Tail asymptotics for extinction times of self-similar fragmentations

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**Goal:** obtain some information on the distribution of some positive random variables, which, depending on the point of view, can be seen as:

- the extinction times of some fragmentation processes
- the heights of continuous compact rooted random trees
- the scaling limits of the heights of sequences of discrete trees (e.g. the scaling limit of the height of a uniform rooted random tree with \( n \) nodes)
Three parts:

1. Self-similar fragmentations, extinction times and connections with random trees
2. Large time asymptotics of the distribution tails of the extinction times; examples
3. Two main steps of the proof
Fragmentation models:

Fragmentation models: describe the evolution of objects that **split repeatedly** as time goes on.

![Diagram of fragmentation models](image)

**Extensive study** in Mathematics since the mid-1900s (both from deterministic and random points of view) explained by:

- **many motivations** coming from biology and population genetics, computer science, polymerization, but also random trees and graphs

- the setting of **fairly general models** that are relatively easy to study
Self-similar fragmentations

We focus on random models where objects are only characterized by their mass and the dynamic is governed by:

- a branching property: different objects evolve independently
- a self-similarity property: an object splits at a rate proportional to a power of its mass

Starting at time 0 with a unique object of mass 1, we let $F(t)$ denote the sequence of masses present at time $t \geq 0$:

$F(t) \in S := \{(s_i)_{i \geq 1} : s_1 \geq s_2 \geq s_3 \ldots ; \infty \sum_{i=1} s_i \leq 1\}$

$F(0) = (1, 0, 0, \ldots)$

The splitting rule depends on two parameters: $\alpha \in \mathbb{R}$ (the index of self-similarity) and a measure $\nu$ on $S$ such that a mass $m$ splits in masses $(m s_1, m s_2, \ldots)$ at rate $m^\alpha d\nu(s_1, s_2, \ldots)$

First ref.: Kolmogorov 41, Filippov 61, Brennan and Durrett 86-87, Bertoin 01-02

Many studies on those models since 2000+.

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The \((\alpha, \nu)\)-model when \(\nu\) is finite:

- A mass \(m\):
  - splits after a time \(\sim \text{Exp}(m^\alpha \nu(S))\)
  - in masses \((mS_1, mS_2, \ldots)\) where \((S_1, S_2, \ldots) \sim \nu(\cdot)/\nu(S)\)

**Remark.** Mean time of splitting of a fragment with mass \(m\): \(m^{-\alpha}/\nu(S)\):

- when \(\alpha > 0\) small fragments splits slower than the large ones
- when \(\alpha < 0\), small fragments splits faster than the large ones.
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When \(\nu\) is infinite: infinitely many fragmentations in any strictly positive interval of times.
Necessity that \(\int_{S} (1 - s_1) \nu(ds) < \infty\) to prevent the system to explode entirely at time 0+.
Hypotheses: $\alpha < 0$ and $\nu(S) > 0 \Rightarrow$ very small objects split very quickly!

Ex.: $\nu = \delta_{(1/2, 1/2, 0, \ldots)}$

for any $x \in (0, 1)$ non-dyadic, the fragment containing $x$ reaches mass $2^{-n}$ at time
$\sum_{i=1}^{n} T_i$, with $T_i \sim \text{Exp}(2^{-\alpha(i-1)})$

hence reaches 0 at time $\sum_{i=1}^{\infty} T_i < \infty \text{ a.s.}$
Extinction time

Hypotheses: \( \alpha < 0 \) and \( \nu(S) > 0 \) \( \Rightarrow \) very small objects split very quickly!

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In general: For any \( (\alpha, \nu) \), \( \alpha < 0 \) and any \( (\alpha, \nu) \) fragmentation \( F \):

\[
\zeta := \inf\{t \geq 0 : F(t) = (0, 0, \ldots)\},
\]

the first time at which the entire initial mass is reduced to dust.
For any \((\alpha, \nu)\), \(\alpha < 0\):

**Proposition** (Filippov 61, McGrady & Ziff 87, Bertoin 02)

The extinction time \(\zeta\) is finite almost surely.

**Proposition** (H. 03)

The tail of \(\zeta\) is exponential or even lighter:

\[
\exists \theta \geq 1 : \mathbb{P}(\zeta > t) \leq \exp(-\text{cst} \cdot t^\theta) \text{ for all } t \text{ large enough.}
\]
Connection with random trees

The r.v. \( \zeta \) may also be seen as the height of a random tree which is the scaling limit of models of discrete trees.

