The small subgraph conditioning method and hypergraphs

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Also establishes the asymptotic distribution of Y and a property called contiguity of two probability spaces.

See Wormald's 1999 regular graphs survey
+ Janson's 1995 paper with "contiguity" in the title.

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Second moment method works.

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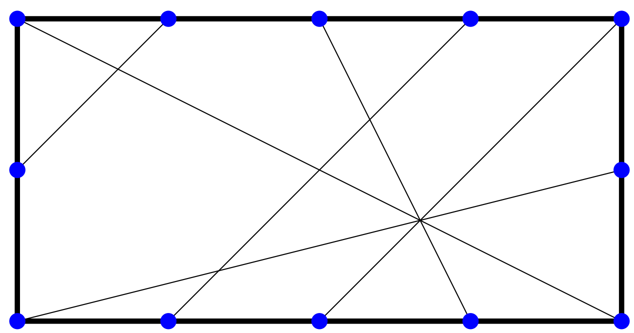
Robinson & Wormald faced exactly this problem when studying Hamilton cycles in random 3-regular graphs.



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$\Rightarrow$ Small cycles can have a **big** effect!

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This paper introduced the small subgraph conditioning method.

Janson (1995) observed that R & W's proof technique also

- gives the asymptotic distribution of Y, and
- establishes a property called "contiguity" between $\mathcal{G}_{n,3}$ and a probability space, denoted $\mathcal{G}_{n,3}^{(Y)}$, where each 3-regular graph G on $[n]$ has probability proportional to $Y(G)$.

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Suppose $(\mathcal{A}_n)$ and $(\mathcal{B}_n)$ are two sequences of probability spaces on the same sequence of underlying sets (Ω_n).

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Say $(\mathcal{A}_n)$ and $(\mathcal{B}_n)$ are (mutually) contiguous if

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Write $\mathcal{A}_n \approx \mathcal{B}_n$ when $(\mathcal{A}_n)$ and $(\mathcal{B}_n)$ are contiguous.

Janson: contiguity is "qualitative asymptotic equivalence".

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Observe, a graph in $\mathcal{G}_{n,3}^{(Y)}$ is Hamiltonian with probability 1.

Then contiguity immediately implies that

$$\Pr(\mathcal{G}_{n,3} \text{ is Hamiltonian}) \to 1.$$

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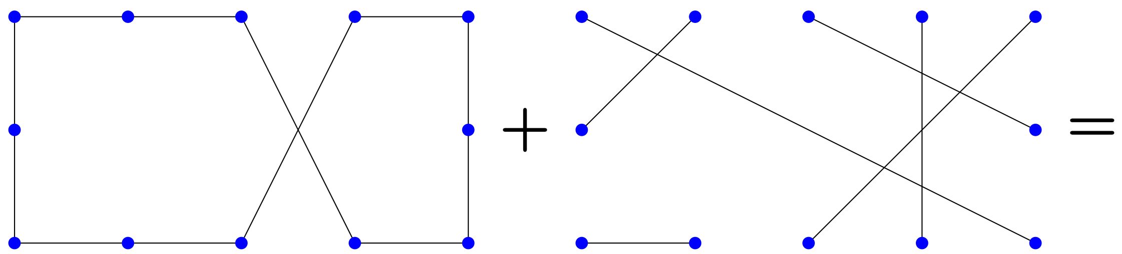

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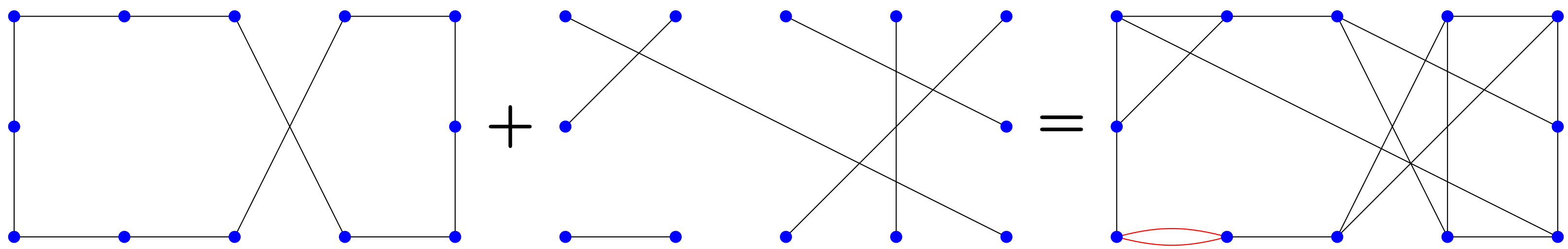

