

Trees and linear anticomplete pairs

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Abstract

We prove a conjecture of Liebenau, Pilipczuk, and the last two authors [8], that for every forest H there exists $\varepsilon > 0$, such that if G has $n \geq 2$ vertices and does not contain H as an induced subgraph, then either

- some vertex has degree at most εn ; or
- there are two disjoint sets $A, B \subseteq V(G)$ with $|A|, |B| \geq \varepsilon n$, such that there are no edges between A, B .

(It is known that no graphs H except forests have this property.) Consequently we prove that for every forest H , there exists $c > 0$ such that for every graph G containing neither H nor its complement as an induced subgraph, there is a clique or stable set of cardinality at least $|V(G)|^c$.

1 Introduction

All graphs in this paper are finite and have no loops or parallel edges. If G, H are graphs, we say G *contains* H if some induced subgraph of G is isomorphic to H , and G is H -free otherwise. We denote by $\alpha(G), \omega(G)$ denote the cardinalities of the largest stable sets and largest cliques in G respectively. Two disjoint sets A, B are *complete* if every vertex in A is adjacent to every vertex in B , and *anticomplete* if no vertex in A has a neighbour in B ; and we say A *covers* B if every vertex in B has a neighbour in A .

The Erdős-Hajnal conjecture [6, 7] asserts that:

1.1 Conjecture: *For every graph H , there exists $c > 0$ such that every H -free graph G satisfies*

$$\alpha(G)\omega(G) \geq |V(G)|^c.$$

One way to try to prove this for appropriate graphs H might be to prove a stronger property, that in every H -free graph there are two disjoint sets of vertices, both of linear size, and complete or anticomplete to each other; and then deduce 1.1 by applying induction to the subgraphs induced on these two sets. Unfortunately this is not true, except for some graphs H with at most four vertices. This method is much more useful if we exclude two graphs H_1, H_2 rather than one; or just exclude a graph H and its complement \overline{H} . For instance, Bousquet, Lagoutte and Thomassé [2] proved that for every path H , every graph which is both H -free and \overline{H} -free (with at least two vertices) does have two linear sets, complete or anticomplete. This was extended by Choromanski, Falik, Liebenau, Patel, and Pilipczuk [3], who proved the same when H is a path with a leaf added adjacent to the third vertex.

Which other graphs have this property? It was proved in [8] that “subdivided caterpillars” have the property; and in the reverse direction, that if H has this property then one of H, \overline{H} is a forest (this follows easily from the random construction by Erdős of graphs with large girth and large chromatic number [5]). They conjectured that this was the characterization, that in fact all forests (and hence their complements) do have the property. That conjecture is a consequence of our main result:

1.2 *For every forest H , there exists $\varepsilon > 0$ such that for every graph G that is both H -free and \overline{H} -free with $n \geq 2$ vertices, there exist disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq \varepsilon n$, complete or anticomplete.*

It follows that

1.3 *For every forest H , there exists $c > 0$ such that every graph G that is both H -free and \overline{H} -free satisfies*

$$\alpha(G)\omega(G) \geq |V(G)|^c.$$

By a theorem of Rödl [9], in order to prove 1.2 in general, it is enough to prove it for “sparse” graphs G , graphs G with n vertices and with maximum degree at most cn (for any convenient constant c); this argument is given in [8]. But for sparse graphs, a stronger statement is true (again, a conjecture of [8]), that we do not need to exclude \overline{H} , and the complete option is no longer needed:

1.4 *For every forest H there exists $\varepsilon > 0$ such that for every H -free graph G with $n \geq 2$ vertices, either*

- some vertex has degree at least εn ; or
- there exist disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq \varepsilon n$, anticomplete.

(And once again, no non-forest H has this property, by the same construction of Erdős.) This is our main result; the derivations of 1.2 and 1.3 from 1.4 are given in [8]. The proof of 1.4 is given at the end of section 6.

Let us say a graph G is ε -coherent, where $\varepsilon > 0$, if (where $n = |V(G)|$):

- $n > 1$;
- every vertex has degree less than εn ; and
- there do not exist disjoint anticomplete subsets $A, B \subseteq V(G)$ with $|A|, |B| \geq \varepsilon n$.

(It follows easily that $n > \varepsilon^{-1}$.) Thus, 1.4 is the assertion that for every forest H , there exists $\varepsilon > 0$ such that every ε -coherent graph contains H .

We mention two other papers on ε -coherence. First, Bonamy, Bousquet and Thomassé [1] proved that for every k , there exists $\varepsilon > 0$ such that every ε -coherent graph has an induced cycle of length at least k ; and second, we proved in [4] that for any graph H there exists ε such that every ε -coherent graph has an induced subgraph that is a subdivision of H .

A *blockade* in G means a sequence (B_1, \dots, B_k) of pairwise disjoint subsets of $V(G)$, all with the same cardinality. Its *length* is k , and $|B_1|/|V(G)|$ is its *width*. We call the sets B_i *blocks* of the blockade. We are interested in blockades of some fixed length and width, independent of $|V(G)|$; thus each block will contain linearly many vertices of G .

Here are two useful ways to make smaller blockades from larger. First, if $\mathcal{B} = (B_1, \dots, B_K)$ is a blockade, let $1 \leq r_1 < r_2 < \dots < r_k \leq K$; then $(B_{r_1}, \dots, B_{r_k})$ is a blockade, of smaller length but of the same width, and we call it a *sub-blockade* of \mathcal{B} . Second, for $1 \leq i \leq K$ let $B'_i \subseteq B_i$, all the same cardinality; then the sequence (B'_1, \dots, B'_K) is a blockade, of the same length but of smaller width, and we call it a *contraction* of \mathcal{B} . A contraction of a sub-blockade (or equivalently, a sub-blockade of a contraction) we call a *minor* of \mathcal{B} .

A minor $(B'_{r_1}, \dots, B'_{r_k})$ of a blockade $\mathcal{B} = (B_1, \dots, B_k)$ is *matching-covered* (in \mathcal{B}) if for each $i \in \{1, \dots, k\}$ there exist $s_i \in \{1, \dots, K\} \setminus \{r_1, \dots, r_k\}$ and $X_{s_i} \subseteq B_{s_i}$ such that X_{s_i} covers B'_{r_i} and X_{s_i} is anticomplete to B'_{r_j} for all $j \in \{1, \dots, k\} \setminus \{i\}$.

A substantial part of the proof of 1.4 is the proof of a theorem that says nothing about being ε -coherent; it (theorem 4.3) says that, given a blockade \mathcal{B} of sufficient length in any graph G , either \mathcal{B} has a matching-covered minor of any desired length, or the graph contains another blockade which is highly uniform in several ways, and in particular has a certain concavity property, and in both cases the width of the new blockade is at least a constant fraction of the width of the old. (The details are too technical to state more precisely here.) If the first always happens then by applying 4.3 recursively we can prove any desired tree T is present (theorem 6.1); and if ever the second happens, then we prove directly (theorem 5.1) that G contains T anyway, assuming G is ε -coherent for small enough ε (depending on T but not on G). Thus both outcomes lead to the conclusion that G contains T , provided that G is ε -coherent for small enough ε .

2 Contraction-invariant blockades

Let $\mathcal{B} = (B_1, \dots, B_k)$ be a blockade, and let $0 \leq \lambda \leq 1/2$. For $1 \leq i \leq k$, a subset X of $V(G) \setminus B_i$ is said to λ -cover B_i if there are at least $\lambda|B_i|$ vertices in B_i with a neighbour in X , and to λ -miss B_i if there are at least $\lambda|B_i|$ vertices in B_i with no neighbour in X . Since $\lambda \leq 1/2$, X either λ -covers B_i or λ -misses B_i , but it might do both.

Let $1 \leq i \leq k$, and let (H, J) be a pair of subsets of $\{1, \dots, k\} \setminus \{i\}$, and let $X \subseteq B_i$. We say X λ -realizes (H, J) (relative to \mathcal{B}) if X λ -covers B_j for each $j \in H$, and X λ -misses B_j for each $j \in J$. The set of all pairs (H, J) that are λ -realized by some subset of B_i is called the λ -pattern of B_i (relative to \mathcal{B}); and if Π_i denotes the λ -pattern of B_i , the sequence (Π_1, \dots, Π_k) is called the λ -spectrum of \mathcal{B} . Let us say the λ -covering-cost of \mathcal{B} is the sum over $1 \leq i \leq k$ of the cardinality of the λ -pattern of B_i . Since there are only 2^{2K} pairs (H, J) of subsets of $\{1, \dots, K\}$, the λ -covering-cost of \mathcal{B} is at most $K2^{2K}$.