- **Ex.1**: \( H_n \): height of a Galton-Watson tree with offspring distribution with mean 1 and variance \( 0 < \sigma^2 < \infty \) conditioned on having total progeny \( n \).

**Aldous 93**: This GW tree, appropriately normalized, converges to the *Brownian continuum tree*. In particular,\[
\frac{H_n}{\sqrt{n}} \xrightarrow{n \to \infty} \frac{2}{\sqrt{\sigma^2}} \cdot \zeta_{Br}
\]
where \( \zeta_{Br} \) is the height of the Brownian tree.
Connection with random trees

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$$\frac{H_n}{\sqrt{n}} \xrightarrow{\text{law}} \frac{2}{\sqrt{\sigma^2}} \cdot \zeta_{\text{Br}}$$

where $\zeta_{\text{Br}}$ is the height of the Brownian tree.

**Bertoin 02:** Aldous’ Brownian tree is the genealogical tree of a self-similar fragmentation with parameters

$$\alpha = -1/2, \quad \nu(s_1 + s_2 < 1) = 0 \quad \text{and} \quad \nu(s_1 \in dx) = \frac{1_{\{x > 1/2\}}}{\sqrt{\pi} x(1-x)^{3/2}} \, dx$$

The r.v. $\zeta_{\text{Br}}$ is its extinction time.
Connection with random trees

- **Ex.2**: When the offspring distribution of the GW tree has a tail $\mathbb{P}(\text{offspring } \geq k) \sim ck^{-\beta}$ for some $\beta \in (1, 2)$, then (Duquesne 03)

\[
\frac{H_n}{n^{1-\frac{1}{\beta}}} \xrightarrow{\text{law}} C(c, \beta) \cdot \zeta_{\beta}
\]

where $\zeta_{\beta}$ is the height of the $\beta$-stable Lévy tree of Duquesne, Le Gall, Le Jan

**Miermont 03**: the $\beta$-stable Lévy tree is the genealogical tree of a self-similar fragmentation with parameters $\beta^{-1} - 1$.

- More generally: models of random discrete trees satisfying a *Markov-Branching property*, were proved to converge in the scaling limit to continuous trees describing the genealogy of $(\alpha, \nu)$-fragmentations

  (H.-Miermont-Pitman-Winkel 08, H.-Miermont 12)

$\Rightarrow$ their rescaled heights converge to the r.v. $\zeta$. 

Kennedy 76 and Duquesne & Wang 17: asymptotic expansions at all orders of $\zeta_{Br}$ and $\zeta_{\beta}$

**Theorem** (Kennedy 76, Duquesne & Wang 17)

\[
\mathbb{P}(\zeta_{Br} > t) \underset{t \to \infty}{\sim} 2t^2 \exp(-t^2) \quad \text{and} \quad \mathbb{P}(\zeta_{\beta} > t) \underset{t \to \infty}{\sim} C(\beta)t^{1+\frac{\beta}{2}} \exp(-(\beta - 1)^{\beta-1}t^{\beta})
\]

for some explicit $C(\beta)$

**Goal:** obtain similar results for general $(\alpha, \nu)$ random variables $\zeta$
Main result: Precise estimate for $\mathbb{P}(\zeta > t)$

The parameters $\alpha < 0$ and $\nu$ are fixed; $\zeta$ denotes the corresponding extinction time.

**Two functions:** we let for $x$ large enough

$$\phi(x) = \int_S (1 - s_1^{x+1})\nu(ds) \quad \text{and} \quad \psi : \frac{\psi(x)}{\phi(\psi(x))} = x$$

**Ex.:** if $\nu(s_1 \leq u) \sim c(1 - u)^{-\gamma}, \gamma \in [0, 1)$ then:

$$\phi(x) \sim c\Gamma(1 - \gamma)x^\gamma \quad \text{and} \quad \psi(x) \sim (c\Gamma(1 - \gamma)x)^{\frac{1}{1-\gamma}}$$

**Brownian frag.:** $\phi(x) \sim 2\sqrt{x}, \quad \psi(x) \sim 4x^2$
Main result: Precise estimate for $\mathbb{P}(\zeta > t)$

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**Notation:** For positive functions $f, g$,

$$f(t) \asymp g(t)$$

means there exists $a, b > 0$ such that $a \cdot g(t) \leq f(t) \leq b \cdot g(t)$ for $t$ large enough.