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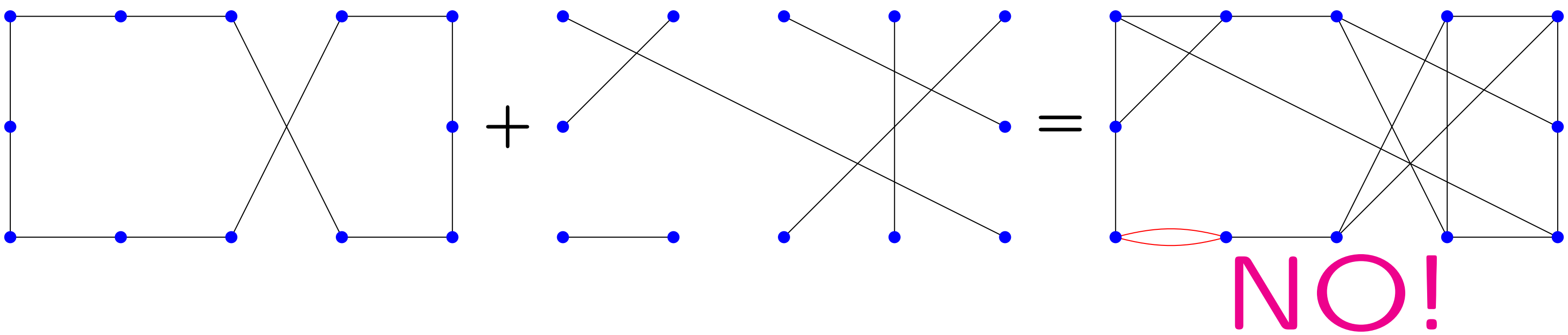

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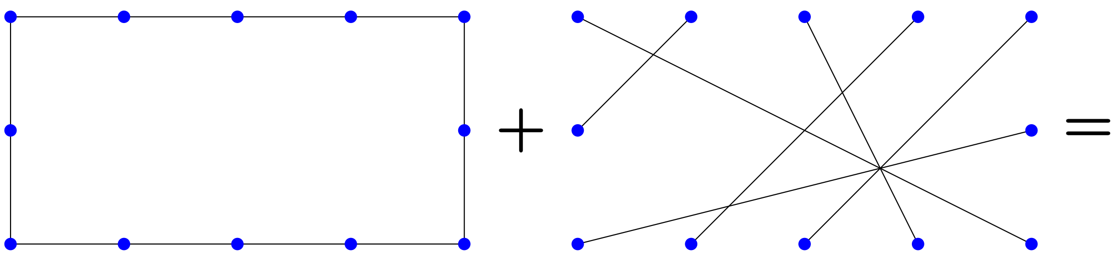

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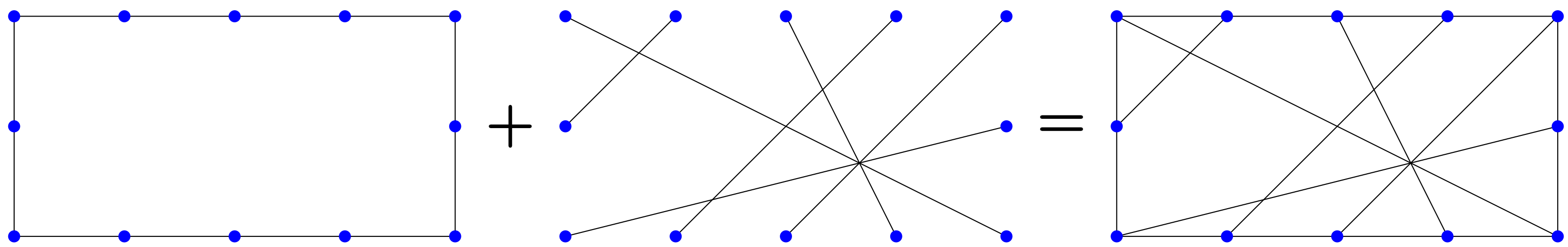

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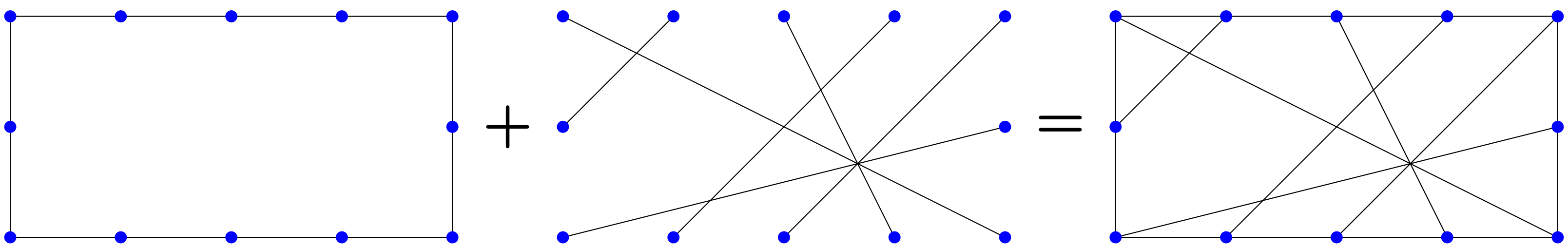

