The percolation density $\theta(p)$ is analytic.

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Joint work with Christoforos Panagiotis
Bernoulli bond Percolation

Each edge
— present with probability $p$

and

— absent with probability $1-p$

independently

$$p_c := \sup\{p \mid \mathbb{P}_p(\text{cluster } C_o \text{ of } o \text{ is infinite}) = 0\}$$
Motivation

Introduced by physicists Broadbent & Hammersley '57 as a toy model of statistical mechanics

Many deep rigorous results by mathematicians

Varying the underlying graph unleashes an interesting interplay between geometry & probability

Rich connections to other models (Ising, GFF, loop $O(n)$)
The 3 regimes

subcritical: $p < p_c$

critical: $p = p_c$

supercritical: $p > p_c$

But is $p_c$ the only phase transition?
Exponential decay

Theorem (Aizenman & Barsky '87/ Menshikov '86)

For every \( p < p_c \) there is \( c_p > 1 \) such that

\[
P_p(|C_0| \geq n) \leq c_p^{-n}.
\]
Analyticity of $\chi(p)$

$\chi(p) := \mathbb{E}_p(|C_o|)$

**Theorem (Kesten '82)**

$\chi(p)$ is an analytic function of $p$ for $p \in [0, p_c)$ when $G$ is a lattice in $\mathbb{R}^d$.

**Proof:**

$\chi(p) = \sum_{0 \in S \subseteq G \text{ finite, connected}} |S| \cdot \mathbb{P}_p(C_o = S) = \sum_{n \in \mathbb{N}} \sum_{1 \leq |S| = n} |S| \cdot \mathbb{P}_p(C_o = S)$

$= \sum_{n \in \mathbb{N}} n \cdot \mathbb{P}_p(1|C_o| = n)$

polynomial: $\sum p^ni(1-p)^{ni}$
Complex analysis basics

**Theorem (Weierstrass):** Let \( f = \sum f_n \) be a series of analytic functions which converges uniformly on each compact subset of a domain \( \Omega \subset \mathbb{C} \). Then \( f \) is analytic on \( \Omega \).

**Weierstrass M-test:** Let \( (f_n) \) be a sequence of functions such that there is a sequence of ‘upper bounds’ \( M_n \) satisfying

\[
|f_n(z)| \leq M_n, \forall x \in \Omega \quad \text{and} \quad \sum M_n < \infty.
\]

Then the series \( \sum f_n(x) \) converges uniformly on \( \Omega \).
Analyticity of $\chi(p)$

$$\chi(p) := \mathbb{E}_p(|C_0|)$$

**Theorem (Kesten '82)**

$\chi(p)$ is an analytic function of $p$ for $p \in [0, p_c)$ when $G$ is a lattice in $\mathbb{R}^d$.

**Proof:**

$$\chi(p) = \sum_{0 \in S \subseteq G, S \text{ finite, connected}} |S| \cdot \mathbb{P}_p(C_0 = S) = \sum_{n \in \mathbb{N}} \sum_{|S| = n} \mathbb{P}_p(C_0 = S)$$

$$= \sum_{n \in \mathbb{N}} \mathbb{P}_p(|C_0| = n)$$

polynomial: $\sum p^n (1-p)^{n_i}$
Analyticity of $\theta(p)$

$\theta(p) := \mathbb{P}_p( |C_0| = \infty)$

Question (Kesten ’81): Is $\theta(p)$ analytic for $p > p_c$?

Appearing in the textbooks

— $\theta(p)$ is infinitely differentiable [Chayes, Chayes & Newman ’87]
— $\theta(p)$ is analytic near $p=1$ [Braga, Proccaci & Sanchis ‘02]
Analytic vs. $C_\infty$ functions

Bob:
  — What was the difference between $C_\infty$ and analytic again?
Analise:
  — The latter has a convergent Taylor series.
Bob:
  — Isn’t almost every $C_\infty$ function analytic?
Analise:
  — Quite the contrary: the nowhere analytic functions are a dense $G_\delta$ subset of the $C_\infty$ functions! [Cater ‘84]
[Griffiths ‘69] introduced models, constructed by applying the Ising model on 2-dimensional percolation clusters, in which the free energy is \textit{infinitely differentiable} but \textit{not analytic}.