**Proposition (H. 03)**

If $\phi$ is regularly varying at $\infty$,

$$\ln(\mathbb{P}(\zeta > t)) \asymp -\psi(t).$$

We want to sharpen this estimate by removing the logarithm
Main result: Precise estimate for $\mathbb{P}(\zeta > t)$

Main hypothesis:

$$\limsup_{x \to \infty} \frac{\phi'(x)x}{\phi(x)} < 1$$  \hspace{1cm} (H)

Not restrictive at all!

**Theorem (H. 21)**

Assume (H). Then

$$\mathbb{P}(\zeta > t) \asymp \left( \frac{\psi(|\alpha|t)}{t} \right)^{\frac{1}{|\alpha|}} (\psi'(|\alpha|t))^{\frac{1}{2}} \exp \left( - \int_1^t \frac{\psi(|\alpha|r)}{|\alpha|r} dr \right)$$

**Corollary**

If $\phi$ is regularly varying at $\infty$,

$$\mathbb{P}(\zeta > t) \asymp \left( \frac{\psi(|\alpha|t)}{t} \right)^{\frac{1}{|\alpha|}} \exp \left( - \int_1^t \frac{\psi(|\alpha|r)}{|\alpha|r} dr \right).$$
Examples with finite splitting rate

Here \( \psi(x) \sim |\nu(S)| x \), hence \( \int_1^t \frac{\psi(|\alpha|r)}{|\alpha|r} \, dr = |\nu(S)| t + o(t) \).

**Ex.1:** Fragmentations into \( k \) identical pieces: a fragment of size \( m \) splits into \( k \) fragments of same sizes \( m/k \). For all indices of self-similarity \( \alpha < 0 \):

\[
P(\zeta > t) \sim c \exp(-t)
\]

for some \( c \in (0, \infty) \).

**Ex.2:** Uniform fragmentation: a fragment of size \( m \) splits into two fragments of sizes \( mU, m(1 - U) \), where \( U \) is uniform on \([0, 1]\). For all indices of self-similarity \( \alpha < 0 \):

\[
P(\zeta > t) \asymp t^{\frac{2}{|\alpha|}} \exp(-t).
\]
Examples with finite splitting rate

Ex.3: Beta fragmentations: a fragment of size $m$ splits into two fragments of sizes $mB, m(1 - B)$, where $B \sim \text{Beta}(a, b)$, $b \geq a > 0$ (density on $(0, 1)$ proportional to $x^{a-1}(1 - x)^{b-1}$). For all indices of self-similarity $\alpha < 0$:

$$
\mathbb{P}(\zeta > t) \approx \begin{cases} 
\exp(-t) & \text{if } b \geq a > 1 \\
t^{\frac{1}{\alpha}} \exp(-t) & \text{if } b > a = 1 \\
t^{\frac{2}{\alpha}} \exp(-t) & \text{if } b = a = 1 \\
\exp \left( -t + \frac{\Gamma(a)}{(1-a)\alpha^a} t^{1-a} \right) & \text{if } b > 1 > a > 1/2 \\
t^{\frac{1}{\alpha}} \exp \left( -t + \frac{\Gamma(a)}{(1-a)\alpha^a} t^{1-a} \right) & \text{if } 1 = b \geq a > 1/2 \\
\exp \left( -t + \frac{\Gamma(a)}{(1-a)\alpha^a} t^{1-a} + \frac{\Gamma(b)}{(1-b)\beta^b} t^{1-b} \right) & \text{if } 1 > b \geq a > 1/2.
\end{cases}
$$

If $a$ (and possibly $b$) is smaller than $1/2$, there will be additional terms.
Examples with infinite splitting rates

Ex.4: **Aldous’ beta-splitting models**: scaling limits of discrete models introduced by Aldous to interpolate between some phylogenetic trees.