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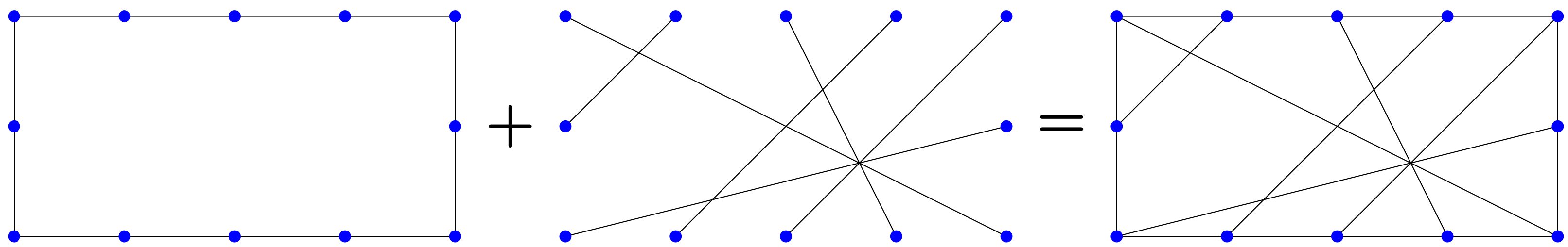

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Further suppose that:

(A1) $X_{in} \xrightarrow{d} Z_i$ as $n \to \infty$, jointly for all $i \geq 1$, where $Z_i \sim \mathrm{Po}(\lambda_i)$ are independent Poisson.

(A2) For any sequence $x_1, \ldots, x_m$ of nonnegative integers,

$$\frac{\mathbb{E}(Y_n \mid X_{1n} = x_1, \ldots, X_{mn} = x_m)}{\mathbb{E}\, Y_n} \to \prod_{i=1}^{m} (1 + \delta_i)^{x_i}\, e^{-\lambda_i \delta_i}$$

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Then (distribution version, Janson 1995):

$$\frac{Y_n}{\mathbb{E}\, Y_n} \xrightarrow{d} W = \prod_{i=1}^{\infty} (1+\delta_i)^{Z_i}\, e^{-\lambda_i \delta_i} \quad \text{as } n \to \infty;$$

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moreover, this and the convergence in (A1) hold jointly.
Also, $W > 0$ almost surely if and only if $\delta_i > -1$ for all i.

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Then (contiguity version, Wormald 1999):

$$\Pr(Y_n > 0) = \exp\left(- \sum_{\delta_i = -1} \lambda_i \right) + o(1)$$

and $\mathcal{G}_n^{(Y_n)} \approx \mathcal{G}_n$ if $\delta_i > -1$ for all i.

Many structural results about regular graphs, regular bipartite graphs, proved using SSCM by various authors:

Delcourt, Frieze, Greenhill, Janson, Jerrum, Kim, Kwan, Molloy, Postle, Prałat, Robalewska, Robinson, Ruciński, Shi, Wind, Wormald.

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Prałat & Wormald (2019)

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Also: Bank, Moore, Neeman, Netrapalli (2016),
community detection in sparse networks.

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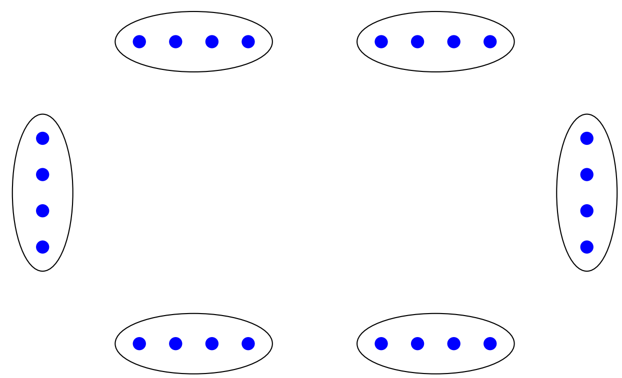
The SSCM works when the variance of Y is well-controlled by the short cycle counts.

