A Topological Turán Problem

Corrine Yap

Rutgers University

Joint work with Jason Long and Bhargav Narayanan

March 9, 2021

Background

Turán Problem: Given a graph H, how many edges can an n-vertex graph have while containing no isomorphic copy of H?

Background

Turán Problem: Given a graph H, how many edges can an n-vertex graph have while containing no isomorphic copy of H?

Q1: What if we replace "graph" with "r-uniform hypergraph?" (r-graph)

Background

Turán Problem: Given a graph H, how many edges can an n-vertex graph have while containing no isomorphic copy of H?

Q1: What if we replace "graph" with "r-uniform hypergraph?" (r-graph)

Q2: What if we replace "isomorphic" with "homeomorphic?"

Topological POV

Definition

- A *simplicial complex* $\mathcal{S}$ on a vertex set V is a family of subsets of V, closed under taking subsets.
- Elements of $\mathcal{S}$ are called *simplices* or *faces*.
- Maximal faces are called *facets*.
- The *dimension of a face* $e \in \mathcal{S}$ is $|e| - 1$.
- The *dimension of* $\mathcal{S}$ is the maximum dimension of its faces.

Topological POV

Definition

- A *simplicial complex* $\mathcal{S}$ on a vertex set V is a family of subsets of V, closed under taking subsets.
- Elements of $\mathcal{S}$ are called *simplices* or *faces*.
- Maximal faces are called *facets*.
- The *dimension of a face* $e \in \mathcal{S}$ is $|e| - 1$.
- The *dimension of* $\mathcal{S}$ is the maximum dimension of its faces.

Topological POV

We think of r-graphs H and G as geometric / topological structures.

Topological Turán Problem

First posed by Nati Linial: Given a k-complex $\mathcal{S}$, how many facets can an n-vertex k-complex have while containing no homeomorphic copy of $\mathcal{S}$?

Known: For a k-complex $\mathcal{S}$, there exists $\lambda(\mathcal{S}) > 0$ such that every k-complex on $n^{k+1-\lambda(S)}$ facets contains a homeomorph of $\mathcal{S}$. ("folklore," or results of Erdős on degenerate Turán problems)

Q: Is there some $\lambda_k > 0$ such that $\lambda(\mathcal{S}) \geq \lambda_k$ for all k-dimensional $\mathcal{S}$?

Topological Turán Problem

First posed by Nati Linial: Given a k-complex $\mathcal{S}$, how many facets can an n-vertex k-complex have while containing no homeomorphic copy of $\mathcal{S}$?

Known: For a k-complex $\mathcal{S}$, there exists $\lambda(\mathcal{S}) > 0$ such that every k-complex on $n^{k+1-\lambda(S)}$ facets contains a homeomorph of $\mathcal{S}$.
("folklore," or results of Erdős on degenerate Turán problems)

Q: Is there some $\lambda_k > 0$ such that $\lambda(\mathcal{S}) \geq \lambda_k$ for all k-dimensional $\mathcal{S}$?

Topological Turán Problem

First posed by Nati Linial: Given a k-complex $\mathcal{S}$, how many facets can an n-vertex k-complex have while containing no homeomorphic copy of $\mathcal{S}$?

Known: For a k-complex $\mathcal{S}$, there exists $\lambda(\mathcal{S}) > 0$ such that every k-complex on $n^{k+1-\lambda(S)}$ facets contains a homeomorph of $\mathcal{S}$.
("folklore," or results of Erdős on degenerate Turán problems)

Q: Is there some $\lambda_k > 0$ such that $\lambda(\mathcal{S}) \geq \lambda_k$ for all k-dimensional $\mathcal{S}$?

Results for 1- and 2-Complexes

- Mader '67: $\lambda_1 = 1$.
- Brown-Erdős-Sós '73: $\lambda(\mathcal{S}) = \frac{1}{2}$ when $\mathcal{S}$ is the 2-sphere.
- Keevash-Long-Narayanan-Scott '20: $\lambda_2 \geq \frac{1}{5}$
- Kupavskii-Polyanskii-Tomon-Zakharov '20: $\lambda(\mathcal{S}) = \frac{1}{2}$ if $\mathcal{S}$ is a triangulation of a closed orientable surface.
- Conjecture: $\lambda_2 = \frac{1}{2}$

3-complexes: ??? General k-complexes: ???

Results for 1- and 2-Complexes

- Mader '67: $\lambda_1 = 1$.
- Brown-Erdős-Sós '73: $\lambda(\mathcal{S}) = \frac{1}{2}$ when $\mathcal{S}$ is the 2-sphere.
- Keevash-Long-Narayanan-Scott '20: $\lambda_2 \geq \frac{1}{5}$
- Kupavskii-Polyanskii-Tomon-Zakharov '20: $\lambda(\mathcal{S}) = \frac{1}{2}$ if $\mathcal{S}$ is a triangulation of a closed orientable surface.
- Conjecture: $\lambda_2 = \frac{1}{2}$

3-complexes: ??? General *k*-complexes: ???