2.1 Let $0 < \lambda \leq \mu \leq 1/2$, let $\mathcal{B} = (B_1, \dots, B_k)$ be a blockade in G , and let $\mathcal{B}' = (B'_1, \dots, B'_k)$ be a contraction, where $\mu|B'_i| \geq \lambda|B_i|$ for $1 \leq i \leq k$. Then for $1 \leq i \leq k$, if $X \subseteq B'_i$ μ -realizes (H, J) relative to \mathcal{B}' , then X λ -realizes (H, J) relative to \mathcal{B} . Consequently the λ -pattern of B_i relative to \mathcal{B} is a subset of the μ -pattern of B'_i relative to \mathcal{B}' , and so the μ -covering-cost of \mathcal{B}' is at most the λ -covering-cost of \mathcal{B} .

Proof. We assume that $X \subseteq B'_i$ is a set that μ -realizes (H, J) relative to \mathcal{B}' . Thus for each $j \in H$, X μ -covers B'_j , that is, there are at least $\mu|B'_j|$ vertices in B'_j with a neighbour in X . But all these vertices belong to B_j , and since $\mu|B'_j| \geq \lambda|B_j|$, there are at least $\lambda|B_j|$ such vertices, and so X λ -covers B_j . Similarly X λ -misses B_j for each $j \in J$, and so X λ -realizes (H, J) relative to \mathcal{B} . This proves the first assertion, and the other two follow immediately. This proves 2.1. \blacksquare

Let $0 < \lambda < \mu \leq 1/2$, and let $\mathcal{B} = (B_1, \dots, B_K)$ be a blockade in a graph G . We say \mathcal{B} is (λ, μ) -cover-invariant if it has the following property: for every contraction $\mathcal{B}' = (B'_1, \dots, B'_K)$ of \mathcal{B} such that $\mu|B'_i| \geq \lambda|B_i|$ for $1 \leq i \leq K$, the λ -spectrum of \mathcal{B} equals the μ -spectrum of \mathcal{B}' .

A rooted graph H is a pair $(H^-, r(H))$, where H^- is a graph and $r(H) \in V(H^-)$; we call $r(H)$ the root. If H_1, H_2 are rooted graphs, by an isomorphism between them we mean an isomorphism between H_1^- and H_2^- that takes root to root; and we say H_1 is an induced subgraph of H_2 if $r(H_1) = r(H_2)$ and H_1^- is an induced subgraph of H_2^- . If one of H_1, H_2 is a rooted graph and the other is a graph (unrooted), we say H_1 is an induced subgraph of H_2 if this is true ignoring the root. We shall normally not bother to distinguish H and H^- in what follows.

If $\delta \geq 2$ is an integer, let $T(\delta, 0)$ be the rooted tree with one vertex (thus, δ is irrelevant, but this will be convenient). If $\eta \geq 1$ and $\delta \geq 2$ are integers, we denote by $T(\delta, \eta)$ the rooted tree with the properties that

- every vertex has degree $\delta + 1$ or 1, except the root, which has degree δ ; and
- for each vertex of degree one, its distance from the root is exactly η .

Thus for $\eta \geq 1$, $T(\delta, \eta)$ is formed by taking the disjoint union of δ copies of $T(\delta, \eta - 1)$, and adding a new vertex adjacent to all the roots, and making this vertex the new root.

Let $\mathcal{B} = (B_1, \dots, B_k)$ be a blockade in G . We say an induced subgraph H of G is *rainbow* relative to \mathcal{B} if each vertex of H belongs to some block of \mathcal{B} , and no two vertices belong to the same block. An induced rooted subgraph H of G is *left-rainbow* relative to \mathcal{B} if

- it is rainbow relative to \mathcal{B} ; and
- if the root of H belongs to B_i , then $i \leq j$ for all $j \in \{1, \dots, k\}$ with $V(H) \cap B_j \neq \emptyset$.

We define *right-rainbow* similarly, requiring $i \geq j$ instead.

Let $\mathcal{B} = (B_1, \dots, B_k)$ be a blockade in G . If H is an induced subgraph that is rainbow relative to \mathcal{B} , its *support* is the set of all $i \in \{1, \dots, k\}$ such that $V(H) \cap B_i \neq \emptyset$. Let $\delta \geq 2$ and $\eta \geq 0$ be integers, fixed for the remainder of this section. For each rooted subtree T of $T(\delta, \eta)$, we define the *left-trace* of T to be the set of supports of all rooted induced subgraphs of G that are isomorphic to T and left-rainbow relative to \mathcal{B} . We define the *right-trace* similarly. We define the *trace-cost* of \mathcal{B} to be the sum, over all rooted subtrees T of $T(\delta, \eta)$, of the sum of the cardinalities of the left-trace and right-trace of T .

The cardinality of the left-trace of any given rooted subtree T of $T(\delta, \eta)$ is at most 2^K , and since $T(\delta, \eta)$ has at most $\delta^{\eta+1} - 1$ vertices, it has at most $2^{\delta^{\eta+1} - 1}$ rooted subtrees. Hence the trace-cost of \mathcal{B} is at most $2^{K + \delta^{\eta+1}}$. We define the λ -cost of \mathcal{B} (with λ as before) to be the sum of the λ -covering-cost and the trace-cost. Thus the λ -cost is at most $K2^{2K} + 2^{K + \delta^{\eta+1}}$.

Let $0 < \kappa \leq 1$, and let $\mathcal{B} = (B_1, \dots, B_K)$ be a blockade in a graph G . We say \mathcal{B} is κ -*support-invariant* if it has the following property: for every contraction $\mathcal{B}' = (B'_1, \dots, B'_K)$ of \mathcal{B} such that $|B'_i| \geq \kappa|B_i|$ for $1 \leq i \leq K$,

- for every rooted subtree T of $T(\delta, \eta)$, the left-trace of T relative to \mathcal{B} equals the left-trace of T relative to \mathcal{B}' ; and
- the same for right-trace.

2.2 Let $K \geq 0$ and $m \geq 1$ be integers, and let $0 < \mu_0 < 1/2$. Let $\nu = K2^{2K} + 2^{K + \delta^{\eta+1}}$, and let $c = \mu_0^{m^{\nu+1}}$. Let $\mathcal{B} = (B_1, \dots, B_K)$ be a blockade in a graph G . Then there is a contraction $\mathcal{B}' = (B'_1, \dots, B'_K)$ of \mathcal{B} , and λ, μ with $\lambda = \mu^m$ and $c \leq \lambda < \mu \leq \mu_0$, such that

- $|B'_i| \geq c|B_i|$ for $1 \leq i \leq K$;
- \mathcal{B}' is (λ, μ) -cover-invariant; and
- \mathcal{B}' is λ/μ -support-invariant.

Proof. The blockade \mathcal{B} has c -cost at most ν . Choose an integer $t \geq 0$ maximum such that there is a contraction $\mathcal{B}' = (B'_1, \dots, B'_K)$ of \mathcal{B} with $|B'_i| \geq c^{m^{-t}}|B_i|$ for $1 \leq i \leq k$, and with $c^{m^{-t}}$ -cost at most $\nu - t$. Define $\lambda = c^{m^{-t}}$ and $\mu = \lambda^{1/m} = c^{m^{-t-1}}$. Since this $c^{m^{-t}}$ -cost is nonnegative, it follows that $t \leq \nu$. Since $\mu = c^{m^{-t-1}} \leq c^{m^{-\nu-1}} = \mu_0$, it follows that $c \leq \lambda < \mu \leq \mu_0$. Let $\mathcal{B}'' = (B''_1, \dots, B''_K)$ be a contraction of \mathcal{B}' such that $\mu|B''_i| \geq \lambda|B'_i|$ for $1 \leq i \leq K$. By 2.1 the μ -covering-cost of \mathcal{B}'' is at most the λ -covering-cost of \mathcal{B}' , and the trace-cost of \mathcal{B}'' is at most the trace-cost of \mathcal{B}' . But from the maximality of t , the λ -cost of \mathcal{B}'' is at least that of \mathcal{B}' , and so we have equality throughout. This proves 2.2. ■

3 Ramsey versus blockades

We have done what we can to make blockades nicer by adjusting λ and moving to contractions, but there is another method to make blockades nicer: move to sub-blockades. This is compatible with what we did earlier, because if a blockade is (λ, μ) -cover-invariant, then so are all its sub-blockades, and the same for support-invariance. Again, we keep δ, η fixed throughout this section. By a *copy* of a graph T in G we mean an induced subgraph of G isomorphic to T . Let $\mathcal{B} = (B_1, \dots, B_K)$ be a blockade in G . We say \mathcal{B} is *support-uniform* if

- for every rooted subtree T of (δ, η) , if the left-trace of T (relative to \mathcal{B}) is nonempty then it consists of all subsets of $\{1, \dots, K\}$ of cardinality $|T|$; and
- the same for right-trace.

