Every tree contains a large induced subgraph with all degrees odd

A.J. Radcliffe

Carnegie Mellon University, Pittsburgh, PA

A.D. Scott

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge, England

Abstract: Caro, Krasikov and Roditty [3] proved that every tree of order n contains an induced subgraph of order at least $\lceil \frac{n}{2} \rceil$ with all degrees odd, and conjectured a better bound. In this note we prove that every tree of order n contains an induced subgraph of order at least $2\lfloor \frac{n+1}{3} \rfloor$ with all degrees odd; this bound is best possible for every value of n.

Gallai (see [4], §5 Problem 17) proved that we can partition the vertices of any graph into two sets, each of which induces a subgraph with all degrees even; we can also partition the vertices into two sets so that one set induces a subgraph with all degrees even and the other induces a subgraph with all degrees odd. As an immediate consequence of this, we see that every graph of order n contains an induced subgraph of order at least $\lceil n/2 \rceil$ with all degrees even.

It is natural to ask whether we can partition every graph into induced subgraphs with odd degrees, but this turns out not to be possible (consider, for instance, C_3). However, the results for induced subgraphs with even degrees suggest the following conjecture, the origin of which is unclear (see [2]).

Conjecture. *There exists $\epsilon > 0$ such that every connected graph G contains some $W \subset V(G)$ with $|W| \geq \epsilon|G|$ such that the graph induced by W has all degrees odd.*

Caro [2] proved that we can demand $|W| \geq c\sqrt{|G|}$, and Scott [5] proved that we can get $|W| \geq c|G|/\log(|G|)$. If the conjecture is true, then an example of Caro shows that we must have $\epsilon \leq \frac{2}{7}$. This can be seen by considering $\mathbb{Z}_7$ with each i joined to $i \pm 1$ and $i \pm 2$.

The conjecture can be proved for some special classes of graph. In particular, Caro, Krasikov and Roditty [3] showed that for trees we can take $|W| \geq \lceil |V(G)|/2 \rceil$, and conjectured a better bound in a slightly incorrect form. The result of this paper is the following best possible bound suggested by B. Bollobás (we use standard notation – see [1]).

Theorem. *Let T be a tree of order n. There is a set $S \subset V(T)$ such that*

$$|S| \geq 2 \left\lfloor \frac{n+1}{3} \right\rfloor$$

and $|\Gamma(x) \cap S|$ is odd for every $x \in S$. This bound is best possible for all n.

Remark. In trees, an induced subgraph with even degrees is exactly the same as an independent set, since any nonempty subgraph of a tree must contain a vertex of degree 1. Since every tree is bipartite, it is obvious that we can always find an independent set of size $\lceil |T|/2 \rceil$; this is easily seen to be best possible by considering any path.

Proof. We note first that if the first part of the theorem is true then it is best possible for all n, as can be seen by considering P_n, the path on n vertices. Now suppose that the theorem is false, and let T be a smallest counterexample. Trivially $|T| > 2$, so $\mathrm{diam}(T) \geq 2$. If $\mathrm{diam}(T) = 2$ then T is a star, in which case one of $V(T)$ or $V(T) \setminus \{v\}$, where v is any endvertex of T, will do for S. If $\mathrm{diam}(T) = 3$ then T consists of two stars with their centres joined by an edge, and is easily seen to satisfy the theorem. Thus we may assume that $\mathrm{diam}(T) \geq 4$.

Let W_0 be the set of endvertices of T, W_1 the set of endvertices of $T \setminus W_0$ and W_2 the set of endvertices of $T \setminus (W_0 \cup W_1)$. We write $\Gamma_i(v)$ for $\Gamma(v) \cap W_i$ and $d_i(v)$ for $|\Gamma_i(v)|$, where $i = 0, 1, 2$. Note that W_2 is non-empty, since $\mathrm{diam}(T) \geq 4$. Also, $d_0(v) > 0$ if $v \in W_1$ and $d_1(v) > 0$ if $v \in W_2$.

If $S \subset V(T)$ induces a graph with all degrees odd, then $|S|$ must be even. Thus if we want to prove that we have chosen S such that $|S| \geq 2\lfloor(|T| + 1)/3\rfloor$, it is in fact enough to prove that $|S| \geq (2|T| - 2)/3$. Let us define

$$f(n) = \frac{2n - 2}{3}.$$

If T' is any tree then we say that $S' \subset V(T')$ has *odd degrees in* T' if $|\Gamma(x) \cap S'|$ is odd for every x in S'; we say that S' is *good in* T' if S' has odd degrees in T' and $|S'| \geq f(|T'|)$. It is enough to prove that for every tree T' there is some $S' \subset V(T')$ that is good in T'.