Main Theorem

Theorem 1 (Long-Narayanan-Y.'20)

$\lambda_k \geq k^{-2k^2}$ *for all* $k \in \mathbb{N}$.

Far from tight: e.g. $\lambda_2 \geq 2^{-8}$
But the proof is purely combinatorial, vs the dimension-specific arguments in previous results.

Our combinatorial/probabilistic tools:

1. Trace-bounded hypergraphs
2. Dependent random choice

Main Theorem

Theorem 1 (Long-Narayanan-Y.'20)

$\lambda_k \geq k^{-2k^2}$ *for all* $k \in \mathbb{N}$.

Far from tight: e.g. $\lambda_2 \geq 2^{-8}$
But the proof is purely combinatorial, vs the dimension-specific arguments in previous results.

Our combinatorial/probabilistic tools:

1. Trace-bounded hypergraphs
2. Dependent random choice

Trace-Bounded Hypergraphs

Definition

The *trace* of an r-graph G on $U \subset V(G)$ is

$$Tr(G, U) = \{S \cap U : S \in E(G)\}$$

For r-partite r-uniform G, let $V(G) = X_1 \cup X_2 \cup \cdots \cup X_r$ and let $Tr_i(G) = Tr(G, X_1 \cup \cdots \cup X_i)$.

Definition

G as above is *d-trace-bounded* if for each i, $deg_{Tr_i(G)}(v) \leq d$ for all $v \in X_i$.

Trace-Bounded Hypergraphs

Definition

The *trace* of an r-graph G on $U \subset V(G)$ is

$$Tr(G, U) = \{S \cap U : S \in E(G)\}$$

For r-partite r-uniform G, let $V(G) = X_1 \cup X_2 \cup \cdots \cup X_r$ and let $Tr_i(G) = Tr(G, X_1 \cup \cdots \cup X_i)$.

Definition

G as above is *d-trace-bounded* if for each i, $deg_{Tr_i(G)}(v) \leq d$ for all $v \in X_i$.

Example

Definition

G as above is *d-trace-bounded* if for each i, $deg_{Tr_i(G)}(v) \leq d$ for all $v \in X_i$.

Main Combinatorial Result

Theorem 2 (L-N-Y)

$\exists\ \alpha_{r,d} \geq (5rd)^{1-r}$ such that for any d-trace-bounded r-partite r-graph H, any r-graph with at least $n^{r-\alpha_{r,d}}$ edges contains a copy of H.

Observe: By itself, this gives a universal lower bound for Turán exponents on degenerate trace-bounded hypergraphs (generalizing a result of Conlon-Fox-Sudakov).

Theorem 1 follows:

Theorem 2 Construction

G: r-graph (WMA r-partite) on $X_1 \cup \cdots \cup X_r$ with at least $n^{r-\alpha_{r,d}}$ edges.
H: d-trace-bounded r-partite r-graph on $Y_1 \cup \cdots \cup Y_r$

We want to embed H into G one part at a time.

Embedding H

Potential obstacles? Not enough "extension" possibilities.

Definition

The *link* $L(v, G)$ of a vertex v in G is the $(r-1)$-graph with edges $\{S : S \cup \{v\} \in E(G)\}$.

Embedding H

Let $\mathcal{H}(t,d)$ be the collection of t-graphs that have at most d edges. For each $J \in \mathcal{H}(t,d)$, we identify the "bad" copies of J in G

Lemma

Given a t-partite t-graph with many edges and few bad copies of each $J \in \mathcal{H}(t,d)$, we can find a $(t-1)$-partite $(t-1)$-graph in $Tr_{t-1}(G)$ with many edges and few bad copies of each $L \in \mathcal{H}(t-1,d)$.

We apply this lemma iteratively to embed H.

Embedding H

Let $\mathcal{H}(t,d)$ be the collection of t-graphs that have at most d edges. For each $J \in \mathcal{H}(t,d)$, we identify the "bad" copies of J in G

Lemma

Given a t-partite t-graph with many edges and few bad copies of each $J \in \mathcal{H}(t,d)$, we can find a $(t-1)$-partite $(t-1)$-graph in $Tr_{t-1}(G)$ with many edges and few bad copies of each $L \in \mathcal{H}(t-1,d)$.

We apply this lemma iteratively to embed H.

Open Problems

- Optimal values for λ_k (in particular, $\lambda_2 = \frac{1}{2}$)?
- $\lambda(S^k)$?
- Optimal values for $\alpha_{r,d}$?

Thank You!