Let $0 \leq \lambda \leq 1/2$, and let (B_1, \dots, B_k) and (B'_1, \dots, B'_k) be blockades in G , of the same length k . We say they are λ -*cospectral* if their λ -spectra are equal. Now let $0 < \lambda \leq 1/2$, let $d \geq 0$ be an integer, and let $\mathcal{B} = (B_1, \dots, B_K)$ be a blockade in G ; we say \mathcal{B} is (λ, d) -*cover-uniform* if all its sub-blockades of length d are λ -cospectral.

By many iterated applications of Ramsey's theorem for uniform hypergraphs (one for each rooted subtree T of $T(\delta, \eta)$ for its left-trace; one more for each such T for its right-trace; and one more to make all the sub-blockades of length d λ -cospectral), we deduce:

3.1 *Let $k, d \geq 0$ be integers; then there exists an integer $K \geq 0$ with the following property. Let $\mathcal{B} = (B_1, \dots, B_K)$ be a blockade in a graph G , and let $0 < \lambda \leq 1/2$. Then \mathcal{B} has a sub-blockade of length k which is support-uniform and (λ, d) -cover-uniform.*

Combining 2.2 and 3.1, and taking $d = 2m + 2$, we obtain:

3.2 *Let $k \geq 0$ and $m \geq 1$ be integers, and $0 < \mu_0 \leq 1/2$; then there exist an integer K and $0 < c < 1$ with the following property. Let $\mathcal{B} = (B_1, \dots, B_K)$ be a blockade in a graph G of width W . Then there is a minor \mathcal{B}' of \mathcal{B} , and λ with $c \leq \lambda < \mu \leq \mu_0$, where $\lambda = \mu^m$, such that*

- \mathcal{B}' has length k and width at least cW ;
- \mathcal{B}' is (λ, μ) -cover-invariant and λ/μ -support-invariant; and
- \mathcal{B}' is support-uniform and $(\lambda, 2m + 2)$ -cover-uniform.

Proof. Let K satisfy 3.1 with d replaced by $2m + 2$, and let c satisfy

$$\log(1/c) = m^{K2^{2K} + 2^{K+\delta\eta+1} + 1} \log(1/\mu_0).$$

Then we claim they satisfy 3.2. For let $\mathcal{B} = (B_1, \dots, B_K)$ be a blockade in a graph G . By 2.2 there is a contraction $\mathcal{B}' = (B'_1, \dots, B'_K)$ of \mathcal{B} , and λ with $c \leq \lambda < \mu \leq \mu_0$, where $\lambda = \mu^m$, such that

- \mathcal{B}' has width at least cW ; and
- \mathcal{B}' is (λ, μ) -cover-invariant and λ/μ -support-invariant.

By 3.1 applied to \mathcal{B}' , the result follows, since both the properties of the bullets above are inherited under taking sub-blockades. This proves 3.2. ■

4 Concavity

The properties of the blockade produced by 3.2 are very powerful in combination, and next we see some consequences. Let \mathcal{B} be a blockade in G of length K , and let $0 < \lambda \leq 1/2$. A minor $(B'_{r_1}, \dots, B'_{r_k})$ is λ -matching-covered if for each $i \in \{1, \dots, k\}$ there exist $s_i \in \{1, \dots, K\} \setminus \{r_1, \dots, r_k\}$ and a subset X_{s_i} of B_{s_i} such that X_{s_i} λ -covers B'_{r_i} and X_{s_i} λ -misses B'_{r_j} for all $j \in \{1, \dots, k\} \setminus \{i\}$.

4.1 *Let $m > 0$ be an integer, let $0 < \mu \leq 1/2$, let $\lambda = \mu^m$, let G be a graph, and let \mathcal{B} be a blockade in G that is (λ, μ) -cover-invariant of width w . If \mathcal{B} has a λ -matching-covered submatching of length m , then \mathcal{B} has a matching-covered minor of length m and width $\geq \lambda w$.*

Proof. Let $\mathcal{B} = (B_1, \dots, B_K)$ say, and suppose that there is a λ -matching-covered submatching of length m and width w , say $(B_{r_1}, \dots, B_{r_m})$; and let s_1, \dots, s_m and X_{s_1}, \dots, X_{s_m} be as in the definition of λ -matching-covered. Without loss of generality we may assume that $r_i = i$ for $1 \leq i \leq m$, and so $s_1, \dots, s_m > m$. Let $n = |V(G)|$.

We claim that for $0 \leq i \leq m$ and $1 \leq j \leq m$, there exists $B_j^i \subseteq B_j$, and for $1 \leq i \leq m$ there exists $Y_i \subseteq B_{s_i}$, such that for $1 \leq i \leq m$:

- for $1 \leq h \leq i$, Y_h covers B_h^i ;
- for $1 \leq h \leq i$, Y_h is anticomplete to B_j^i for all $j \in \{1, \dots, m\} \setminus \{h\}$;
- the sets B_j^i ($1 \leq j \leq m$) all have the same cardinality, say $w_i n$, and $w_i \geq \mu^i w$.

This is true for $i = 0$, setting $B_j^0 = B_j$ for $1 \leq j \leq m$, so we proceed by induction on i . We assume $1 \leq i \leq m$, and B_j^1, \dots, B_j^{i-1} are defined for $1 \leq j \leq m$, and Y_1, \dots, Y_{i-1} are defined, satisfying the three bullets above. For $m < j \leq K$ choose $B_j^{i-1} \subseteq B_j$ of cardinality $w_{i-1} n$. Now the blockade $\mathcal{B}^{i-1} = (B_1^{i-1}, B_2^{i-1}, \dots, B_K^{i-1})$ has width $w_{i-1} \geq \mu^{i-1} w$; and so, since \mathcal{B} is (λ, μ) -cover-invariant, and $\mu |B_j^{i-1}| \geq \lambda |B_j|$ for each $j \leq m$, the λ -pattern of B_{s_i} relative to \mathcal{B} equals the μ -pattern of B_{s_i} relative to \mathcal{B}^{i-1} . Since the pair $(\{i\}, \{1, \dots, m\} \setminus \{i\})$ belongs to the λ -pattern of B_{s_i} relative to \mathcal{B} , it also belongs to the μ -pattern of B_{s_i} relative to \mathcal{B}^{i-1} . Consequently there exists $Y_i \subseteq B_{s_i}$ that μ -covers B_i^{i-1} and μ -misses B_j^{i-1} for $1 \leq j \leq m$ with $j \neq i$. Choose w_i such that $w_i n = \lceil \mu w_{i-1} n \rceil$. Define B_i^i to be a set of $w_i n$ vertices in B_i^{i-1} with a neighbour in Y_i ; and for $1 \leq j \leq m$ with $j \neq i$, define B_j^i to be a set of $w_i n$ vertices in B_j^{i-1} with no neighbour in Y_i . Hence $|B_j^i| = w_i n \geq \mu^i w n$, for $1 \leq j \leq m$; and for $h < i$, Y_h covers B_h^i since it covers B_h^{i-1} , and Y_h is anticomplete to B_j^i for $1 \leq j \leq m$ with $j \neq h$, since Y_h is anticomplete to B_j^{i-1} . This completes the inductive definition. But then (B_1^m, \dots, B_m^m) is a matching-covered minor of \mathcal{B} of width at least λw . This proves 4.1. \blacksquare

4.2 *Let $m > 0$ be an integer, let $0 < \lambda \leq 1/2$, let G be a graph, and let \mathcal{B} be a $(\lambda, 2m + 2)$ -cover-uniform blockade in G of length K , where $K \geq 4m + 1$. Suppose that for some $i \in \{1, \dots, K\}$, there exist $X \subseteq B_i$ and $2m + 1$ values of j different from i , say j_1, \dots, j_{2m+1} with $1 \leq j_1 < j_2 < \dots < j_{2m+1} \leq K$, such that X λ -covers $B_{j_{m+1}}$, and λ -misses B_j for all $j \in \{j_1, \dots, j_{2m+1}\} \setminus \{j_{m+1}\}$. Then \mathcal{B} has a λ -matching-covered sub-blockade of length m .*

Proof. The relative value of i compared to j_1, \dots, j_{2m+1} matters; certainly i is different from j_1, \dots, j_{2m+1} , and from the symmetry we may assume that $i > j_{m+1}$. Let s be maximum such that $j_s < i$; thus either $s = 2m + 1$, or $j_s < i < j_{s+1}$. Then:

(1) For every choice of $2m + 2$ distinct integers $r_0 < r_1 < \dots < r_{2m+1}$ between 1 and K , there is a subset of B_{r_s} that λ -covers B_{r_m} and λ -misses B_j for all $j \in \{r_0, \dots, r_{2m+1}\} \setminus \{r_m, r_s\}$.

