrow{P}^k_\ell$ on the same vertex set containing a directed edge $v_i v_j$ if and only if $i < j \leq i + k$. The $k$-th power of a directed cycle is defined analogously. An old result of Bollobás and Häggkvist [1] says that, for large $n$, every $n$-vertex tournament with all indegrees and outdegrees at least $(1/4 + \varepsilon)n$ contains the $k$-th power of a Hamilton cycle.

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(the constant $1/4$ is optimal). However, we cannot expect to find powers of directed cycles in general, as the transitive tournament contains no cycles at all.

What about powers of directed paths? A classical result, which appears in every graph theory book (see, e.g., [7]), says that every tournament contains a directed Hamilton path. On the other hand, Yuster [6] recently observed that some tournaments are quite far from containing the square of a Hamilton path. In particular, there is an $n$-vertex tournament that does not even contain the square of $P_{2n/3}$, and more generally, for every $k \geq 2$, there are tournaments with $n$ vertices and no $k$-th power of a path with more than $nk/2^{k/2}$ vertices. In the other direction, Yuster proved that every tournament with $n$ vertices contains the square of a path of length $n^{0.205}$. This was improved very recently by Girão [4], who showed that for fixed $k$, every tournament on $n$ vertices contains the $k$-th power of a path of length $n^{1-o(1)}$. Both papers noted that no sublinear upper bound is known. Our main result shows that the maximum length is in fact linear in $n$.

**Theorem 1.** For $n \geq 2$, every $n$-vertex tournament contains the $k$-th power of a directed path of length $n/2^{2k+6k}$.

The proof of this theorem combines Kővári–Sós–Turán style arguments, used for the bipartite Turán problem, and median orderings of tournaments. A median ordering is a vertex ordering that maximizes the number of forward edges. Theorem 1 and Yuster’s construction show that an optimal bound on the length has the form $n/2^{\Theta(k)}$. It would be interesting to find the exact value of the constant factor in the exponent. Optimizing our proof can yield a lower bound of $n/2^{c+o(k)}$ with $c \approx 3.9$, but is unlikely to give the correct bound.

We also improve the exponential constant in the upper bound from $1/2$ to $1$.

**Theorem 2.** Let $k \geq 5$ and $n \geq k(k+1)2^k$. There is an $n$-vertex tournament that does not contain the $k$-th power of a directed path of length $k(k+1)n/2^k$.

Note that this theorem also holds trivially for $k \leq 4$, when $k(k+1)n/2^k > n$.

Finally, we can solve the problem completely in the special case of $k = 2$. Once again, the proof uses certain properties of median orderings.

**Theorem 3.** For $n \geq 1$, every $n$-vertex tournament contains the square of a directed path of length $\ell = \lceil 2n/3 \rceil - 1$, but not necessarily of length $\ell + 1$.

Theorems 1, 2 and 3 are proved in Sections 2, 3 and 4 respectively.

## 2 Lower bound

We will need the following Kővári–Sós–Turán style lemma.

**Lemma 4.** Let $G$ be a directed graph with disjoint vertex subsets $A$ and $B$ with $|A| = 2k + 1$, $|B| \geq 2^{4k+4k}$, and every vertex in $A$ has at least $(1 - \frac{1}{2k+1})|B|/2$ outneighbours in $B$. Then $A$ contains a subset $A'$ of size $k$ that has at least $(2k+1)2^{2k}$ common outneighbours in $B$.  


Proof. Suppose there is no such set $A'$. Then every $k$-subset of $A$ appears in the inneighbourhood of less than $(2k+1)2^{2k}$ vertices in $B$. So if $d^-(v)$ denotes the number of inneighbours a vertex $v \in B$ has in $A$, then we have

$$\binom{2k+1}{k} \cdot (2k+1)2^{2k} = \binom{|A|}{k} \cdot (2k+1)2^{2k} > \sum_{v \in B} \binom{d^-(v)}{k}.$$  \hfill (1)

On the other hand, $\sum_{v \in B} d^-(v) \geq |A|(1 - \frac{1}{2k+1})|B|/2 = k|B|$. By Jensen’s inequality, $\sum_{v \in B} \binom{d^-(v)}{k} \geq |B| \cdot \left( \frac{\sum_{v \in B} d^-(v)/|B|}{k} \right) = |B| \geq 2^{2k+4} k$. This contradicts (1). \hfill $\square$

One more ingredient we need for the proof of Theorem [1] is the folklore fact that every tournament on $2^m$ vertices contains a transitive subtournament of size $m+1$. This is easily seen by taking a vertex of outdegree at least $2^{m-1}$ as the first vertex of the subtournament, and then recursing on the outneighbourhood.