This phenomenon is now called a \textit{Griffiths singularity}. 
Interlude: Peierls’s argument

\[ 1 - \Theta(p) \leq \sum_{n \in \mathcal{N}} \sum_{\text{cycles } c \text{ around } o \in c \cap n} (1-p)^n \]

\[ = \sum_n c_n (1-p)^n \]

\[ \Rightarrow \ldots \rho_c(\mathbb{Z}^2) \leq \frac{2}{3} \]
Theorem (Hardy & Ramanujan 1918)

The number of partitions of the integer n is of order \( \exp(\sqrt{n}) \).

Theorem (G & Panagiotis ’18+)

\( \theta(p) \) is analytic for \( p > p_c \) on any planar lattice.
Analyticity of $\theta(p)$

**Theorem (G & Panagiotis ’18+)**

$\theta(p)$ is analytic for $p > p_c$ on any planar lattice.

Ingredients:
- elementary complex analysis
- better interfaces
- Inclusion-Exclusion Principle
- Weak Hardy-Ramanujan
- BK inequality
- Exponential decay (in dual)
- More combinatorics
Further results

- \( \theta \) analytic for \( p > p_c \) for continuum percolation – asked by [Last et al. ’17]
- \( \theta \) analytic for \( p > p_c \) on regular trees, and on almost every Galton-Watson tree. – asked by [Michelen, Pemantle & Rosenberg]
- \( \theta \) analytic for \( p \) near 1 on all finitely presented Cayley graphs.
- \( \theta \) analytic for \( p \) near 1 on all non-amenable graphs. –Extended to \( p \in (p_c, 1] \) by [Hermon & Hutchcroft ’19+]
- For certain families of planar triangulations for which [Benjamini et al. ’96, ’15, ’18] conjectured that \( p_c^{site} \leq 1/2 \), we prove \( p_c^{bond} \leq 1/2 \) (and analyticity of \( \theta \)).
Chapter II: Polyominoes and growth rates of interfaces
A *polyomino*, aka. *lattice animal*, is a connected, induced, subgraph of $\mathbb{Z}^2$.

Their exponential growth rate

$$a(\mathbb{Z}^2) := \lim_{n \to \infty} \left( \#\{ \text{polyominoes of size } n \} \right)^{1/n}$$

is not known.
Kesten's argument

\[ 1 - \Theta(p) = \sum_{0 \in SCG} P(C_0=S) \]

\[ = \sum_{n \in \mathbb{N}} \sum_{|S|=n} p^n (1-p)^{|S|} = \sum_{n \in \mathbb{N}} a_n p^n (1-p)^{\Delta n} \geq \sum_{n} a_n p^n (1-p)^{\Delta n} \]

\[ \Rightarrow \alpha(G) \leq \frac{1}{p(1-p)^{\Delta}} \leq ... \leq \Delta e \]

Can we do better?
The growth rates $b_r$

\[ c_{n,r,\varepsilon} : = \# \left\{ \text{interfaces of size } n \ \text{and boundary size 'roughly' } r n \right\} \]

i.e. \( \text{in} \{(r-\varepsilon)n,(r+\varepsilon)n\} \)

\[ b_{r,\varepsilon} : = \text{their growth rate} = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} c_{n,r,\varepsilon}^{1/n} \]

\[ b(G) : = \max_r b_r = \text{growth rate of all interfaces} \]
The growth rates $b_r$

... refining Kesten's argument, we obtain:

$$b_{r(p)}(G) \leq f(r(p))$$

where $f(r) := \frac{(1 + r)^{1+r}}{r^r}$ and $r(p) := \frac{1-p}{p}$ are universal.

Equality holds iff exponential decay fails!

