SUBDIVISIONS OF TRANSITIVE TOURNAMENTS

A.D. SCOTT

ABSTRACT. We prove that, for $r \geq 2$ and $n \geq n(r)$, every directed graph with n vertices and more edges than the r-partite Turán graph $T(r, n)$ contains a subdivision of the transitive tournament on $r+1$ vertices. Furthermore, the extremal graphs are the orientations of $T(r, n)$ induced by orderings of the vertex classes.

1. Introduction

A *subdivision* of a graph G is any graph obtained by replacing some of the edges of G by paths. A graph G with at least $c(r)|G|$ edges contains a subdivision of K_{r+1} (see Mader [7], Bollobás and Thomason [3], [4] and Komlós and Szemerédi [6]). A subdivision of a directed graph D is any graph obtained by replacing directed edges by directed paths (in the same direction as the corresponding edges). Jagger [5] proved a variety of extremal results concerning subdivisions of digraphs, and asked for the maximal number of edges in a directed graph of order n that does not contain a subdivision of T_{r+1}, the transitive tournament on $r+1$ vertices. (For further discussion on definitions, and for related problems for directed graphs, see Jagger [5].)

A lower bound is given by $t(r, n) = e(T(r, n))$, where $T(r, n)$ is the complete r-partite Turán graph on n vertices, in which each vertex class has size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Indeed, any orientation of $T(r, n)$ induced by an ordering of the vertex classes (thus we order the vertex classes $V_1 < \cdots < V_r$ and an edge is oriented from $v \in V_i$ to $w \in V_j$ if $i < j$) contains no directed path with more than r vertices and therefore no subdivision of T_{r+1}. Jagger proved an upper bound of form

$$(1 - \frac{1}{r} + o(1))\binom{n}{2} = t(r, n) + o(n^2)$$

on the size of a directed graph of order n that contains no subdivision of T_{r+1}, and asked whether in fact $t(r, n)$ is the correct bound for sufficiently large n. To this end he proved that, if there is $n \geq 3r$ such that any extremal graph of order n is obtained by an orientation of $T(r, n)$ (induced by an ordering of the vertex classes) then any extremal graph of order $n' \geq n$ is obtained by orienting $T(r, n')$. (Jagger claimed that there is then a unique extremal graph. However, if $n = pr + q$, where $0 \le q \le r$, then there are $\binom{r}{q}$ distinct oriented graphs that can be induced by ordering the vertex classes, since the vertex classes may be of two different sizes.)

The aim of this paper is to answer Jagger's question in the affirmative.

Theorem 1. *For every $r \ge 2$ there is $N(r)$ such that every digraph with $n \ge N(r)$ vertices and more than $t(r,n)$ edges contains a subdivision of T_{r+1}. The extremal graphs are the orientations of $T(r,n)$ induced by ordering the vertex classes.*

2. Proof of Theorem 1

A subdivision of T_{r+1} in a directed graph D consists of $r+1$ vertices $v_1, \dots, v_{r+1}$ and internally disjoint directed paths P_{ij} from v_i to v_j for $1 \le i < j \le r+1$. We shall refer to this as *a subdivision of T_{r+1} with vertices $v_1, \dots, v_{r+1}$*. Thus in order to demonstrate the presence of a subdivision of T_{r+1} with vertices $v_1, \dots, v_{r+1}$, we need to specify paths P_{ij} from v_i to v_j for each $1 \le i < j \le r+1$ for which the edge v_iv_j is not present.

We begin with two straightforward lemmas, which are implicitly stated in [5]. We write $K_r(s)$ for the complete r-partite graph with s vertices in each vertex class.