Proof of Theorem [2] Order the vertices as $0, 1, \ldots, n - 1$ to maximize the number of forward edges, i.e., the number of edges $ij$ such that $i < j$. As was mentioned in the introduction, we will refer to such a sequence as a median ordering of the vertices. We denote an “interval” of vertices with respect to this ordering by $[i, j] = \{i, \ldots, j - 1\}$, where $0 \leq i < j \leq n$.

We will embed $P_{\ell}^k$ inductively using the following claim.

Claim. Let $t = 2^{k+4}k$ and $t \leq i \leq n - (2k+1)t$. For every subset $A^* \subseteq [i - t, i)$ of size $2^k$, there is an index $i + t \leq j \leq i + (2k+1)t$ and a set $A' \subseteq A^*$ of size $k$ such that $A'$ induces a transitive tournament and its vertices have at least $2^k$ common outneighbours in $[j - t, j)$.

Proof. There is a subset $A \subseteq A^*$ of size $2k+1$ that induces a transitive tournament. Let $B = [i, i + (2k+1)t)$. Then every vertex $v \in A$ has at least $kt = \left( 1 - \frac{1}{2k+1} \right) |B|/2$ outneighbours in $B$. Indeed, otherwise $v$ would have more than $(k + 1)t$ inneighbours in the interval $B$, so moving $v$ to the end of this interval would increase the number of forward edges in the ordering, contradicting our choice of the vertex ordering.

We can thus apply Lemma 3 to find a $k$-subset $A' \subseteq A$ with at least $(2k+1)2^k$ common outneighbours in $B$. Partition $B$ into $2k+1$ intervals of size $t$, and we can choose $j$ accordingly so that $A'$ has at least $2^k$ common outneighbours in the interval $[j - t, j)$. \hfill $\square$

The theorem trivially holds for $n < 2^{2k}$, so assume $n \geq 2^{2k}$. Let $i_0 = 2^{2k}$ and $A_0 = [0, 2^{2k})$, and apply the Claim with $i = i_0$ and $A^* = A_0$. We get a set $A' \subseteq A_0$ of size $k$ that induces a transitive tournament, i.e., the $k$-th power of some path $v_0 \ldots v_{k-1}$. Moreover, this $A'$ has at least $2^k$ common outneighbours in some interval $[j - t, j]$ with $i_0 + t \leq j \leq i_0 + (2k+1)t$.

Let us define $i_1 = j$, and choose $A_1$ to be any $2^k$ of the common outneighbours.

At step $s$, we apply the Claim again with $i = i_s$, $A^* = A_s$ to find the $k$-th power of some path $v_{sk} \ldots v_{(s+1)k-1}$ in $A_s$ with $2^k$ common outneighbours in some $[i_{s+1} - t, i_{s+1}]$ with $i_s + t \leq i_{s+1} \leq i_s + (2k+1)t$, and repeat this process until some step $\ell$ with $i_\ell > n - (2k+1)t$. Note that intervals $[i_s - t, i_s]$ and $[i_{s+1} - t, i_{s+1}]$ are always disjoint. Finally, $A_\ell$ must also contain a transitive tournament of size $2k+1$. Call these vertices $v_{\ell k}, \ldots, v_{(\ell+2)k}$. Observe that $n - (2k+1)t < i_\ell \leq 2^{k+\ell}(2k+1)t$, so $n < (\ell + 2)(2k+1)t$. 

Then \( v_0 \ldots v_{(\ell+2)k} \) is a directed path of length \((\ell+2)k \geq kn/(2k+1)t \geq n/(2^{4k+6}k)\) whose \( k \)-th power is contained in the tournament. In fact, we proved a bit more: the tournament contains all edges of the form \( v_a v_b \) with \( a < b \) and \( \lfloor a/k \rfloor + 1 \geq \lfloor b/k \rfloor \).

\[ \square \]

## 3 Upper bound

Let \( \ell_k(n) \) denote the smallest integer \( \ell \) such that there is an \( n \)-vertex tournament that does not contain \( \overrightarrow{P}_k^\ell \), or in other words, the largest integer such that every \( n \)-vertex tournament contains the \( k \)-th power of a directed path on \( \ell \) vertices.