Since \mathcal{B} is $(\lambda, 2m + 2)$ -cover-uniform, every sub-blockade of length $2m + 2$ has the same λ -spectrum, and in particular the λ -spectrum of $\mathcal{C} = (B_{r_0}, \dots, B_{r_{2m+1}})$ equals that of

$$\mathcal{C}' = (B_{j_1}, \dots, B_{j_s}, B_i, B_{j_{s+1}}, \dots, B_{j_{2m+1}}).$$

From the hypothesis, the pair

$$(\{j_{m+1}\}, \{j_1, \dots, j_m, j_{m+2}, \dots, j_{2m+1}\})$$

belongs to the λ -pattern of B_i relative to \mathcal{C}' ; and so

$$(\{r_m\}, \{r_0, \dots, r_{m-1}, r_{m+1}, \dots, r_{s-1}, r_{s+1}, \dots, r_{2m+1}\})$$

belongs to the λ -pattern of B_{j_s} relative to \mathcal{C} . This proves (1).

Take a set I of m even integers between $m + 1$ and $K - m$. Consequently, for each $i \in I$ there is a set J of $2m + 2$ distinct integers in $\{1, \dots, K\}$, say $\{j_0, \dots, j_{2m+1}\}$, numbered in increasing order, such that

- $I \subseteq J$;
- $j_m = i$;
- j_s is odd; and
- for each $j \in J$, either $j \in I$, or $j = j_s$, or j is less than each member of I , or j is greater than each member of I .

From (1) applied to J , we deduce that for each $i \in I$, some subset of some B_j (where $j \in \{1, \dots, K\} \setminus \{i\}$) λ -covers B_i and λ -misses $B_{i'}$ for each $i' \in I \setminus \{i\}$; that is, the sub-blockade formed by the blocks B_i ($i \in I$) in order is λ -matching-covered. This proves 4.2. ■

A blockade $\mathcal{B} = (B_1, \dots, B_k)$ is λ -concave if it has the following very strong property: for all i with $1 \leq i \leq k$, and for every $X \subseteq B_i$, there do not exist h_1, h_2, h_3 with $1 \leq h_1 < h_2 < h_3 \leq k$, all different from i , such that X λ -covers B_{h_2} and λ -misses B_{h_1} and B_{h_3} . We deduce:

4.3 For all integers $k \geq 0$ and $m \geq 3$, and all λ with $0 < \lambda < 1/(2m)$, there exist an integer $K \geq 0$ and $0 < c \leq 1$ with the following property. Let \mathcal{B} be a blockade of length K and width W in a graph G . Then either:

- \mathcal{B} has a matching-covered minor of length m and width at least c^2W , or

- there exists a λ -concave blockade in G of length k and width at least $2mcW/\lambda$, that is support-uniform and λ -support-invariant.

Proof. Let $r = \lceil 2m/\lambda \rceil$, and let K, c satisfy 3.2 with k, μ_0 replaced by rk, λ respectively. We claim that K satisfies 4.3. For let \mathcal{B} be a blockade of length K and width W in a graph G . By 3.2, there is a minor $\mathcal{B}' = (B'_1, \dots, B'_{rk})$ of \mathcal{B} , and λ_0 with $c \leq \lambda_0 < \mu \leq \lambda$, where $\lambda_0 = \mu^m$, such that

- \mathcal{B}' has width at least cW ;
- \mathcal{B}' is (λ_0, μ) -cover-invariant and λ_0/μ -support-invariant; and
- \mathcal{B}' is support-uniform and $(\lambda_0, 2m + 2)$ -cover-uniform.

(1) We may assume that, for $i \in \{1, \dots, rk\}$, and all $X \subseteq B'_i$ there do not exist $2m + 1$ values of j different from i , say i_0, \dots, i_{2m} with $1 \leq i_0 < i_1 < \dots < i_{2m} \leq rk$, such that X λ_0 -covers $B'_{i_{m+1}}$, and λ_0 -misses B'_j for all $j \in \{i_1, \dots, i_{2m+1}\} \setminus \{i_{m+1}\}$.

Suppose that such i_0, \dots, i_{2m} exist. Then by 4.2, \mathcal{B}' has a λ_0 -matching-covered sub-blockade of length m , and therefore of width $\geq cW$; and hence by 4.1, \mathcal{B}' has a matching-covered minor of length m and width at least $\lambda_0 cW \geq c^2W$. But this is also a minor of \mathcal{B} , and the theorem holds. This proves (1).

For $1 \leq i \leq k$, let C_i be the union of the sets B'_j for all $j \in \{(r-1)i + 1, \dots, ri\}$. Then $\mathcal{C} = (C_1, \dots, C_k)$ is a blockade, of width at least rcW . We claim it satisfies the second outcome of the theorem.

(2) \mathcal{C} is λ -concave.

First, note that since $\mu^m = \lambda_0$, and $\mu \leq \lambda$, and $m \geq 3$, and $\lambda \leq 1/(2m)$, it follows that $2m\lambda_0 \leq \lambda^2$, and so $\lambda \geq r\lambda_0$. Moreover, $\lambda \geq 2m/r$, and $\lambda \geq 2\lambda_0$, and so $\lambda \geq m/r + (1 - m/r)\lambda_0$; and it follows that $(1 - m/r)(1 - \lambda_0) > 1 - \lambda$. (We use both these facts below.) Let $1 \leq i \leq k$ and let $X \subseteq C_i$. Let $X_j^i = B'_{(r-1)i+j} \cap X$ for $1 \leq j \leq r$; thus X is the union of the sets X_j^i ($1 \leq j \leq r$). We claim that for $h_1 < h_2 < h_3 \in \{1, \dots, k\} \setminus \{i\}$, either X does not λ -miss C_{h_1} , or X does not λ -miss C_{h_3} , or X does not λ -cover C_{h_2} . Suppose then that X λ -covers C_{h_2} . Hence at least $\lambda|C_{h_2}|$ vertices in C_{h_2} have a neighbour in X , and so for one of X_1^i, \dots, X_r^i , say $X_{j_2}^i$, there are at least $\lambda|C_{h_2}|/r$ vertices in C_{h_2} that have a neighbour in $X_{j_2}^i$. Consequently for some $j_2 \in \{1, \dots, r\}$, there exist at least $\lambda|B'_{(r-1)h_2+j_2}|/r$ vertices in $B'_{(r-1)h_2+j_2}$ that have a neighbour in $X_{j_2}^i$, and so $X_{j_2}^i$ λ_0 -covers $B'_{(r-1)h_2+j_2}$, since $\lambda_0 \leq \lambda/r$. By (1), applied to $X_{j_2}^i$, either there do not exist m distinct values of $j' \in \{1, \dots, (r-1)h_2+j_2-1\}$ such that $X_{j_2}^i$ λ_0 -misses $B'_{j'}$, or there do not exist m distinct values of $j' \in \{(r-1)h_2+j_2+1, \dots, rk\}$ such that $X_{j_2}^i$ λ_0 -misses $B'_{j'}$, and from the symmetry we may assume the first. In particular, there are fewer than m values of $j' \in \{1, \dots, r\}$ such that $X_{j_2}^i$ λ_0 -misses $B'_{(r-1)h_1+j'}$, and so for at least $r - m$ values of $j' \in \{1, \dots, r\}$, $X_{j_2}^i$ does not λ_0 -miss $B'_{(r-1)h_1+j'}$, that is, there are more than $(1 - \lambda_0)|B'_{(r-1)h_1+j'}|$ vertices in $B'_{(r-1)h_1+j'}$ with a neighbour in $X_{j_2}^i$ and hence in X . Since each $B'_{(r-1)h_1+j'}$ has cardinality r^{-1} times the cardinality of C_{h_1} , it follows that there are at least $(1 - m/r)(1 - \lambda_0)|C_{h_1}|$ vertices in

C_{h_1} with a neighbour in X . Since $(1 - m/r)(1 - \lambda_0) > 1 - \lambda$, it follows that X does not λ -miss C_{h_1} . This proves (2).