Lemma 1. *For $r, s \ge 1$ there is an integer $q = q(r,s)$ such that every orientation of $K_r(q)$ contains a copy of $K_r(s)$ with the edges between any two classes all oriented in the same direction.*

Proof. Let G be an r-partite oriented complete graph with vertex classes $V_1, \dots, V_r$. For $1 \le i < j \le r$, colour an edge between V_i and V_j red if it is oriented from V_i to V_j and blue otherwise. The result follows easily by repeatedly choosing monochromatic bipartite graphs between vertex classes. □

Lemma 2. *For every integer $r \ge 2$ there is an integer s such that every orientation of $K_{r+1}(s)$ contains a subdivision of T_{r+1}.*

Proof. Let $t = \binom{r+1}{2}$. By Lemma 1, if s is large enough then every orientation of $K_{r+1}(s)$ contains an oriented $K_{r+1}(t)$ in which the edges between any two classes are all oriented in the same direction. If this orientation is transitive, then picking one vertex from each class gives a copy of T_{r+1}. Otherwise, there are distinct vertex classes V_i, V_j and V_k such that edges are oriented from V_i to V_j, from V_j to V_k, and from V_k to V_i. Pick vertices $v_1, \dots, v_{r+1}$ in V_i, $w_1, \dots, w_t$ in V_j, and $x_1, \dots, x_t$ in

V_k. Then, for each i and j with $1 \leq i < j \leq r+1$, we can join v_i to v_j by a path of form $v_i w_l x_l v_j$, where each w_l and x_l is used exactly once. Thus we obtain a subdivision of T_{r+1} with vertices $v_1, \ldots, v_{r+1}$. □

Now for the proof of the main result.

Proof. Fix r; let $\epsilon > 0$ be small and $n > n(\epsilon)$ (we shall not attempt to determine appropriate values of ϵ and n: we need only that ϵ is smaller than a constant dependent on r, while $n(\epsilon)$ depends on r and ϵ). Suppose D is a digraph of order n with no subdivision of T_{r+1} and the maximal number of edges. We shall prove that D is the Turán graph, with a transitive orientation induced by an ordering of the vertex classes.

As noted in [5], there is a constant $K = K(r)$ such that D has at most Kn pairs $\{x, y\}$ of vertices for which both xy and yx are edges. Otherwise, let G_0 be the graph with vertex set $V(D)$, where two vertices x and y are adjacent in G_0 if and only if there are edges both from x to y and from y to x in D. Then G_0 has at least Kn edges and so contains a subdivision of K_{r+1}, which implies that D contains a subdivision of T_{r+1}.

Let G be the underlying graph of D: we define $V(G) = V(D)$, and vertices x and y are adjacent in G if either xy or yx is present in D. Then $e(G) \geq t_r(n) - Kn$, and G does not contain a copy of $K_{r+1}(s)$, where s is given by Lemma 2. It follows from a result of Bollobás, Erdős, Simonovits and Szemerédi ([2], see also [1]) that, provided $n(\epsilon)$ is sufficiently large, there is a vertex partition $V(G) = V_0 \cup \cdots \cup V_r$, with $|V_0| < \epsilon n$ and $(1-\epsilon)\frac{n}{r} < |V_i| < (1+\epsilon)\frac{n}{r}$ for $i \geq 1$, such that $e(G[V_i]) < \epsilon n^2$, $e(V_i, V_j) > (1-\epsilon)\frac{n^2}{r^2}$ and every vertex in V_i has at least $(1-\epsilon)\frac{n}{r}$ neighbours in V_j for i and j distinct and nonzero.

Pick a set R_1 of q vertices in V_1, where $q = q(r, r^2)$ is the minimal integer satisfying Lemma 1. These have at least $(1 - \frac{1}{r} - (2q-1)\epsilon)n$ common neighbours in V_2. Let $R_2 \subset V_2$ be a set of q common neighbours of R_1. Continuing in the same way, providing ϵ is sufficiently small, for $1 \leq i \leq r$ we can pick sets $R_i \subseteq V_i$ for $1 \leq i \leq r$ such that $|R_i| = q$ for each i and $R_1, \ldots, R_r$ span a complete r-partite graph. Orient each of the edges to agree with an edge of D (there may be two choices). It follows from Lemma 1 that, for $1 \leq i \leq r$, we can find $S_i \subset R_i$ with $|S_i| = r^2$, such that, for $1 \leq i < j \leq r$, the edges between S_i and S_j are all oriented in the same direction. Let $S = \bigcup_{i=1}^r S_i$. Now we claim that, permuting subscripts if necessary, we may assume that, for $1 \leq i < j \leq r$, all edges between S_i and S_j are oriented from S_i

to S_j. Otherwise we obtain a subdivision of T_{r+1} as in the proof of Lemma 2, which gives a subdivision of T_{r+1} in D.