To prove Theorem 3, we first note that \( \ell_k(n) \) is subadditive.

**Lemma 5.** For any \( k, n, m \geq 1 \), we have \( \ell_k(n + m) \leq \ell_k(n) + \ell_k(m) \).

**Proof.** Let \( T_1 \) and \( T_2 \) be extremal tournaments on \( n \) and \( m \) vertices, respectively, not containing the \( k \)-th power of any directed path of length \( \ell_k(n) \) and \( \ell_k(m) \). Let \( T \) be the tournament on \( n + m \) vertices, obtained from the disjoint union of \( T_1 \) and \( T_2 \) by adding all remaining edges directed from \( T_1 \) to \( T_2 \). Then any \( k \)-th power of a path in \( T \) must be the concatenation of the \( k \)-th power of a path in \( T_1 \) and the \( k \)-th power of a path in \( T_2 \), and hence it must have length at most \((\ell_k(n) - 1) + (\ell_k(m) - 1) + 1 < \ell_k(n) + \ell_k(m) \).

Our improved upper bound is based on the following construction.

**Lemma 6.** For every \( k \geq 5 \), we have \( \ell_k(2^{k-1}) < \frac{k(k+1)}{2} \).

**Proof.** Let \( n = 2^{k-1} \) and \( \ell = \frac{k(k+1)}{2} \), and note that \( \overrightarrow{P}_{\ell-1}^k \) has \( k\ell - \ell \) edges.

Let \( T \) be a random \( n \)-vertex tournament obtained by orienting the edges of \( K_n \) independently and uniformly at random. The probability that a fixed sequence of \( \ell \) vertices \( v_0 \ldots v_{\ell-1} \) forms a copy of \( \overrightarrow{P}_{\ell-1}^k \) is \( 2^{-(k-1)\ell} \). There are \( \binom{n}{\ell} \cdot \ell! \) such sequences, so the probability that \( T \) contains the \( k \)-th power of a path of length \( \ell - 1 \) is at most \( \binom{n}{\ell} \cdot \ell! \cdot 2^{-(k-1)\ell} < n^{\ell} \cdot 2^{-(k-1)\ell} = 1 \). So with positive probability \( T \) does not contain \( \overrightarrow{P}_{\ell-1}^k \), therefore \( \ell_k(2^{k-1}) \leq \ell - 1 \).

Combining Lemmas 5 and 6 and using the monotonicity of \( \ell_k(n) \), we get

\[
\ell_k(n) \leq \left\lfloor \frac{n}{2^{k-1}} \right\rfloor \cdot \ell_k(2^{k-1}) \leq \left( \frac{n}{2^{k-1}} + 1 \right) \left( \frac{k(k+1)}{2} - 1 \right) \leq \frac{k(k+1)n}{2^k}
\]

for \( n \geq k(k+1)2^k \), establishing Theorem 3.

## 4 The square of a path

**Proof of Theorem 3** Recall that \( \ell_2(n) \) is the largest integer such that every \( n \)-vertex tournament contains the square of a path on \( \ell \) vertices. Proving Theorem 3 is therefore equivalent to showing \( \ell_2(n) = \lceil 2n/3 \rceil \) for every \( n \geq 1 \).

It is easy to check that \( \ell_2(1) = 1 \) and \( \ell_2(2) = \ell_2(3) = 2 \), so \( \ell_2(n) \leq \lceil 2n/3 \rceil \) follows from Lemma 5 by induction, as \( \ell_2(n) \leq \ell_2(n-3) + \ell_2(3) = \ell_2(n-3) + 2 \) holds for every \( n > 3 \). For the lower bound we need to take a closer look at median orderings.
Claim. Every median ordering \( x_1, \ldots, x_n \) of a tournament has the following properties:

(a) All edges of the form \( x_ix_{i+1} \) are in the tournament.

(b) If \( x_ix_{i-2} \) is an edge of the tournament, then “rotating” \( x_{i-2}x_{i-1}x_i \) gives two other median orderings \( x_1, \ldots, x_{i-3}, x_{i-1}, x_i, x_{i-2}, x_{i+1}, \ldots, x_n \) and \( x_1, \ldots, x_{i-3}, x_i, x_{i-2}, x_{i-1}, x_{i+1}, \ldots, x_n \).

(c) If \( x_ix_{i-2} \) is an edge of the tournament, then each of \( x_{i-2}, x_{i-1}, x_i \) is an inneighbour of \( x_{i+1} \), and at most one of them is an outneighbour of \( x_{i+2} \).