Configuration model (Bollobás, 1980)

Start with n cells, each containing d points. Take a uniformly random perfect matching of dn points into $dn/2$ pairs.



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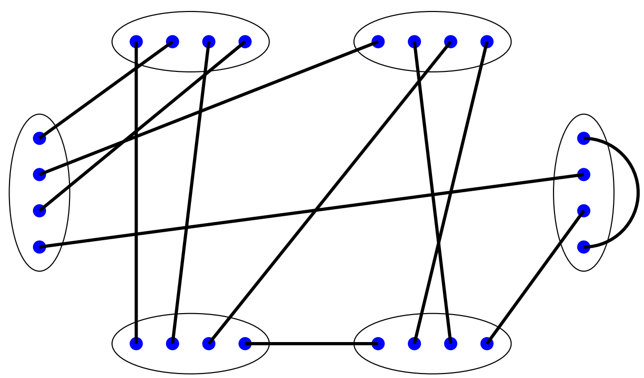

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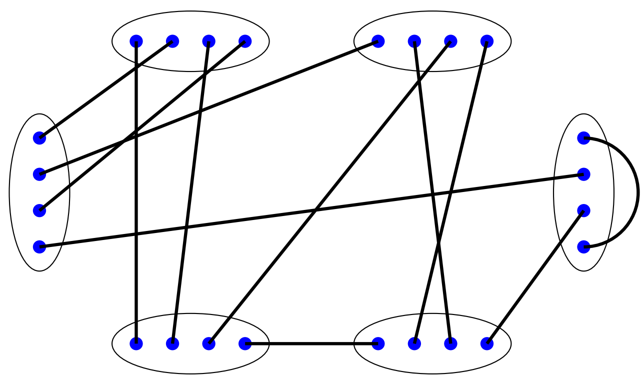
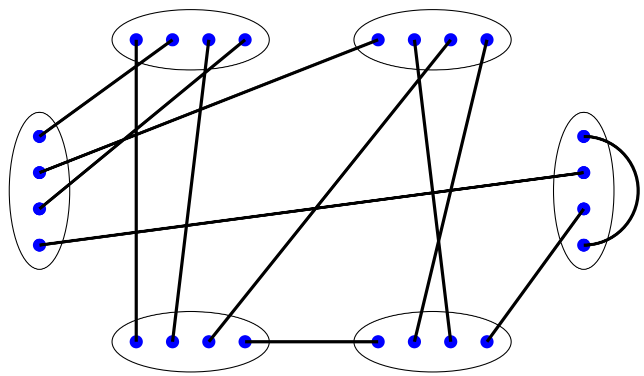


Shrink each cell to a vertex to get a d-regular multigraph. Every simple graph is equally likely.

Configuration model (Bollobás, 1980)

Start with n cells, each containing d points. Take a uniformly random perfect matching of dn points into $dn/2$ pairs.



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Bender & Canfield (1978): Pr(simple) $\sim e^{-(d^2-1)/4}$.

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Also, if the uniform and Y-weighted configuration models are contiguous then $\mathcal{G}_{n,d}^{(Y)} \approx \mathcal{G}_{n,d}$.

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$$\frac{Y}{\mathbb{E}\,Y} \xrightarrow{d} W = \prod_{i=1}^{\infty} (1+\delta_i)^{Z_i}\, e^{-\lambda_i \delta_i}$$

for some constants λ_i, δ_i, and where the random variables $Z_i \sim \mathsf{Po}(\lambda_i)$ are independent.

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Now (A2) implies that

$$\frac{\mathbb{E}\,Y_{\mathcal{G}}}{\mathbb{E}\,Y} = \frac{\mathbb{E}(Y \mid X_{1n} = X_{2n} = 0)}{\mathbb{E}\,Y} \to e^{-\lambda_1\delta_1 - \lambda_2\delta_2}.$$

(A configuration gives a simple graph iff $X_{1n} = X_{2n} = 0$.)

By the joint convergence of $(\frac{Y}{\mathbb{E}(Y)}, X_{1n}, X_{2n})$ to (W, Z_1, Z_2), we conclude that

$$\begin{aligned}\mathcal{L}\left(\frac{Y_{\mathcal{G}}}{\mathbb{E}\,Y}\right) &= \mathcal{L}\left(\frac{Y}{\mathbb{E}\,Y}\middle| X_{1n} = X_{2n} = 0\right) \\ &\xrightarrow{d} \mathcal{L}(W \mid Z_1 = Z_2 = 0) \\ &= \mathcal{L}\left(e^{-\lambda_1\delta_1 - \lambda_2\delta_2} \prod_{i=3}^{\infty} (1+\delta_i)^{Z_i}\, e^{-\lambda_i\delta_i}\right).\end{aligned}$$

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Hence

$$\frac{Y_{\mathcal{G}}}{\mathbb{E}\,Y_{\mathcal{G}}} \sim e^{\lambda_1\delta_1 + \lambda_2\delta_2} \frac{Y_{\mathcal{G}}}{\mathbb{E}\,Y} \xrightarrow{d} \prod_{i=3}^{\infty} (1+\delta_i)^{Z_i}\, e^{-\lambda_i\delta_i}.$$

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TL;DR Delete $i = 1, 2$ factors to get result for regular graphs!