Now consider $V_0, \ldots, V_r, S_1, \ldots, S_r$ as sets of vertices in D. For $1 \leq i < j \leq r$, there is a directed edge from every vertex of S_i to every vertex of S_j. Let $S = \bigcup_{i=1}^r S_i$. Suppose that there is a directed path P from v to w where $v \in S_i$, and $w \in S_j$ for some $1 \leq j \leq i \leq r+1$, and all the internal vertices of P lie outside S. For $1 \leq p \leq r$, pick $s_p \neq v, w$ in S_p. Then if $j < i$, we obtain a subdivision of T_{r+1} with vertices $s_1, \ldots, s_j, w, s_{j+1}, \ldots, s_r$, where all edges are present, except that s_j and w are joined by the path $s_j v P w$. If $j = i$ then we obtain a subdivision of T_{r+1} with vertices $s_1, \ldots, s_{j-1}, v, w, s_{j+1}, \ldots, s_r$, where v and w are joined by P and all other edges are present.

Now suppose that D contains a directed path P from $v \in S_i$ to $w \in S_j$, where $1 \leq j \leq i \leq r+1$. We may assume that P, v, w have been chosen such that P is of minimal length. If all internal vertices of P lie outside S then we can find a subdivision of T_{r+1}. Otherwise, we can write the path as vP_1xP_2w, where x is the first vertex on P after v that belongs to S. But x cannot belong to $S_1 \cup \cdots \cup S_j$, since vP_1x then contradicts the minimality of P; while x cannot belong to $S_{j+1} \cup \cdots \cup S_r$, since xP_2w then contradicts the minimality of P. Therefore D contains no paths from $v \in S_i$ to $w \in S_j$ for $1 \leq j \leq i \leq r+1$.

For $i = 1, \ldots, r$, we let C_i be the set of common neighbours of $\bigcup_{j \neq i} S_i$ in $V_i \setminus S_i$:

$$C_i = (V_i \setminus S_i) \cap \bigcap_{j \neq i} \bigcap_{v \in S_j} (\Gamma^+(v) \cup \Gamma^-(v)).$$

Clearly $|C_i| \geq (1 - 2r^3\epsilon)\frac{n}{r}$. Pick $v \in C_i$. Then, for $j \neq i$, all edges between v and S_j must be oriented in the same direction, or we obtain a path between two vertices of S_j. Furthermore, there must be some k such that edges are oriented from S_j to v for $j \leq k$ and from v to S_j for $j > k$ $(j \neq i)$: otherwise, we obtain a directed path from S_p to S_q, where $p > q$.

We claim that in fact edges are oriented from S_j to v for $j < i$ and from v to S_j for $j > i$. If there is $j \neq i, i-1$ such that edges are oriented from S_j to v and from v to S_{j+1} then pick vertices $s_l \in S_l$ for $1 \leq l \leq r$, and distinct vertices $t_1, \ldots, t_{j-1} \in S_j$ and $t_{j+2}, \ldots, t_r \in S_{j+1}$. We obtain a subdivision of T_{r+1} with vertices $s_1, \ldots, s_j, v, s_{j+1}, \ldots, s_r$, where the subdivided edges between s_l and v are given by $s_l t_l v$ for $l < j$ and $v t_l s_l$ for $l > j+1$; all other edges are present. It follows that all edges between v and $\bigcup_{j<i} S_j$ are oriented in the same direction, and similarly all edges between v and $\bigcup_{j>i} S_j$ are oriented in the same direction.

Now suppose that either edges are oriented from v to S_1, or edges are oriented from S_r to v. In the first case, pick $s_l \in S_l$ for $1 \leq l \leq r$ and distinct vertices $t_2, \dots, t_r \in S_1$. We obtain a subdivision of T_{r+1} with vertices $v, s_1, \dots, s_r$, where the subdivided edges are $vt_2s_2, \dots, vt_rs_r$; all other edges are present. In the second case, pick $s_l \in S_l$ for $1 \leq l \leq r$ and distinct vertices $t_1, \dots, t_{r-1} \in S_r$. We obtain a subdivision of T_{r+1} with vertices $s_1, \dots, s_r, v$, where the subdivided edges are $s_1t_1v, \dots, s_{r-1}t_{r-1}v$; all other edges are present. It follows that, for $v \in C_i$, all edges are oriented from S_j to v for $j < i$ and from v to S_j for $j > i$.