Parametrized by $\beta \in (-2, -1)$; binary splitting ($\nu(s_1 + s_2 < 1) = 0$) and

\[
\nu(s_1 \in du) = \frac{-\beta - 1}{\Gamma(2 + \beta)} (u(1 - u))^{\beta}, \quad u \in (1/2, 1) \quad \text{and} \quad \alpha = 1 + \beta.
\]

Then for $\beta \in (-2, -3/2]$:

\[
P(\zeta > t) \lessapprox t \frac{-2\beta - 1}{2(\beta + 2)} \exp \left(-a_{\beta} t^\frac{1}{\beta + 2} + b_{\beta} t\right)
\]

where $a_{\beta} = (\beta - 1) \frac{-\beta - 1}{\beta + 2} (\beta + 2)$ and $b_{\beta} = \frac{(2\beta + 3)\Gamma(\beta + 2)}{(\beta + 2)\Gamma(2\beta + 4)}$.

For $\beta \in (-3/2, 1)$: additional power terms in the exponential.
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For $\beta \in (-3/2, 1)$: additional power terms in the exponential.

Ex.5: **Height of stable Lévy trees.** Then $\phi(x) = \beta x^{1 - \frac{1}{\beta}} \left(1 - \frac{\beta - 1}{2\beta^2} x^{-1} + O(x^{-2})\right)$

So we retrieve, for all $\beta \in (1, 2)$:

$$
P(\zeta > t) \asymp t^{1 + \frac{\beta}{2}} \exp \left(-(\beta - 1)^{-1} t^{\beta}\right).
$$
Outline of the proof of the theorem

An intermediate tool: the extinction time of a typical point

\[ U \sim \text{Unif}(0, 1) \]

\( I \): time at which \( U \) is reduced to dust

**Proposition** (Bertoin 02)

\[ I = \int_0^\infty \exp(\alpha \xi_t) dt \]

where \( \xi \) is a subordinator (increasing Lévy process) with Laplace exponent \( \bar{\phi} \) (i.e. \( \mathbb{E}[\exp(-x\xi_t)] = \exp(-t\bar{\phi}(x)), \forall x, t \geq 0 \)) where \( \bar{\phi}(x) = \int_S (1 - \sum_i s_i^{x+1}) \nu(ds) \).

**Rk.** \( \bar{\phi}(x) = \phi(x) + O(2^{-x}) \) as \( x \to \infty \).
Two main steps

Step 1. Link between the tails of $\zeta$ and $I$

Proposition 1 (H. 21)
Assume (H). Then,

$$\mathbb{P}(\zeta > t) \asymp \left( \frac{\psi(|\alpha|t)}{t} \right)^{1/|\alpha|} \cdot \mathbb{P}(I > t)$$

Step 2. Asymptotics of the tail of $I$

Proposition 2 (H. 21)
Assume (H). Then there exists $c \in (0, \infty)$ such that

$$\mathbb{P}(I > t) \underset{t \to \infty}{\sim} c \cdot \frac{t(\psi'(|\alpha|t))^{1/2}}{\psi(|\alpha|t)} \cdot \exp \left( - \int_{1}^{t} \frac{\psi(|\alpha|r)}{|\alpha|r} \, dr \right).$$
Some hints for Step 1

Remark: $I < \zeta$ and it is a priori not obvious how to compare their tails

**Step 1. a) Connections with moments of typical fragments.**

$U_1, U_2$ uniformly distributed on $(0, 1)$, independent

$\Lambda(i)(t)$: mass of the fragment containing $U_i$ at time $t$, $i = 1, 2$

**Proposition** (H. 21)

There exists $c \in (0, \infty)$ such that for all $t$ large enough

$$\frac{\mathbb{E} [\Lambda(1)(t)^2]}{\mathbb{E} [\Lambda(1)(t)\Lambda(2)(t)]} \leq \mathbb{P}(\zeta > t) \leq c \left( \frac{\psi(|\alpha|t)}{t} \right)^{\frac{2}{|\alpha|}} \mathbb{E} [\Lambda(1)(t)]$$

Idea: Introduce $S(t) := \sum_{i \geq 1} (F_i(t))^2$ and use the first and second moments methods.
**Step 1. b) Asymptotics of moments of 1 and 2 typical fragments**

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<th>Proposition (H.- Rivero 12)</th>
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Some references

- **On fragmentation models and the existence of shattering:**

- **On the tail of the random variables $\zeta$:**
  - **T. Duquesne, M. Wang**, *Decomposition of Lévy trees along their diameter*, Ann. IHP 2017
  - **B. Haas**, *Tail asymptotics for extinction times of self-similar fragmentations*, In preparation.