What about hypergraphs?

Let $\mathcal{G}_{n,r,s}$ denote a uniformly random r-regular s-uniform hypergraph on $[n]$. Here r, s are fixed constants. Assume $s|rn$.

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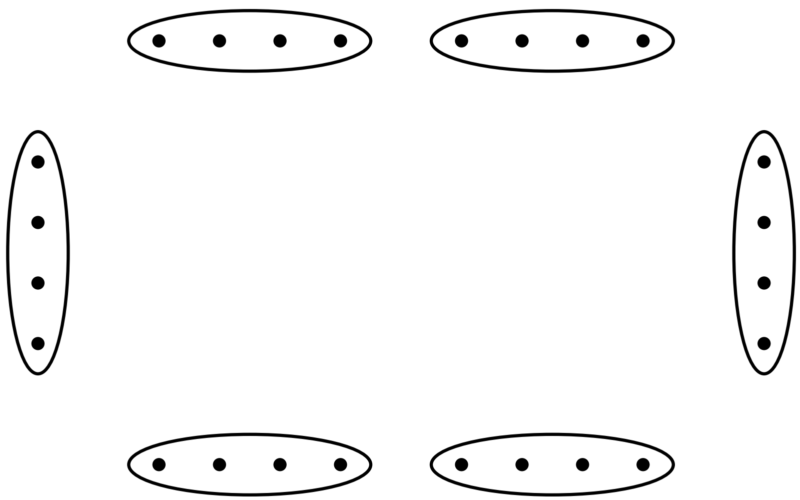

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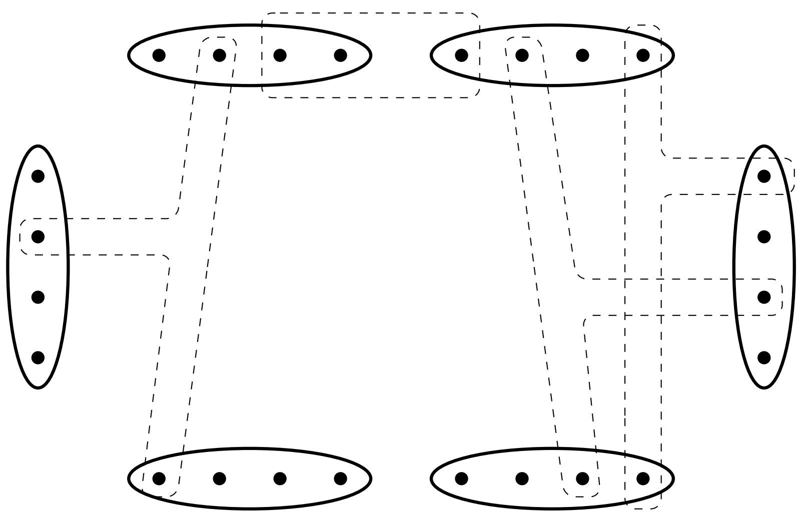

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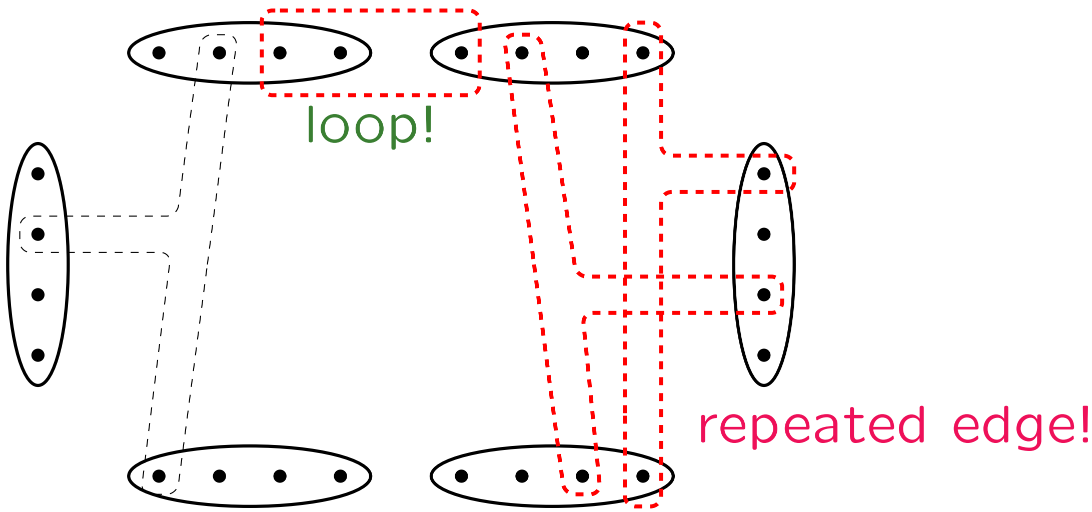