(3) \mathcal{C} is support-uniform and λ -support-invariant.

The first statement is clear, and we only need prove the second. Let T be a rooted subtree of $T(\delta, \eta)$, such that there is a copy of T in G that is left-rainbow relative to \mathcal{C} , and let $t = |V(T)|$. Let \mathcal{B}' have width w , so \mathcal{C} has width rw . Let $\mathcal{C}' = (C'_{i_1}, \dots, C'_{i_t})$ be a contraction of a sub-blockade of \mathcal{C} , of width at least $\lambda rw \geq \mu^{m-1}rw$, of length t . We must show that there is a copy of T that is left-rainbow relative to \mathcal{C}' . To simplify notation we assume without loss of generality that $i_j = j$ for $1 \leq j \leq t$. Now for $1 \leq j \leq t$, $|C'_j| \geq \mu^{m-1}rwn$, where $n = |V(G)|$, and so there exists h_j with $(r-1)j+1 \leq h_j \leq rj$ such that $|C'_j \cap B'_{h_j}| \geq \mu^{m-1}|B'_{h_j}| = \mu^{m-1}wn$. Choose $B''_{h_j} \subseteq C'_j \cap B'_{h_j}$ for $1 \leq j \leq t$, all the the same cardinality, and all with cardinality at least $\mu^{m-1}wn$. Since \mathcal{B} is support-uniform and μ^{m-1} -support-invariant, it follows that there is a copy of T that is left-rainbow relative to $(B''_{h_1}, \dots, B''_{h_t})$, and hence relative to (C'_1, \dots, C'_t) . This proves (3).

From (2) and (3), this proves 4.3. ▀

This completes the first step of the programme outlined at the end of section 1. Now there are separate arguments to exploit the two possible outcomes of 4.3.

5 Using a concave blockade

Let $\mathcal{B} = (B_1, \dots, B_K)$ be a blockade in G , and let S, T be rooted subtrees of $T(\delta, \eta)$. (Again, δ, η are fixed throughout this section.) Let $\mathcal{B}' = (B'_i : i \in I)$ be a minor of \mathcal{B} (and so $B'_i \subseteq B_i$ for each $i \in I$). We say \mathcal{B}' is (S, T) -anchored in \mathcal{B} if there exist h_1, Y such that:

- h_1 is an integer with $1 \leq h_1 \leq K$ such that $I = \{1, \dots, h_1 - 1\} \cup \{K\}$;
- Y is a subset of $\bigcup_{h_1 \leq j \leq K-1} B_j$;
- Y is anticomplete to B'_i for all $i \in I - \{1, K\}$; and
- for every $v \in B'_1$ there is a copy of S in $G[Y \cup \{v\}]$ with root v , left-rainbow relative to \mathcal{B} ; and for every $v \in B'_K$ there is a copy of T in $G[Y \cup \{v\}]$ with root v , right-rainbow relative to \mathcal{B} .

5.1 *Let $0 < \lambda \leq 2^{-9\delta}\delta^{-1-\eta}$, and let $\varepsilon > 0$. Let G be an ε -coherent graph with a blockade \mathcal{B} of length at least $6\delta^{\eta+2}$ and width at least $2^{9\delta}\varepsilon$, such that \mathcal{B} is λ -concave, support-uniform and $2^{-9\delta}$ -support-invariant. Then G contains $T(\delta, \eta)$.*

Proof. Let $n = |V(G)|$. Let $\mathcal{B} = (B_1, \dots, B_K)$, and let W be its width. Choose $\alpha \geq 0$ maximum such that there is a copy of $T(\delta, \alpha)$ that is left-rainbow relative to \mathcal{B} , and define β similarly for right-rainbow. We suppose for a contradiction that G does not contain $T(\delta, \eta)$, and so $\alpha, \beta < \eta$; and by reversing the blockade if necessary we may assume that $\alpha \leq \beta$. We need three special rooted trees:

- For $0 \leq \gamma \leq \delta$, let $Q(\gamma)$ be obtained from the disjoint union of γ copies of $T(\delta, \alpha)$ by adding a new root adjacent to the old roots.
- Let $R(0)$ be the rooted tree with only one vertex; for $1 \leq \gamma \leq \delta$ let $R(\gamma) = Q(\gamma)$; and for $\delta < \gamma \leq 2\delta$ let $R(\gamma)$ be obtained from the disjoint union of $2\delta - \gamma$ copies of $T(\delta, \alpha)$ and $\gamma - \delta$ copies of $T(\delta, \beta)$ by adding a new root adjacent to all the old roots.
- For $0 \leq \gamma \leq \delta$, let $S(\gamma)$ be obtained from the disjoint union of $\gamma + 1$ copies of $T(\delta, \alpha)$ by making the root v of the first copy adjacent to all other roots, and making v the new root.

Choose $\gamma_0 \geq 0$ maximum such that there exist $\gamma_1, \gamma_2 \geq 0$ with $\gamma_1 + \gamma_2 = \gamma_0$ and a $(Q(\gamma_1), R(\gamma_2))$ -anchored minor \mathcal{B}' of \mathcal{B} of length at least $K - 2\delta^{\eta+1}\gamma_0$ and width at least $W2^{-3\gamma_0}$. (This is possible, because for $\gamma_0 = 0$ we can take $\mathcal{B}' = \mathcal{B}$ and $h_1 = 1, h_2 = K$.) Let $W' \geq W2^{-3\gamma_0}$ be its width. Choose γ_3 maximum such that there is a copy of $S(\gamma_3)$ that is left-rainbow relative to \mathcal{B} .

(1) $\gamma_1, \gamma_3 \leq \delta - 1$, and $\gamma_2 \leq 2\delta - 1$. Consequently $\gamma_0 \leq 3\delta - 2$, and so $W' \geq W2^{-9\delta+6} \geq 64\varepsilon$.

For $Q(\delta)$ is isomorphic to $T(\delta, \alpha + 1)$, and so from the choice of α , there is no copy of $Q(\delta)$ that is left-rainbow relative to \mathcal{B} . On the other hand there is a copy of $Q(\gamma_1)$ that is left-rainbow relative to \mathcal{B} , from the definition of “ $(Q(\gamma_1), R(\gamma_2))$ -anchored”; so $\gamma_1 < \delta$. Similarly $\gamma_2 < 2\delta$, since $R(2\delta)$ is isomorphic to $T(\delta, \beta + 1)$. Also $\gamma_3 < \delta$ from the maximality of α , since $S(\delta)$ contains $T(\delta, \alpha + 1)$. This proves (1).

Now \mathcal{B}' is $(Q(\gamma_1), R(\gamma_2))$ -anchored; let Y, h_1 be as in the definition of “anchored”, and let $\mathcal{B}' = (B'_i : i \in \{1, \dots, h_1 - 1\} \cup \{K\})$. Let $S(\gamma_3)$ have s vertices, and let $T(\delta, \beta)$ have t vertices. Define $h = h_1 - s - t$.

(2) $1 \leq h \leq K - 2\delta^{\eta+1}(\gamma_0 + 1)$.

Since \mathcal{B}' has length h_1 , it follows that $h_1 \geq K - 2\delta^{\eta+1}\gamma_0$. Hence $h = h_1 - s - t \geq K - 2\delta^{\eta+1}(\gamma_0 + 1)$ since $s, t \leq \delta^{\eta+1}$. Since by (1)

$$K \geq 6\delta^{\eta+2} > 2\delta^{\eta+1}(3\delta - 1) \geq 2\delta^{\eta+1}(\gamma_0 + 1),$$

it follows that $h > 0$. This proves (2).

Let $r = \lceil (W' - 2^{-9\delta}W)n \rceil$. By (1), $W' \geq W2^{-9\delta+6}$, and so $r \geq 63 \cdot 2^{-9\delta}Wn$. Since $W \geq 2^{9\delta}\varepsilon$, this implies that $r \geq 63\varepsilon n$.

