Matching Random Points

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Fairness: how to share resources between individuals?

Uneven setting: individuals have different locations/desires/abilities.

Maximize “overall happiness”…?

Should we make $X$ happier but $Y$ less happy?

...even if $X$ already happier than $Y$?

Maximize $\sum_{\text{individuals}} f(\text{happiness})$.

Given such a metric, what are the consequences? Can individuals agree on a solution?
Simple but rich mathematical setting: **Matching**

\[ R = \{\text{red points}\} \]
\[ B = \{\text{blue points}\} \]
\[ M = \text{red–blue matching} \]

Minimize \[
\sum_{(r,b) \in M} |r - b|^\gamma.
\]

“fairness” \(0 < \gamma < \infty\)

But then \(= \infty!\)

Independent intensity-1 Poisson point processes in \(\mathbb{R}^d\)

# of points in \(A \sim\) Poisson(vol(A))

Independent #s of points in disjoint sets

Countably infinite # of points

Euclidean norm
Solution: minimize locally.

\( M \) is \( \gamma \)-minimal if \( \forall \) finite \( \{(r_1, b_1), \ldots, (r_m, b_m)\} \subseteq M \),

\[
\sum_i |r_i - b_i|^\gamma = \min_{\sigma \in S_m} \sum_i |r_i - b_{\sigma(i)}|^\gamma
\]

To handle \( \gamma \) in different cases:

- \( -\infty < \gamma < 0 \): replace \( | \cdot |^\gamma \) with \( -| \cdot |^\gamma \)
- \( \gamma = 0 \): replace \( | \cdot |^\gamma \) with \( \log | \cdot | \)
- \( \gamma = -\infty \) (selfish): lexicographically minimize \( \uparrow \) ordering of \( |r_1 - b_1|, \ldots, |r_m - b_m| \)
- \( \gamma = +\infty \) (altruistic): lexicographically minimize \( \downarrow \) ordering of \( |r_1 - b_1|, \ldots, |r_m - b_m| \)
\gamma = 1
\( \gamma = -\infty \)  
(stable)
> $\gamma = \infty$

(altruistic)
Questions:
Does a $\gamma$-minimal matching exist?
Is it unique?
Is every point matched?

Allow unmatched points! Then $\gamma$-minimal means:

$$\forall (r_1, b_1), \ldots, (r_m, b_m) \in M$$
and unmatched $x_1, \ldots, x_k \in R \cup B$,

$$\left(\#\text{unmatched}, \sum_i |r_i - b_i|^\gamma\right)$$
is lexicographically minimized among matchings of $\{r_i, b_i, x_i\}$

(in particular, cannot have both red and blue unmatched points)
Questions:
Does a $\gamma$-minimal matching exist?
Is it unique?
Is every point matched?
Can we decide on a matching by a local algorithm?
Edge lengths?

Also: 1-colour matching
(all definitions analogous).
\[ \gamma = -\infty: \quad \text{fairly complete picture (especially 1-colour)} \]

\[ d = 1: \quad \text{fairly complete picture (especially 2-colour)} \]

\[ d \geq 2, \gamma > -\infty: \quad \text{existence in some cases} \]

Open (e.g.): existence for 2 colours, \( d = 2, \gamma = 1, \infty? \)
Case $\gamma = -\infty$: stable matching

(Holroyd, Pemantle, Peres, Schramm, 2008)

**Theorem:** For any $d \geq 1$, and for $R$ (and $B$) independent intensity-1 Poisson processes on $\mathbb{R}^d$, a.s. there is a unique $(-\infty)$-minimal 1- (2-)colour matching, and it is perfect (i.e. no unmatched points).

In fact, it is the unique stable matching (marriage) in the sense of (Gale, Shapley, 1962): each point *prefers* a partner as close as possible; matching is unstable if there exist a pair (of opposite colours) that both prefer each other over their current situation.
Original formulation (Gale, Shapley, 1962): 
n girls, n boys have arbitrary preference orders over 
those of opposite sex.