Cooper, Frieze, Molloy & Reed (1996): Pr(simple) $\sim e^{-\frac{(r-1)(s-1)}{2}}$.

Let X_{in} be the number of loose i-cycles in $\mathcal{G}_{n,r,s}$, for $i \geq 2$, and let X_{1n} be the number of 1-cycles.

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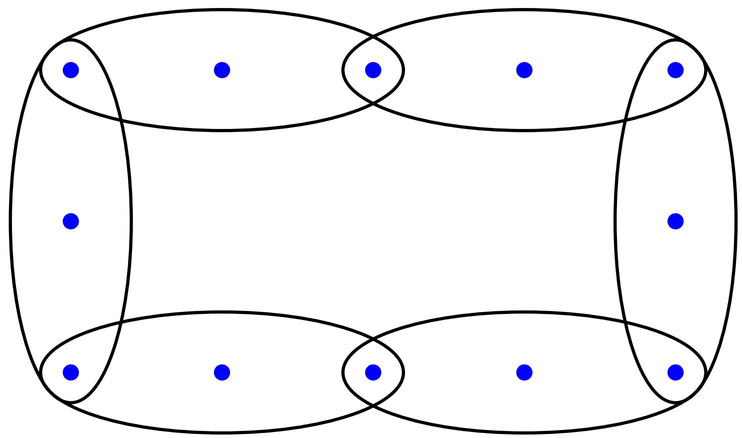

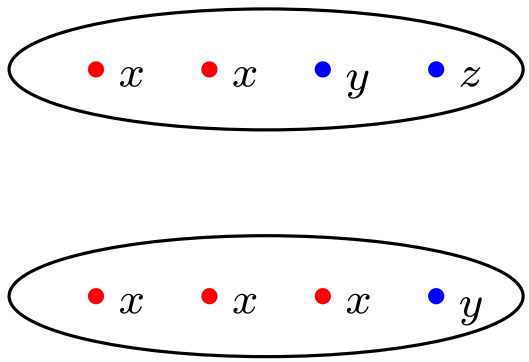

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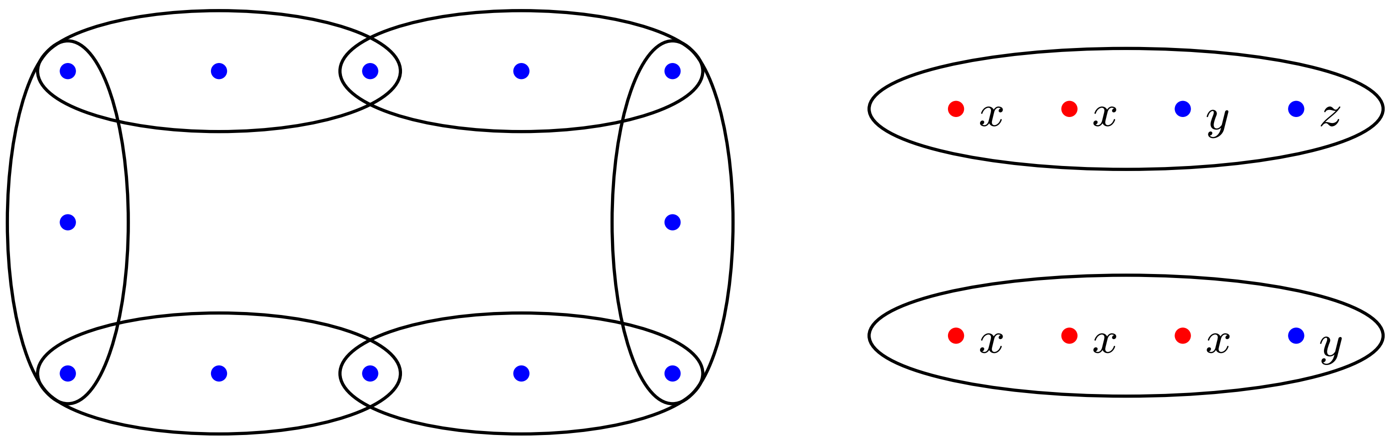

Cooper et al. (1996) proved that the X_{in} are asymptotically independent Poisson random variables, with

$$\mathbb{E}\, X_{in} \to \lambda_i = \frac{((r-1)(s-1))^i}{2i}.$$

So (A1) holds.