(3) *There are r copies E_1, \dots, E_r of $S(\gamma_3)$, pairwise vertex-disjoint and each left-rainbow relative to $(B'_i : h \leq i \leq h + s - 1)$; and there are r copies F_1, \dots, F_r of $T(\delta, \beta)$, pairwise vertex-disjoint and right-rainbow relative to $(B'_i : h + s \leq i \leq h + s + t - 1)$.*

Since there is a copy of $S(\gamma_3)$ that is left-rainbow relative to \mathcal{B} , and \mathcal{B} is support-uniform, there is such a copy that is left-rainbow relative to $(B_i : h \leq i \leq h + s - 1)$. Choose $r' \leq r$ maximum such that there are r' pairwise disjoint copies of $S(\gamma_3)$, pairwise vertex-disjoint and each left-rainbow

relative to $(B'_i : h \leq i \leq h + s - 1)$. By removing the vertices of these copies from the blocks B'_i ($i \in \{h, \dots, h + s - 1\}$), we obtain a contraction of $(B_i : h \leq i \leq h + s - 1)$ of width $W' - r'/n$ in which there is no left-rainbow copy of $S(\gamma_3)$. But $(B_i : h \leq i \leq h + s - 1)$ is $2^{-9\delta}$ -support-invariant, and so $W' - r'/n < 2^{-9\delta}W$, that is, $r' = r$. This proves the first assertion, and the second follows similarly. This proves (3).

For $v \in B'_1 \cup B'_K$, and $1 \leq i \leq r$, we say

- v meets $E_i \cup F_i$ if v is adjacent to some vertex of $E_i \cup F_i$;
- v meets $E_i \cup F_i$ *internally* if v is adjacent to some vertex of $E_i \cup F_i$ that is not the root of E_i or F_i (and possibly v is also adjacent to one or both roots);
- v meets $E_i \cup F_i$ *properly* if v is adjacent to one or both of the roots of E_i, F_i , but to no other vertices of $E_i \cup F_i$, that is, if v meets $E_i \cup F_i$ and does not meet $E_i \cup F_i$ internally.

For $X \subseteq B'_1 \cup B'_K$, let $a(X)$ be the number of $i \in \{1, \dots, r\}$ such that some vertex in X meets $E_i \cup F_i$, and let $b(X)$ be the number of $i \in \{1, \dots, r\}$ such that some vertex in X meets $E_i \cup F_i$ internally. Choose $X_1 \subseteq B'_1$ maximal such that $a(X_1) \leq r/2$ and $b(X_1) \geq a(X_1)/2$.

(4) $|X_1| < \varepsilon n$, and $a(X_1) \leq r/2 - \varepsilon n$.

There are at least $r/2$ vertices in B_h with no neighbour in X_1 (the roots of the trees E_i such that no vertex in X_1 meets $E_i \cup F_i$). Since $r/2 \geq \varepsilon n$ and G is ε -coherent, it follows that $|X_1| \leq \varepsilon n$; and since $r/2 \geq \lambda W n$, it follows that X_1 λ -misses B_h . Similarly it λ -misses $B_{h+s+t-1}$, and since \mathcal{B} is λ -concave, X_1 does not λ -cover any of the sets $B_{h+1}, \dots, B_{h+s+t-2}$. Hence there are at most $\lambda(s+t-2)W n$ vertices in $B_{h+1} \cup \dots \cup B_{h+s+t-2}$ that have neighbours in X_1 . Since there are at least $b(X_1)$ such vertices in total, it follows that $\lambda(s+t)W n \geq b(X_1) \geq a(X_1)/2$, and so

$$a(X_1) \leq 2\lambda(s+t)W n \leq 2(2^{-9\delta}\delta^{-1-\eta})(2\delta^{\eta+1})(2^{9\delta}r/63) = 4r/63,$$

since $\lambda \leq 2^{-9\delta}\delta^{-1-\eta}$, and $s+t \leq 2\delta^{\eta+1}$, and $r \geq 63 \cdot 2^{-9\delta}W n$. Since $r \geq 63\varepsilon n$, it follows that $4r/63 \leq r/2 - \varepsilon n$. This proves (4).

Let C be the set of all $i \in \{1, \dots, r\}$ such that X_1 is anticomplete to $V(E_i \cup F_i)$. Thus $|C| = r - a(X_1)$. By renumbering, we may assume that C consists of the first $|C|$ positive integers. Let X_2 be the set of vertices in $(B'_1 \cup B'_K) \setminus X_1$ that meet one of $E_1 \cup F_1, \dots, E_r \cup F_r$.

(5) $|X_2| \geq 2(W' - \varepsilon)n$.

Since $r \geq \varepsilon n$, and G is ε -coherent, there are fewer than εn vertices in $B'_1 \cup B'_K$ that have no neighbour in any of $E_1 \cup F_1, \dots, E_r \cup F_r$. All the other vertices in $B'_1 \cup B'_K$ belong to either X_1 or X_2 , so $|X_1| + |X_2| + \varepsilon n \geq |B'_1 \cup B'_K| = 2W'n$. From (4), this proves (5).

(6) For each $v \in X_2$, the number of $i \in C$ such that v meets $E_i \cup F_i$ internally is at most half the number of $i \in C$ such that v meets $E_i \cup F_i$.

Since $a(X_1) \leq r/2 - \varepsilon n$ by (4), it follows that $a(X_1 \cup \{v\}) \leq r/2$, and the maximality of X_1 implies that $b(X_1 \cup \{v\}) < a(X_1 \cup \{v\})/2$. Since $b(X_1) \geq a(X_1)/2$, this proves (6).

(7) *We may assume that there is a subset $X \subseteq X_2$ of cardinality at least $|X_2|/2$, such that for each $v \in X$, if $i \in C$ is minimum such that v meets $E_i \cup F_i$, then v meets $E_i \cup F_i$ properly.*

Let $v \in X_2$, and take a linear order of C ; and let $i \in C$ be the first member of C (under this order) such that v meets $E_i \cup F_i$ (there is such a member i from the definition of X_2 .) We say v is *happy* (under this order), if v meets $E_i \cup F_i$ properly. If we choose the linear order uniformly at random, the probability that v is happy is at least $1/2$, by (6); and so there is a linear order of C such that at least $|X_2|/2$ vertices in X_2 are happy. By renumbering, we may assume that this order is the natural order of C as a set of integers. This proves (7).

For $v \in X$, we call this value of i in (7) the *happiness* of v . Let $v \in X$ and let i be its happiness. Since v meets $E_i \cup F_i$ properly, it is adjacent to one or both of the roots of E_i, F_i , and has no other neighbours in $E_i \cup F_i$. Also, v belongs to one of B_1, B_K . Let us say v has

- type $(1, E)$ if $v \in B_1$ and v is adjacent to the root of E_i ;
- type $(1, F)$ if $v \in B_1$ and v is adjacent to the root of F_i ;
- type (K, E) if $v \in B_K$ and v is adjacent to the root of E_i ; and
- type (K, F) if $v \in B_K$ and v is adjacent to the root of F_i .

Every vertex in X has one of these four types (some have more than one type). Now $|X_2| \geq 2(W' - \varepsilon)n$ by (5), and so $|X| \geq (W' - \varepsilon)n \geq 63W'n/64$ by (1). Thus one of $B'_1 \cap X, B'_K \cap X$ has cardinality at least $63W'n/128 \geq W'n/4$, so we may choose $m \leq |C|$ minimum such that one of $B'_1 \cap X, B'_K \cap X$ contains at least $W'n/4$ vertices with happiness at most m . Consequently there is a set $U \subseteq X$, such that $|U| \geq W'n/8$, and all vertices in U have happiness at most m , and they all have the same type (which, from now on, we call the “type of U ”). Let

$$Y' = V(E_1) \cup \dots \cup V(E_m) \cup V(F_1) \cup \dots \cup V(F_m).$$

(8) *For each $i \in \{2, \dots, h-1\}$, there is a subset of B'_i anticomplete to Y' , of cardinality at least $W'n/8$.*

Let $j \in \{h, \dots, h+s+t-1\}$. From the choice of m , fewer than $W'n/4$ vertices in $B'_1 \cap X$ have happiness less than m ; and so at most $W'n/4 + 2\varepsilon n$ have happiness at most m , since those with happiness exactly m are adjacent to one of the roots of E_m, F_m . Since $|X| \geq 3W'n/2$, it follows that $|X \cap B'_1| \geq W'n/2$, and so there are at least $W'n/4 - 2\varepsilon n$ vertices in $X \cap B'_1$ that have no neighbour in Y' , and in particular have no neighbour in $B'_j \cap Y'$. Since $W'n/4 - 2\varepsilon n > \lambda Wn$, $B'_j \cap Y'$ λ -misses B_1 . By the same argument it λ -misses B_K , and so does not λ -cover any of B_2, \dots, B_{h-1} , since \mathcal{B} is λ -concave. In other words, for $i \in \{2, \dots, h-1\}$ and $j \in \{h, \dots, h+s+t-1\}$, there are at most λWn vertices in B'_i with a neighbour in $B'_j \cap Y'$; and consequently there are at most $(s+t)\lambda Wn$ vertices in B'_i with a neighbour in Y' . Since $|B'_i| = W'n$ and $W'n - (s+t)\lambda Wn \geq W'n/8$, this proves (8).