Theorem: there exists a stable set of n heterosexual 
maririages (and an algorithm...)

Not necessarily unique; may not exist in 
1-colour / same-sex marriage / “roommates” version.

2012 Nobel memorial prize in Economics: 
Roth and Shapley.
Simple algorithm to construct the stable matching in our setting:

match all mutually closest pairs
Simple algorithm to construct the stable matching in our setting:

match all mutually closest pairs
remove them
repeat for countably many steps
**Theorem:** (HPPS 2008) For the stable matching, the matching distance $X$ from a typical point to its partner satisfies $\mathbb{E} X^{\alpha - \epsilon} < \infty$ but $\mathbb{E} X^{\beta} = \infty$, where:

<table>
<thead>
<tr>
<th>Colour Type</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-colour</td>
<td>$d$</td>
<td>$d$</td>
</tr>
<tr>
<td>2-colour, $d = 1$</td>
<td>$1/2$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>2-colour, $d = 2$</td>
<td>0.496</td>
<td>1</td>
</tr>
<tr>
<td>2-colour, $d \geq 3$</td>
<td>$\Theta(1 / \log d)$</td>
<td>$d$</td>
</tr>
</tbody>
</table>

**Theorem:** (Eccles, Holroyd, Liggett, 2020+) For the 1-colour stable matching in $d = 1$,

$$
\mathbb{P}(X > r) \sim \frac{c}{X}
$$

where $c = e^{2\gamma}$ where $\gamma$ is the Euler-Mascheroni constant.
Thomas M. Liggett, 1944 - 2020
Many variants...
Case $d = 1$
Theorem (Janson, Holroyd, Wästlund, 2020+)
For $d = 1$, any $\gamma$, a.s. every $\gamma$-minimal 1- or 2-colour matching is perfect. Similarly on the strip $\mathbb{R} \times [0,1]$.

Note: Much stronger conclusion than:

Theorem: for any $d$, any stationary $\gamma$-minimal matching is perfect.

Proof:
1-colour: $\leq 1$ unmatched point.
2-colour: all unmatched points same colour + ergodic decomposition.
$\gamma = 1$ is special:

Introduce:

$\gamma = 1^+$

$\gamma = 1^-$

break ties
The picture for $d = 1$ : classification

**Theorem (JHW 2020+)** For $d=1$, a.s. the set of 2-colour $\gamma$-minimal matchings is:

$\gamma \geq 1^+: \text{countable family } (M^k)_{k \in \mathbb{Z}}; \text{ no stationary matching locally finite}$

$\gamma = 1$: uncountable; uncountably many stationary matchings locally finite and locally infinite

$\gamma = 1^-: (M_k)_{k \in \mathbb{Z}}, M_\infty, M_{-\infty}; \text{ only stationary matchings are mixtures of } M_\infty, M_{-\infty}$

$\gamma < 1^-$: unique $M; \therefore \text{stationary.}$
Theorem (JHW 2020+): $d=1$, 2-colour.

For all $\gamma \geq 1^+$, the matchings are the same.

For any $\gamma < 1^-, \gamma' \leq 1^-$, $M, M'$ have finite differences.
Theorem (JHW 2020+):
\( d=1, \ 2\)-colour, \( \gamma < 1^{-} \) (or \( M_{\infty} \) or \( M_{-\infty} \))

Matching distance \( X \) satisfies \( \mathbb{E} X^{\alpha} < \infty \) iff \( \alpha < \frac{1}{2} \).