PROBLEM: In the configuration model, when $s \geq 3$, the event "is simple" is NOT captured by conditioning on the event $X_{1n} = X_{2n} = 0$, or on the event $X_{1n} = 0$.

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$\Rightarrow$ The SSCM can't tell us $\frac{\mathbb{E}\, Y_{\mathcal{G}}}{\mathbb{E}\, Y}$.

Cooper, Frieze, Molloy & Reed (1996): existence threshold for perfect matchings, which a.a.s. exist in $\mathcal{G}_{n,r,s}$ when $s < \sigma_r$, where

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Altman, Greenhill, Isaev, Ramadurai (2020):
existence threshold for loose Hamilton cycles, which a.a.s. exist in $\mathcal{G}_{n,r,s}$ when $r > \rho(s)$, where

$$\rho(s) \approx \frac{e^{s-1}}{s-1} - \frac{s-2}{2} + o_s(1).$$

(The $o_s(1)$ term tends to zero exponentially fast as $s \to \infty$.)

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If $s \geq 5$ then spanning trees a.a.s. exist in $\mathcal{G}_{n,r,s}$ when $r > \rho(s)$, where

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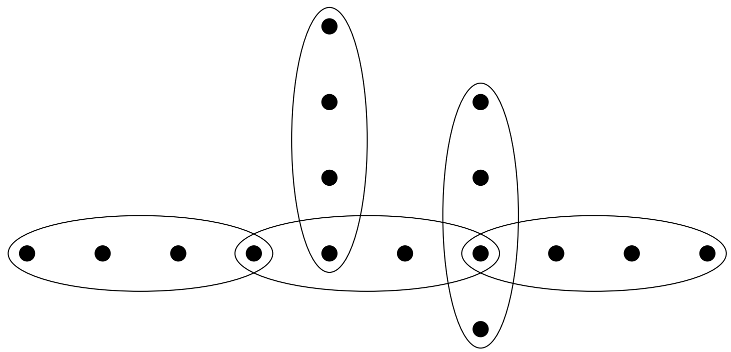
We build on earlier work by Greenhill, Kwan, Wind (2014) for graphs, which

- found expected number of spanning trees in $\mathcal{G}_{n,d}$ for $d \geq 3$,
- gave asymptotic distribution for cubic graphs.

A tree is connected and acyclic, where these terms are defined using Berge cycles and Berge paths. No 2-cycles means that edges overlap in at most 1 vertex (linear).



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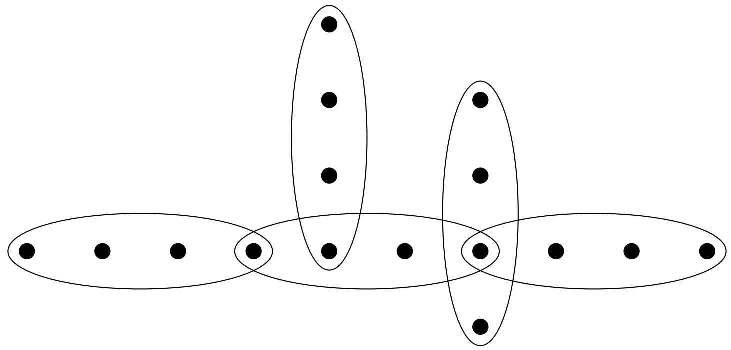

A necessary condition for an s-uniform hypergraph on $[n]$ to contain a spanning tree is that

$$n = (s-1)t + 1$$

where $t = \frac{n-1}{s-1} \in \mathbb{Z}^{+}$ is the number of edges in the spanning tree.

Trees in uniform hypergraphs

Suppose that $n = (s-1)t + 1$ for some $t \in \mathbb{Z}^+$.

Bolian (1988) The number of s-uniform trees on $[n]$ is

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When $s = 2$ we recover Cayley's formula (here $t = n - 1$).

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This generalises the result of Moon (1970) in the graph case. These results can be proved using a hypergraph analogue of Prüfer codes.