Now there are four cases, depending on the four possible types of U . First, suppose U has type $(1, E)$. Let Q' be the rooted tree obtained from the disjoint union of $Q(\gamma_1)$ and $S(\gamma_3)$ by adding an

edge between the roots, and making the root of $Q(\gamma_1)$ the root of the new tree. Thus Q' contains $Q(\gamma_1 + 1)$. Each $v \in U$ is adjacent to the root, and to no other vertices, of a copy of $S(\gamma_3)$ that is rainbow relative to \mathcal{B} and contains in $G[Y']$. But from the definition of “anchored”, v is the root of a copy of $Q(\gamma_1)$ that is left-rainbow relative to \mathcal{B} and contained in $G[Y \cup \{v\}]$. Since Y is anticomplete to Y' , the union of these two rooted trees contains a copy of $Q(\gamma_1 + 1)$, so v is the root of a copy of $Q(\gamma_1 + 1)$ that is left-rainbow relative to \mathcal{B} and contained in $G[Y \cup Y' \cup \{v\}]$. Choose $B''_1 \subseteq U$ of cardinality $\lceil W'n/8 \rceil$ and for $h+1 \leq i \leq h'+t-1$ choose $B''_i \subseteq B'_i$ of cardinality $\lceil W'n/8 \rceil$, anticomplete to Y' (this is possible by (8)), and choose $B''_K \subseteq B_K$ of cardinality $\lceil W'n/8 \rceil$ anticomplete to Y' (this is possible since at least $W'n/4 - \varepsilon n$ vertices in $B'_K \cap X$ have no neighbour in Y'). By (2), $(B''_i : i \in \{1, \dots, h-1\} \cup \{K\})$ is a $(Q(\gamma_1 + 1), R(\gamma_2))$ -anchored minor of \mathcal{B} of width at least $W'/8$, contrary to the maximality of γ_0 .

Next, suppose U has type $(1, F)$. Let Q' be the rooted tree obtained from the disjoint union of $Q(\gamma_1)$ and $T(\delta, \beta)$ by adding an edge between the roots, and making the root of $Q(\gamma_1)$ the root of the new tree. Again, Q' contains $Q(\gamma_1 + 1)$, since $\beta \geq \alpha$. Each $v \in U$ is adjacent to the root, and to no other vertices, of a copy of $T(\delta, \beta)$ that is rainbow relative to \mathcal{B} and contained in $G[Y']$. But v is the root of a copy of $Q(\gamma_1)$ that is left-rainbow relative to \mathcal{B} and contained in $G[Y \cup \{v\}]$. The union of these two rooted trees is a copy of Q' , so v is the root of a copy of $Q(\gamma_1 + 1)$ that is left-rainbow relative to \mathcal{B} and contained in $G[Y \cup Y' \cup \{v\}]$. Then we obtain a contradiction as in the first case.

Next suppose U has type (K, F) . Then similarly we obtain a $(Q, R(\gamma_2 + 1))$ -anchored minor of \mathcal{B} of width at least $W'/8$, again a contradiction.

Finally, suppose U has type (K, E) . We recall that $0 \leq \gamma_2 \leq 2\delta$. If $\gamma_2 < \delta$, then as in the previous case we obtain a $(Q, R(\gamma_2 + 1))$ -anchored minor of \mathcal{B} of width at least $W'/8$, contrary to the maximality of γ_0 . So we may assume that $\gamma_2 \geq \delta$. Choose $v \in U$, and let i be its happiness; then E_i is a copy of $S(\gamma_3)$. Let u be the root of E_i . Since $v \in B'_K$, v is the root of a copy of $R(\gamma_2)$, rainbow relative to \mathcal{B} and contained in $G[Y \cup \{v\}]$. But $R(\gamma_2)$ contains $T(\delta, \alpha)$ (it even contains $T(\delta, \alpha + 1)$, but we do not need that); and consequently v is the root of a copy of $T(\delta, \alpha)$, rainbow relative to \mathcal{B} and contained in $G[Y \cup \{v\}]$. The union of this tree with E_i , rooted at u , gives a copy of $S(\gamma_3 + 1)$, left-rainbow relative to \mathcal{B} , contrary to the choice of γ_3 . This proves 5.1. \blacksquare

6 Using matching-covered blockades

In this section we complete the proof of 1.4. First, by combining 4.3 and 5.1, we obtain:

6.1 *For all $\delta \geq 2$ and $\varepsilon \geq 0$, and for every integer $m \geq 0$ there exist an integer $K \geq 0$ and $0 < d < 1$, such that the following holds. Let $\varepsilon > 0$, and let G be a $T(\delta, \eta)$ -free ε -coherent graph; then for every blockade \mathcal{B} in G of length at least K and width $W \geq \varepsilon/d$, there is a matching-covered minor of length m and width $\geq dW$.*

Proof. We may assume that $m \geq 3$. Let $k = 6\delta^{\eta+2}$, and let $\lambda = \min(2^{-9\delta}\delta^{-1-\eta}, 1/(2m))$. Choose K, c to satisfy 4.3, and let $d = \min((2mc/\lambda)2^{-9\delta}, c^2)$. We claim that K, d satisfy 6.1.

Suppose that G is a $T(\delta, \eta)$ -free ε -coherent graph, and \mathcal{B} in G of length at least K and width $W \geq \varepsilon/d$. By 4.3, either:

- \mathcal{B} has a matching-covered minor of length m and width at least c^2W , or

- there exists a λ -concave blockade \mathcal{B}' in G of length k and width at least $2mcW/\lambda$, that is support-uniform and λ -support-invariant.

In the first case, the theorem holds, since $c^2W \geq dW$. In the second case, \mathcal{B}' has width at least

$$2mcW/\lambda \geq 2mc(\varepsilon/d)2^{9\delta}\delta^{1+\eta} \geq 2^{9\delta}\varepsilon,$$

since $W \geq \varepsilon/d$ and $\lambda \leq 2^{-9\delta}\delta^{-1-\eta}$. But \mathcal{B}' is λ -concave, support-uniform and $2^{-9\delta}$ -support-invariant (since $\lambda \leq 2^{-9\delta}$); and it follows that G contains $T(\delta, \eta)$, a contradiction. This proves 6.1. \blacksquare

This yields:

6.2 *For all $\delta \geq 2$ and $\varepsilon \geq 0$, and for every tree T , there exist an integer $K \geq 0$ and d with $0 < d \leq 1$ such that for all $\varepsilon > 0$, if G is ε -coherent and $T(\delta, \eta)$ -free, and (B_1, \dots, B_K) is a blockade in G of width at least ε/d , then there is a copy of T in G that is rainbow relative to \mathcal{B} .*

Proof. We proceed by induction on $|V(T)|$ (keeping δ, η fixed). Certainly 6.2 holds when $|V(T)| = 1$, so we may assume that T has a vertex v of degree one, and the theorem holds for $T \setminus v$. In particular there exist an integer $k \geq 0$ and $0 < d' \leq 1$ satisfying 6.2 with T, K, d replaced by $T \setminus v, k, d'$ respectively. Let K, d satisfy 6.1 (for T , and with m replaced by k). We claim that K, dd' satisfy 6.2 for T . Let G be ε -coherent and $T(\delta, \eta)$ -free, and let (B_1, \dots, B_K) be a blockade in G of width $W \geq \varepsilon/(dd')$. By 6.1, there is a matching-covered minor \mathcal{B}' of length k and width $\geq dW \geq \varepsilon/d'$. Let $\mathcal{B}' = (B'_{r_1}, \dots, B'_{r_k})$; and let s_1, \dots, s_k and X_{s_1}, \dots, X_{s_k} be as in the definition of “matching-covered”.