Now suppose that there is a directed path P, with more than one vertex, from $C_j \cup S_j$ to $C_i \cup S_i$, where $i \leq j$. As before, we may pick i, j, P such that P is of minimal length. We may then assume that P runs from $v \in C_j \cup S_j$ to $w \in C_i \cup S_i$, where $i \leq j$, and that all interior vertices of P lie outside $\bigcup_{k=1}^{r}(C_k \cup S_k)$. If $i < j$ then pick vertices $s_i \neq v, w$ in S_i for $i = 1, \dots, r$. We obtain a subdivision of T_r with vertices $s_1, \dots, s_i, w, s_{i+1}, \dots, s_r$, where we have the path s_ivPw from s_i to w, and all other edges are present. If $i = j$ then, for each $l \neq i$, pick $s_l \in S_l$. We obtain a copy of T_{r+1} with vertices $s_1, \dots, s_{i-1}, v, w, s_{i+1}, \dots, s_r$, where v and w are joined by P and all other edges are present. It follows in particular that $C_i \cup S_i$ is an independent set for every i and that edges are oriented from $C_i \cup S_i$ to $C_j \cup S_j$ for $1 \leq i < j \leq r$.

Now $|\bigcup_{i=1}^{r} C_i \cup S_i| \geq (1 - 2r^3\epsilon)n$, so there are at most $2r^3\epsilon n$ vertices in $X = V(D) \setminus \bigcup_{i=1}^{r}(C_i \cup S_i)$. Suppose first that a vertex $x \in X$ is adjacent to at least $2\epsilon n$ vertices in $C_i \cup S_i$ for every i. If there is $i \leq j$ such that there is an edge oriented from x to $C_i \cup S_i$ and an edge oriented from $C_j \cup S_j$ to x then we obtain a directed path from $C_j \cup S_j$ to $C_i \cup S_i$, which we have already ruled out. So there is i with $0 \leq i \leq r$ such that edges are oriented from $C_j \cup S_j$ to x for $j \leq i$ and from x to $C_j \cup S_j$ for $j > i$. Since every vertex in $C_i \cup S_i$ is adjacent to all but at most $2\epsilon n/r$ vertices in $C_j \cup S_j$ for $j \neq i$, it is straightforward to find a copy of T_r that has one vertex s_i in $\Gamma(x) \cap (C_i \cup S_i)$ for each i: pick one vertex at a time, and pick each new vertex from the common neighbours of the vertices already chosen. Then the vertices $s_1, \dots, s_i, x, s_{i+1}, \dots, s_r$ span a copy of T_{r+1}. It follows that every vertex of X has fewer than $2\epsilon n$ neighbours in some $C_i \cup S_i$.

For $1 \leq i \leq r$, let X_i be the set of vertices in X with fewer than $2\epsilon n$ neighbours in $C_i \cup S_i$ and at least $(1 - 2r^4\epsilon)\frac{n}{r}$ neighbours in $C_j \cup S_j$ for every $j \neq i$. Let $X_0 = X \setminus \bigcup_{i=1}^{r} X_i$. Note that $X_0 \subseteq V_0$ and

$C_i \cup S_i \cup X_i \supseteq V_i$ for $1 \le i \le r$, so $|X_0| \le \epsilon n$ and $|C_i \cup S_i \cup X_i| \ge (1-\epsilon)n/r$.