Expected number

By summing over all tree degree sequences $\boldsymbol{x}$, we showed that the expected number of spanning trees in the configuration model is

$$\mathbb{E}\, Y = \frac{(s-1)(n-2)!}{((s-1)!)^t} \sum_{\boldsymbol{x}} \prod_{j=1}^{n} \frac{(r)_{x_j}}{(x_j-1)!} \frac{p(rn-st)}{p(rn)}$$

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$$= \frac{r^n\,(s-1)\,(n-2)!}{((s-1!)^t} \binom{(r-1)n}{t-1} \frac{p(rn-st)}{p(rn)}$$

where $p(sN)$ is the number of ways to partition sN points into N subsets (parts) of s points.

Apply Stirling's formula:

$$\mathbb{E}\,Y$$

$$\sim \frac{(r-1)^{1/2}(s-1)}{n(rs-r-s)^{\frac{s+1}{2(s-1)}}} \left(\frac{(s-1)^r\,(r-1)^{r-1}}{r^{(rs-r-s)}\,(rs-r-s)^{(rs-r-s)/(s-1)}} \right)^{n/s}.$$

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The behaviour is dominated by the base of the exponential: taking the logarithm, let

$$L_s(r) = \tfrac{r}{s}\log(s-1) + (r-1)\log(r-1) \\ - \tfrac{rs-r-s}{s}\log(r) - \tfrac{rs-r-s}{s(s-1)}\log(rs-r-s).$$

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For $s \geq 5$ there is a unique threshold $\rho(s) \in (2,\infty)$ so that

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s	5	6	7	8	9	10	11
$\rho(s)$	3.03	8.71	22.14	54.61	133.59	327.25	805.84

Short cycles

We calculated that in the configuration model,

$$\frac{\mathbb{E}(Y\,X_j)}{\mathbb{E}\,Y} \longrightarrow \lambda_j(1+\delta_j)$$

where

$$\delta_j = \frac{\left(\frac{r}{r-1} - s + 1\right)^j - 2}{((r-1)(s-1))^j}.$$

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Then we showed that (A2) and (A3) hold, and

$$\exp\left(\sum_{k=2}^{\infty} \lambda_k \delta_k^2\right) = \frac{r^2\sqrt{s-1}}{\sqrt{(r^2 - rs + r + s - 1)(rs - r - s)(r-1)}}.$$

Second moment

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We express the second moment as, up to a $(1 + o(1))$ factor,

$$\sum_{(k,b)\in D} \psi(k/n, b/n)\ \mathrm{exp}(n\,\varphi(k/n, b/n))$$

where k, b are two parameters arising from the combinatorics and D is the natural domain of these parameters. The function $\psi(\alpha, \beta)$ is relatively unimportant . . .

. . . and

$$\begin{aligned}\varphi(\alpha,\beta) = &(\alpha+\beta)\log(r-1) + g(\alpha+\beta) + g(r-1-\alpha-\beta)\\ &- \tfrac{2}{s-1}g(\beta) - g(\alpha) - \tfrac{1}{s(s-1)}g(rs-r-s-s\beta)\\ &- \tfrac{1}{s-1}g(1-(s-1)\alpha-\beta)\end{aligned}$$

where $g(x) = x\log x$ for $x > 0$, and $g(0) = 0$.

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Lemma: Assume that $r, s \geq 2$ such that $r > \rho(s)$ when $s \geq 5$, or $r \geq 3$ when $s \in \{2, 3, 4\}$. Then φ has a unique maximum in the relevant domain at the point

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This implies that (A4) holds $\Rightarrow$ can apply SSCM to Y.

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This leads to the existence threshold result, and gives us the asymptotic distribution: if $\mathbb{E}\, Y_{\mathcal{G}} \to \infty$ then

$$\frac{Y_{\mathcal{G}}}{\mathbb{E}\, Y_{\mathcal{G}}} \xrightarrow{d} \prod_{j=2}^{\infty} (1+\delta_j)^{Z_j}\, e^{\lambda_j \delta_j} \qquad \text{as} \quad n \to \infty.$$

Some ingredients in the proof

We used Generalised Jensen's identity: for $b \geq 2$,

$$\sum_{\substack{k_1+\cdots+k_b=m,\\ k_j\geq 0}} \prod_{i=1}^{b} \binom{x_i+ck_i}{k_i}$$

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Also generating functions (for short cycles) and a Laplace summation theorem from Greenhill, Janson and Ruciński (2010) to help with the second moment calculations.

Greenhill, Janson, Ruciński (2010), Laplace summation tool.

Say you want to evaluate

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$K \subset \mathbb{R}^m$ is a compact convex set with non-empty interior,

$a_n(\boldsymbol{\ell})$ is a product of factorials and powers.

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where

$\det(\mathcal{L})$ is the determinant of the lattice $\mathcal{L}$,

and H_0 is the Hessian of φ at $\boldsymbol{x}_0$.