We shall use the following method repeatedly. We pick $V_0 \subset V(T)$ so that $T' = T \setminus V_0$ is connected. Then, since T is minimal, we can find some set $S' \subset V(T')$ that is good in T'. If we can now find $S_0 \subset V(T) \setminus S'$ such that $S = S' \cup S_0$ has odd degrees in T and $|S_0| \geq 2|V_0|/3$, then S is good in T, since $|S| \geq f(|T'|) + 2|V_0|/3 = f(|T|)$.

This method is used to prove the next three lemmas. Each lemma limits the structure of T by eliminating configurations that would enable us to exhibit a subset $S \subset V(T)$ good in T. We successively refine our understanding of the structure of the minimal counterexample T until we are ready to show that no such T can exist, thus establishing the theorem.

Lemma 1. *If $x \in W_2$ then $d_0(x) = 0$.*

Proof. Suppose $x \in W_2$ and $d_0(x) > 0$, say $v \in \Gamma_0(x)$. Let w be any vertex in $\Gamma_1(x)$.

If $d_0(w) = 1$, say $\Gamma_0(w) = \{y\}$, then consider $T' = T \setminus \{v, w, y\}$. Since $|T'| < |T|$, and T' is connected, we can find $S' \subset V(T')$ that is good in T'. If $x \in S'$ then let

$S = S' \cup \{v, w\}$; if $x \notin S'$ then let $S = S' \cup \{w, y\}$. In both cases, S has odd degrees in T, and $|S| = |S'| + 2 \geq f(|T'|) + 2 = f(|T|)$, so S is good in T.

If $d_0(w) > 1$ then pick $y, z \in \Gamma_0(w)$. Consider $T' = T \setminus \{v, y, z\}$. We can find some $S' \subset V(T')$ that is good in T'. Then if $w \in S'$ let $S = S' \cup \{y, z\}$; if $w \notin S'$ and $x \in S'$ then let $S = S' \cup \{v, w\}$; and if $w \notin S'$ and $x \notin S'$ then let $S = S' \cup \{w, y\}$. In each case it is easily seen that S is good in T.

Thus if $x \in W_2$ we must have $d_0(x) = 0$. ■

Lemma 2. *If $x \in W_2$ then $d_1(x) = 1$; and if $\Gamma_1(x) = \{v\}$, say, then $d_0(v) = 2$.*

Proof. Let x be any vertex in W_2. We know from Lemma 1 that $d_0(x) = 0$. Suppose first that $d_1(x) > 1$.

If $d_0(v) = 1$ for every v in $\Gamma_1(x)$ then let $V_0 = \{x\} \cup \Gamma_1(x) \cup \{\Gamma_0(v) : v \in \Gamma_1(x)\}$, so $|V_0| = 2d_1(x) + 1 \geq 5$. We can find S' good in $T \setminus V_0$. Setting $S_0 = V_0 \setminus \{x\}$, we find that $S = S' \cup S_0$ has odd degrees in T and $|S_0| = |V_0| - 1 > 2|V_0|/3$, since $|V_0| \geq 5$, so S is good in T.

Thus if $d_1(x) > 1$ we cannot have $d_0(v) = 1$ for every v in $\Gamma_1(x)$. Pick two vertices v, w in $\Gamma_1(x)$ to maximise $d_0(v) + d_0(w)$.

If $d_0(v) + d_0(w) = 3$, say $\Gamma_0(v) = \{y_0\}$ and $\Gamma_0(w) = \{y_1, y_2\}$, then set $V_0 = \{v, w, y_0, y_1, y_2\}$ and $T' = T \setminus V_0$. We can find S' good in T'. If $x \in S'$ then let $S_0 = V_0 \setminus \{y_0\}$; if $x \notin S'$ then let $S_0 = V_0 \setminus \{y_1\}$. In either case, $S = S' \cup S_0$ has odd degrees in T and $|S| = |S'| + 4 \geq f(|T| - 5) + 4 > f(|T|)$, so S is good in T.

If $d_0(v) + d_0(w) \geq 4$ then pick $y_0 \in \Gamma_0(v)$ and $y_1 \in \Gamma_0(w)$. Let $V_0 = \{v, w\} \cup \Gamma_0(v) \cup \Gamma_0(w)$ and $T' = T \setminus V_0$. We can find S' good in T'. Let $S_0 = V_0 \setminus Y$, where Y is some subset of $\{y_1, y_2\}$ chosen to ensure that $|\Gamma(v) \cap (S' \cup S_0)|$ and $|\Gamma(w) \cap (S' \cup S_0)|$ are odd. Then $S = S' \cup S_0$ has odd degrees in T, and $|V_0| \geq 6$, so $|S| \geq f(|T| - |V_0|) + |V_0| - 2 = f(|T|) + |V_0|/3 - 2 \geq f(|T|)$. Therefore S is good in T.