By the choice of k, d' , there is a copy S of $T \setminus v$ in G that is rainbow relative to \mathcal{B}' . Let u be the neighbour of v in T , and let y be the vertex of S that corresponds to u under the isomorphism between S and $T \setminus v$; thus $y \in B'_{r_i}$ for some j with $1 \leq i \leq k$. Since X_{s_i} covers B'_{r_i} , there exists $x \in X_{s_i}$ adjacent to y ; and since X_{s_i} is anticomplete to B'_{r_j} for all $j \in \{1, \dots, k\} \setminus \{i\}$, it follows that the subgraph of G induced on $V(S) \cup \{x\}$ is isomorphic to T , and rainbow relative to \mathcal{B} . This proves 6.2. \blacksquare

Finally we can prove 1.4, which we restate:

6.3 *For every forest T there exists $\varepsilon > 0$ such that every ε -coherent graph contains T .*

Proof. Every forest is an induced subgraph of a tree, so it suffices to prove 1.4 for trees. Let T be a tree, and choose $\delta \geq 2$ and $\eta \geq 0$ such that $T(\delta, \eta)$ contains T . Let K, d satisfy 6.2, and choose $\varepsilon > 0$ such that $2K\varepsilon \leq d$. We claim that every ε -coherent graph contains T . For let G be ε -coherent, and let $n = |V(G)|$; it follows that $n \geq \varepsilon^{-1} \geq 2K/d \geq 2K$. Hence $n/K \geq \lceil n/(2K) \rceil$, and so we may choose K subsets of $V(G)$, pairwise disjoint and each of cardinality $\lceil n/(2K) \rceil \geq \varepsilon n/d$. These sets, in any order, form a blockade of length K and width at least ε/d , and so by 6.2, if G is $T(\delta, \eta)$ -free then G contains T . On the other hand, if G is not $T(\delta, \eta)$ -free then G contains T anyway. This proves 6.3. \blacksquare

7 Remarks

There are some final points we would like to make. First, while the results of [4, 8] concerned ε -coherent graphs, they were capable of generalization in the natural way to ε -coherent “massed graphs”, graphs in which each subset $X \subseteq V(G)$ had a mass $\mu(X)$, where μ was increasing and subadditive (and also satisfied a nontriviality condition); such as, for instance, the function $\mu(X) = \chi(G[X])/\chi(G)$, where χ denotes chromatic number. The proof of 1.4 does not seem to extend to massed graphs; for instance, the method in the proof of 5.1 of pulling out “parallel” rainbow copies of a graph, relies on the fact that we are removing the same number of vertices from each block.

Second, the following question was proposed in [4], and remains open, although it might be amenable to a similar proof method:

7.1 Conjecture: *For every forest T there exists $\varepsilon > 0$ with the following property. Let G be a T -free bipartite graph with bipartition (A, B) , where $|A| = |B| = n$. Then either some vertex has degree at least εn , or there is an anticomplete pair of subsets $A' \subseteq A$ and $B' \subseteq B$ with $|A'|, |B'| \geq \varepsilon n$.*

Third, the proof in this paper actually proves something a little stronger; that for any tree T , there exist an integer K and $d > 0$, such that in every ε -coherent graph with a blockade \mathcal{B} of length at least K and width w say, if G is ε -coherent where ε is at most w/d , then there is a copy of T that is rainbow relative to \mathcal{B} . To see this, observe that the blockade selected in the final proof of 1.4 might as well be \mathcal{B} ; and thereafter, all we do is get a sequence of derived blockades from the initial one, until we find a copy of T that is rainbow relative to the final blockade. It follows that this copy is also rainbow relative to the initial blockade. We omitted this refinement to simplify the proof a little.

Fourth, here is a nice question: for which tournaments H does there exist $\varepsilon > 0$ such that in every tournament G not containing H as a sub-tournament, there are two linear sets A, B where A is complete to B ? One can show that if H is such a tournament, then

- $V(H)$ can be ordered as $\{v_1, \dots, v_n\}$ such that the backedge digraph (the digraph formed by the pairs $v_i v_j$ where v_i is adjacent from v_j in G) is transitive;
- $V(H)$ can be ordered such that the backedge digraph has no induced outdirected 3-star;
- $V(H)$ can be ordered such that the backedge digraph has no induced indirected 3-star; and
- $V(H)$ can be ordered such that the backedge graph (the graph underlying the backedge digraph) is a forest.

But such tournaments exist; for instance, the eulerian orientation of K_5 is such a tournament. Does ε exist for this tournament?

Fifth, for a graph H , define $d(H)$ to be the minimum of $(|V(J)| - 1)/|E(J)|$ over all induced subgraphs J of H that have at least one edge. It is tempting to conjecture that for all H , there exists $\varepsilon > 0$ such that in every H -free graph G with $n \geq 2$ vertices and maximum degree at most εn , there are two disjoint subsets $A, B \subseteq V(G)$, anticomplete, with $|A|, |B| \geq \varepsilon n^{d(H)}$. When $d(H) = 1$ this is our theorem, and one can modify Erdős’ random graph construction to show that the bound would be sharp for all $d(H)$. Unfortunately it is false; for instance, when $H = K_3$, one can show (using

Lovasz’ local lemma) that there is an H -free graph G with n vertices, in which every anticomplete pair of sets A, B satisfies $\min(|A|, |B|) \leq \tilde{O}(n^{1/2})$. (The \tilde{O} notation means “up to a polylog factor”.)

Finally, as with many Ramsey-type theorems, there is a multicolouring version of our result. Take a complete graph, and partition its edge-set into k sets; and let G_i be the subgraph with edge-set the i th of these sets (and all the vertices). We call (G_1, \dots, G_k) a k -multicolouring. Then the following holds, generalizing 1.2:

7.2 *For all $k \geq 1$ and every forest H there exists $\varepsilon > 0$, such that if (G_1, \dots, G_k) is a k -multicolouring of a complete graph K_n with at least two vertices, then for some $i \in \{1, \dots, k\}$, either G_i contains H as an induced subgraph, or there are two disjoint subsets $X, Y \subseteq V(G_i)$, with $|X|, |Y| \geq \varepsilon n$, anticomplete in G_i .*

Proof. (Sketch.) The proof is easy. Choose $\varepsilon' > 0$ such that 1.4 holds, and choose $c > 0$ such that $kc \leq \varepsilon'$. A straightforward modification of the proof of the theorem of [9] shows that there exists $\delta > 0$ (independent of n and G_1, \dots, G_k), such that if no G_i contains H as an induced subgraph, then there is a subset A of the vertex set of K_n , with $|A| \geq \delta n$, such that $|E(G_i[A])| \leq c|A|(|A| - 1)/2$ for all values of $i \in \{1, \dots, k\}$ except one, say all except $i = k$. Let $\varepsilon = \varepsilon'\delta$. By removing vertices of degree at least $kc\delta n$ in one of G_1, \dots, G_{k-1} , we deduce that there exists $B \subseteq A$ with $|B| \geq (\delta/k)n$, such that for $1 \leq i \leq k - 1$, every vertex of $G_i[B]$ has degree at most $kc\delta n \leq \varepsilon'|B|$ in $G_i[B]$. By 1.4 applied to $G_1[B]$, there are two disjoint subsets $X, Y \subseteq B$ with $|X|, |Y| \geq \varepsilon'|B| \geq \varepsilon n$, anticomplete in $G_1[B]$, as required. ■

There is in fact a stronger result:

7.3 *For all $k \geq 2$ and every forest H there exists $\varepsilon > 0$, such that if (G_1, \dots, G_k) is a k -multicolouring of a complete graph K_n with at least two vertices, then for some distinct $i, j \in \{1, \dots, k\}$, either*

- *there is a subset $X \subseteq V(K_n)$ such that every edge of $G[X]$ belongs to $G_i \cup G_j$, and $G_i[X]$ is isomorphic to H ; or*
- *there are two disjoint subsets $X, Y \subseteq V(G_i)$, with $|X|, |Y| \geq \varepsilon n$, complete in G_j .*

We do not know how to prove this as a consequence of 1.4, and it seems necessary to modify the proof of 1.4 in several places; all straightforward, but too many to sketch here, and we omit further details.

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