\( M \) is a finitary factor of \((R,B)\) with coding radius \( L \) satisfying
\[
\mathbb{E} L^{\beta} < \infty
\]

Can determine partner of a point by looking at points within (random) radius \( L \).
d=1, 1-color:

similar, but some proofs missing
- in particular, uniqueness for $\gamma < 1$
Higher dimensions:

**Theorem (JHW 2020+):**
\[ \exists \text{ a stationary (hence perfect) } \gamma\text{-minimal matching for:} \]

- 2-colour, \( d \leq 2, \gamma < 1; \)
- 2-colour, \( d \geq 3, \gamma < \infty; \)
- 1-colour, \( d \geq 2, \gamma < \infty. \)

Uniqueness open. Perfectness in general open. Existence open for other cases (note \( d \leq 2, \gamma \in \{1, \infty\} \) !)

Is there a case with no existence?

**Theorem (Holroyd 2010):** \( \exists \) no 1-minimal 2-colour matching on the strip.
**Note:** 1-minimal matchings in $d=2$ have *no crossings*

**Open:** is there a stationary perfect 2-colour matching of independent Poisson processes with no crossings?

**Theorem (Holroyd 2010):** Yes if we drop stationarity.
Proofs

\( d=1 \) classification

\[ \gamma > 1 \]

\[ \gamma < 1 \]
$\gamma \geq 1^+$

ordering of red points = ordering of blue partners

random walk
\( \gamma \leq 1^- \)

must match on same level of walk

\( \gamma = 1^- \)

\( M_k \) (locally finite)

\( M_\infty, M_{-\infty} \) (locally infinite)
Perfectness: 2-colour, $d = 1$ (or strip), any $\gamma$.

potentially unmatched = unmatched in some $\gamma$-minimal $M$
a.s. exist red and blue potentially unmatched points $r, b$

comp$(b)$ finite $\Rightarrow$ ∙∙∙

comp$(b)$ infinite: 

\begin{align*}
  c_0 &\geq c_1 \\
  c_1 &\geq c_2 \\
  c_0 + c_2 &\geq c_1 + c_3 \\
  c_1 + c_3 &\geq c_2 + c_4 \\
  \ldots
\end{align*}

$\Rightarrow c_0 \geq c_i \ \forall i$

˚˚˚ (percolation)
Existence: $\gamma < 1$, all $d$.

Quasi-stability: 
$\exists \ c(\gamma) \geq 1$:

Stationary subsequential limit

All unmatched points same colour...
**Existence:** provided $0 < \gamma < \infty$ and there exists a stationary (not necessarily minimal) $M$ with finite “average cost”:

$$
\mathbb{E} X^{\gamma} < \infty
$$

subsequential limit of matchings $M_n$ with

$$
\mathbb{E} X_n^{\gamma} \downarrow \inf_M \mathbb{E} X^{\gamma}
$$

is $\gamma$-minimal.

E.g. true for 2 colours:

- $d \leq 2, \quad \gamma < \frac{d}{2}$
- $d \geq 3, \quad \gamma < \infty$
Uniqueness, finite differences, finitariness:
\[ d = 1, \gamma < 1. \]

Random walk / Brownian motion estimates \( \Rightarrow \)
\[ \exists L \text{ (random, with } \mathbb{E} L^\beta < \infty) \]

Quasi-stability + same-level matching \( \Rightarrow \)
matching “trapped” in \([-L, L]\)
Tail bound:
\[ d = 1, \ \gamma < 1. \]

**Bad point:** distance to partner > \((2c + 1)L\)

\[ \mathbb{E} \# \text{bad red in } [0,L] \leq \mathbb{E} \text{ range of random walk in } [0,L] \]

\[ L \mathbb{P}[X > (2c + 1)L] \leq C/\sqrt{L} \]

\[ \mathbb{E}X^{\frac{1}{2} - \epsilon} < \infty \]
Open Questions

Any setting (colours, \(d, \gamma\)) with non-existence?
(e.g. 2 colours, \(d = 2, \gamma = 1, \infty\))

Uniqueness for \(d = 1, \gamma < 1, 2\) colours?

Perfectness for \(d \geq 2\)?

Better tail bounds for \(d = 1\)?

Tail bounds, uniqueness, phase transitions ... for \(d \geq 2\)?