Proof. Property (a) holds, as otherwise we could swap \( x_i \) and \( x_{i+1} \) to get an ordering with more forward edges, contradicting our assumption. Property (b) holds because rotating \( x_{i-2}x_{i-1}x_i \) has no effect on the number of forward edges.

These two properties together imply that each of \( x_{i-2}, x_{i-1}, x_i \) is an inneighbour of \( x_{i+1} \). Suppose, to the contrary of (c), that two of them are outneighbours of \( x_{i+2} \). By rotating \( x_{i-2}x_{i-1}x_i \) if needed, we may assume that these are \( x_{i-1} \) and \( x_i \). But then we can also rotate \( x_ix_{i+1}x_{i+2} \) so that \( x_{i+2} \) comes right after \( x_{i-1} \) in a median ordering. This contradicts (a).

Let us now say that \( i \) is a bad index in a median ordering \( x_1, \ldots, x_n \) if \( x_ix_{i-2} \) is an edge, and at least one of \( x_{i+2}x_i \) and \( x_{i+1}x_{i+2} \) is also an edge.

Lemma 7. Every tournament has a median ordering without any bad indices.

Proof. Suppose this fails to hold for some tournament, and take a median ordering \( x_1, \ldots, x_n \) that minimizes the largest bad index \( i \). As \( i \) is a bad index, \( x_ix_{i-2} \) is an edge, and \( x_i \) or \( x_{i-1} \) is an outneighbour of \( x_{i+2} \). By (b), \( x_{i-2}x_{i-1}x_i \) can be rotated so that \( x_{i+2}x_i' \) is an edge in the new median ordering \( x_1, \ldots, x_{i-3}, x_i', x_{i-2}, x_i', x_{i+1}, \ldots, x_n \). Then neither \( x_{i+2}x_i' \) nor \( x_{i+2}x_i' \) is an edge, since by (b), only one of \( x_{i-2}, x_{i-1}, x_i' \) is an outneighbour of \( x_{i+2} \). Also by (b), \( x_i'-x_{i+1} \) and \( x_i'-x_{i+2} \) are edges, so both of \( x_{i+1} \) and \( x_{i+2} \) are outneighbours of \( x_i' \) and \( x_i' \). This means that none of \( i, i+1, i+2 \) is a bad index in this new ordering, and hence the largest bad index is smaller than \( i \). This is a contradiction.

Now we are ready to prove \( \ell_2(n) \geq \lceil 2n/3 \rceil \). Take an \( n \)-vertex tournament with median ordering \( x_1, \ldots, x_n \) as in Lemma 7 and let \( I = \{i_1 < i_2 < \cdots < i_k\} \) be the set of indices \( i \) such that \( x_ix_{i-2} \) is not an edge (in particular, \( i_1 = 1 \) and \( i_2 = 2 \)). We claim that \( x_{i_1} \ldots x_{i_k} \) is a directed path on \( k \geq \lceil 2n/3 \rceil \) vertices whose square is contained in the tournament.

To see this, first observe that if the index \( i + 2 \) is not in \( I \), then both \( i \) and \( i + 1 \) are in \( I \). Indeed, if \( x_{i+2}x_i \) is an edge, then \( x_{i+1}x_{i-1} \) cannot be one because of (b), and \( x_ix_{i-2} \) cannot be one because \( i \) is not a bad index. This immediately implies \( k \geq \lceil 2n/3 \rceil \).

It remains to check that \( x_{i_j-2}x_{i_j} \) and \( x_{i_j-1}x_{i_j} \) are all edges in the tournament. By the above observation, we know that \( i_j - 3 \leq i_j - 2 < i_j - 1 < i_j \). Here \( x_{i_j-1}x_{i_j} \) is an edge by (b), and \( x_{i_j-2}x_{i_j} \) is an edge by the definition of \( I \). So the only case left is to show that \( x_{i_j-2}x_{i_j} \) is an edge when \( i_j - 2 = i_j - 3 \).

In this case there is an index \( i_j - 3 < i < i_j \) that is not in \( I \), i.e., \( x_ix_{i-2} \) is an edge in the tournament. But then if \( i = i_j - 1 \), then \( x_{i_j-2}x_{i_j} \) is an edge because of (b), while otherwise \( i = i_j - 2 \), and \( x_{i_j-2}x_{i_j} \) is an edge because \( i \) is not a bad index. This concludes our proof.
References