We have proved that we must have $d_1(x) = 1$, say $\Gamma_1(x) = \{v\}$. Now suppose that $d_0(v) \neq 2$. Let $V_0 = \{v, x\} \cup \Gamma_0(v)$ and $T' = T \setminus V_0$; we can find S' good in T'. If $d_0(v)$ is odd then take $S_0 = V_0 \setminus \{x\}$ and if $d_0(v)$ is even take $S_0 = V_0 \setminus \{x, y\}$, where y is any element of $\Gamma_0(v)$. In either case $|S_0| \geq 2|V_0|/3$, and we see that $S = S' \cup S_0$ is good in T.

Thus we have shown that $\Gamma_1(x) = \{v\}$, for some v, and $d_0(v) = 2$. ■

The final lemma, which follows, and the proof of the theorem each proceed by considering a longest path in T. Let $x_0, \dots, x_m$ be such a path, where $m = \mathrm{diam}(T) \geq 4$. Note that $x_i \in W_i$ for $i = 0, 1, 2$. The following lemma limits the possibilities for the neighbours of x_3.

Lemma 3. *The vertex x_3 satisfies $d_0(x_3) = 0$.*

Proof. Suppose that $d_0(x_3) > 0$, say $v \in \Gamma_0(x_3)$. We know by Lemma 1 that $d_0(x_2) = 0$, and by Lemma 2 we know that $\Gamma_1(x_2) = \{x_1\}$ and $\Gamma_0(x_1) = \{y, z\}$, say. Let $T' = T \setminus \{v, y, z\}$. We can find some S' good in T'. If $x_1 \in S'$ then let $S = S' \cup \{y, z\}$; if $x_1 \notin S'$ and $x_2 \in S'$ then $x_3 \in S'$, so let $S = (S' \setminus \{x_2\}) \cup \{v, x_1, y\}$; if $x_1 \notin S'$ and $x_2 \notin S'$ then let $S = S' \cup \{x_1, y\}$. In each case S is good in T. ■

We are now ready to complete the proof of the theorem. We claim that by deleting x_3, and taking from each resulting component a large set that has odd degrees in that component (and thus in T), we can find a set that is good in T. Now $T \setminus \{x_3\}$ has $d(x_3)$ components, say $T_1, \dots, T_k$. It is enough to find sets $S_i \subset V(T_i)$ such that S_i has odd degrees in T_i, $i = 1, \dots, k$, and $\sum_{i=1}^k |S_i| \geq f(|T|)$, for then $S = \bigcup_1^k S_i$ is good in T.

Each component of $T \setminus \{x_3\}$ contains one neighbour of x_3. We may assume that T_1 contains x_4 and T_2 contains x_2. The remaining components are all stars. Indeed, we know from Lemma 3 that $d_0(x_3) = 0$, thus each remaining T_i contains a vertex from $\Gamma_1(x_3) \cup \Gamma_2(x_3)$. If T_i contains $v \in \Gamma_1(x_3)$ it is clearly a star; if T_i contains $v \in \Gamma_2(x_3)$ then Lemma 2 tells us that it must be a star. It is then trivial to verify that, since T_i is a star, we can find $S_i \subset V(T_i)$ having odd degrees in T_i such that $|S_i| \geq 2|T_i|/3$, for $i > 2$. Now we can find S_1 good in T_1, and by Lemma 2 we have that T_2 has all degrees odd, so setting $S_2 = V(T_2)$ we have

$$\begin{aligned}\sum_{i=1}^k |S_i| &\geq |S_1| + 4 + \sum_{i>2} |S_i| \\ &\geq f(|T_1|) + 4 + \frac{2}{3}(|T| - |T_1| - 5) \\ &> f(|T_1|) + \frac{2}{3}(|T| - |T_1|) \\ &= f(|T|).\end{aligned}$$

Thus if we set $S = \bigcup_{i=1}^k S_i$, then S is good in T. This contradicts the supposition that T contains no good set, thereby establishing the theorem. ■

Let us note that there are many extremal graphs for the theorem. Indeed, let $P_1, \dots, P_k$ be paths, with $|P_i| \equiv 1 \bmod 3$. Then the tree T obtained by taking an endvertex x_i in P_i for each i and then identifying $x_1 \dots x_k$ gives equality in the theorem.

Acknowledgment. *ADS would like to thank Dr Bollobás for his helpful advice.*

References

[1] B. Bollobás, Graph Theory, An Introductory Course (Springer-Verlag, New York, Heidelberg, Berlin, 1979).

[2] Y. Caro, *On induced subgraphs with odd degrees*, Discrete Mathematics. To appear.

[3] Y. Caro, I. Krasikov and Y. Roditty, *On induced subgraphs of trees with restricted degrees*. To appear.

[4] L. Lovász, Combinatorial Problems and Exercises (North-Holland, Amsterdam, 1979).

[5] A.D. Scott, *Large induced subgraphs with all degrees odd*, Combinatorics, Probability and Computing. To appear.