We claim that edges are oriented from $C_j \cup S_j$ to X_i, for $j < i$, and from X_i to $C_j \cup S_j$ for $j > i$. Otherwise, pick $x \in X_i$ for which this is not true. Since there is no directed path from $C_j \cup S_j$ to $C_k \cup S_k$ for $k \le j$, there must be j with $0 \le j \le r$ and $j \ne i, i-1$ such that edges are oriented from $C_k \cup S_k$ to x for $k \le j$ and from x to $C_k \cup S_k$ for $k > j$. If $j < i-1$ then we can find a copy of T_{r-1} that contains a vertex s_l in $(C_l \cup S_l) \cap \Gamma(x)$ for each $l \ne i$. Pick a vertex $w \ne s_{j+1}$ in $C_{j+1} \cup S_{j+1}$ that has x as a neighbour and a vertex $s_i \in S_i$. Then we have a copy of T_{r+1} with vertices $s_1, \dots, s_{j-1}, x, s_j, \dots, s_r$, where there is a path xws_i from x to s_i and all other edges are present. Similarly, if $j \ge i+1$ then pick a vertex w in $C_j \cup S_j$ that has x as a neighbour and a vertex $v_i \in S_i$: we can find a copy of T_{r-1} that, for each $l \ne i$, contains a vertex $v_l \ne w$ in $C_l \cup S_l$ that is adjacent to x. Once again, we have a subdivision of T_{r+1} with vertices $v_1, \dots, v_{j-1}, x, v_j, \dots, v_r$, with a path v_iwx from v_i to x and all other edges present. Thus $j = i-1$ or $j = i$, and in particular edges are oriented from $C_l \cup S_l$ to X_i for $l < i$ and from X_i to $C_l \cup S_l$ for $l > i$.

If $X_i \cup C_i \cup S_i$ contains an edge vw then, since v and w are both adjacent to all but at most $3r^4\epsilon n/r$ vertices in $C_j \cup S_j$ for each $j \ne i$, we can find a copy of T_{r-1} with vertices $s_j \in C_j \cup S_j$ for each $j \ne i$, among the common neighbours of v and w: adding v and w gives a copy of T_{r+1}. Thus we may assume that $X_i \cup C_i \cup S_i$ contains no edges. If there is an edge x_jx_i where $i < j$ and $x_i \in X_i$, $x_j \in X_j$, then pick a vertex $v_i \in C_i \cup S_i$ that has x_j as a neighbour. Among the common neighbours of x_i and v_i we can find a copy of T_{r-1} that does not contain x_j and has a vertex in $C_l \cup S_l$ for each $l \ne i$. Adding v_i and w_i gives a subdivision of T_{r+1} with a path $v_ix_jx_i$ from v_i to x_i and all other edges present.

We have shown that $X_i \cup C_i \cup S_i$ contains no edges and, for $i < j$, edges are oriented from $X_i \cup C_i \cup S_i$ to $X_j \cup C_j \cup S_j$. Now each vertex v in X_0 has at most $2\epsilon n$ neighbours in one class $C_i \cup S_i$ and at most $(1-2r^4\epsilon)n/r$ neighbours in some other class. Furthermore, v has at most one double edge to any class, or else we obtain a directed path between two vertices in $C_i \cup S_i$. Thus v has degree less than $(1-\frac{1}{r}-r^3\epsilon)n$. If X_0 is nonempty then, deleting all edges incident with vertices in X_0 and adding edges from every vertex in X_0 to every vertex in $\bigcup_{i=2}^{r}(X_i \cup C_i \cup S_i)$ gives a graph with more edges than D and no subdivision of T_{r+1}. Thus we must have $X_0 = \emptyset$. Finally, since we can now see that our graph is a subgraph of a complete r-partite graph, it follows that D is a Turán graph with vertex classes $W_1, \ldots, W_r$, say, and edges oriented from W_i to W_j for $1 \leq i < j \leq r$.

□

References

[1] B. Bollobás, *Extremal Graph Theory*, Academic Press, New York, 1978

[2] B. Bollobás, P. Erdős, M. Simonovits and E. Szemerédi, Extremal graphs without large forbidden subgraphs, in Advances in Graph Theory, B. Bollobás, ed., North-Holland, 1978, 29–41

[3] B. Bollobás and A.G. Thomason, Highly linked graphs, *Combinatorica* **16** (96), 313–320

[4] B. Bollobás and A.G. Thomason, Proof of a conjecturte of Mader, Erdős and Hajnal on topological complete subgraphs, *European J. Comb.*, *to appear*

[5] C. Jagger, Extremal digraph results for topological complete tournaments, *European J. Comb.* **19** (98), 687–694

[6] J. Komlós and E. Szemerédi, Topological cliques in graphs, II, *Combin., Prob. and Comput.* **5** (96), 79–90

[7] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, *Math. Annalen* **174** (67), 265–268

Department of Mathematics, University College London, Gower Street, London WC1E 6BT, England

E-mail address: scott@math.ucl.ac.uk