For lattice animals obtained by [Delyon '80] and [Hammond '05]
More on $b_r$

$$b_{r(p)}(G) \leq f(r(p))$$

$$b_r = (b_{1/r})^r$$

$$a(G) \geq b(G) \geq f(r(p_c(G)))$$
Further results

- $p_c(\mathbb{Z}^3) > 0.2522$
  - using bounds of [Barequet & Shalah '19+]

- $a(\mathbb{Z}^d) \leq 2de - 5e/2 + O(1/\log(d))$
  - improves on bounds of [Barequet & Shalah '19+]

- as a result, we obtain
  $$p_c(\mathbb{Z}^d) \geq \frac{1}{2d} + \frac{2}{(2d)^2} - O(1/d^2 \log(d))$$

Using upper bounds on $p_c(\mathbb{Z}^d)$ from [Heydenreich & Matzke '19+], we obtain $a(\mathbb{Z}^d) \geq 2de - 3e$
  - asked by [Barequet, Barequet & Rote '10], nonrigorously obtained by [Peard & Gaunt '95]

- $p_c < 1/2$ for plane graphs of minimum degree $\geq 7$
  [Haslegrave & Panagiotis '19+]
  - answers a question of [Benjamini & Schramm '96]
Analyticity of $\theta(p)$

**Figure 1:** An approximate visualisation of $b_r(G)$ when $G$ is a lattice in $\mathbb{R}^d$, $d \geq 3$. The graph of $b_r(G)$ (depicted in black, if colour is shown) lies below the graph of $f(r) = (1 + r)^{-1}$. The fact that $f(r)$ plots (in Mathematica, in this instance) almost like a straight line can be seen by rewriting it as $(1 + r)(1 + 1/r)$. The fact that $b_r = f(r)$ for $r$ in the interval $(r_{1-pc}, r_{pc})$, where $r_{pc} = 1/p_c$, follows by combining a theorem of Kesten & Zhang [27], saying that exponential decay of $P_{p_{pc}}(|S_o| = n)$ fails in that interval, with our Theorem 1.1. That $b_r < f(r)$ for $r > r_{pc}$ follows from the well-known exponential decay of $P_{pc}(|C_o| = n)$ for $p < p_c$ [1].

We also know that $b_r$ is continuous and log-concave. The continuity of $b_r$, combined with Theorem 1.1 again, implies failure of the exponential decay at $p = 1 - p_c$, which was not obtained in [27].

If the cycle space of $G$ is generated by its triangles, then (4) determines the subcritical branch $r > r_{pc}$ given the branch $r < r_{1-pc}$ and vice-versa.

For the planar triangular lattice the picture degenerates as $p_c = 1/p_c$, and so $b_r = f(r)$ for $r = r_{1/2} = 1$ only.

Note that $b_r(G)$ is an invariant of $G$ defined without reference to any random experiment. The connection to percolation is established by Theorem 1.1 via the above transformation $r_{pc}$. Since $r_{pc}$ is monotone decreasing in $p$, the right hand side of Figure 1 corresponds, somewhat annoyingly, to the subcritical percolation regime, and the left hand side to the supercritical.

Using the transformation $r_{1/r}$ (from volume-to-surface into surface-to-volume ratio) we could reverse the picture to have the 'subcritical' interval on the left. For 'triangulated' lattices the picture would look exactly the same due to (4), only the positions of $r_{pc}$ and $r_{1-pc}$ would be interchanged.

Equality (3) means that in principle we could answer any question about $\mathbb{E}_{p_{pc}}(\{M\})$, $\mathbb{E}_{p_{pc}}(\{\xi_x\})$, $\mathbb{E}_{p_{pc}}(\{\varphi_x\})$, $\mathbb{E}_{p_{pc}}(\{\xi_x\} \cap \{\varphi_x\})$, $\mathbb{E}_{p_{pc}}(\{\xi_x\} \cup \{\varphi_x\})$, and so forth. The functions $P_{p_{pc}}(\{M\})$ occur as polynomials of the form $p_{pc}^n$. The formula is proved using the inclusion-exclusion principle, which explains the use of multi-interfaces and the signs ($(-1)$) for more. See [16] for more.

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Kesten’s question

Question (Kesten ’81): Is $\theta(p)$ analytic for $p > p_c$?

Theorem (Panagiotis & G ’20+)

Yes.


Thanks to: Μεγερία

and: