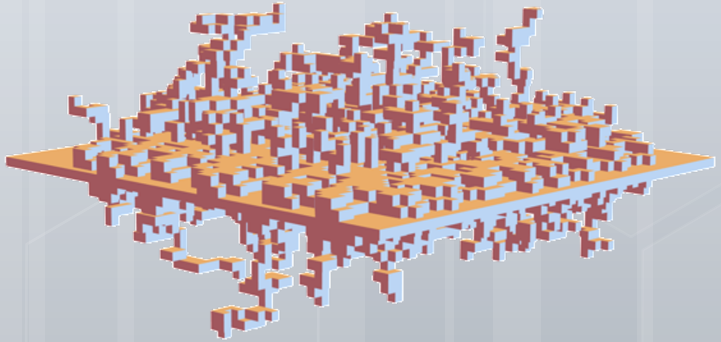
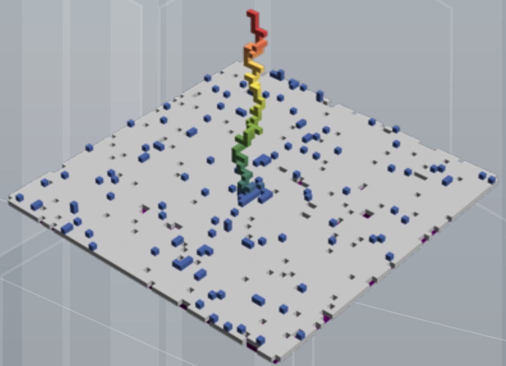
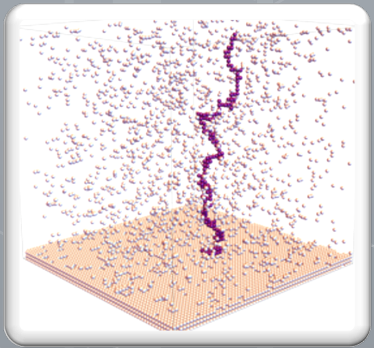


Oxford

May 2020

Maximum height of 3D Ising interfaces





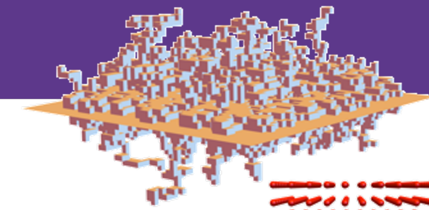
Eyal Lubetzky

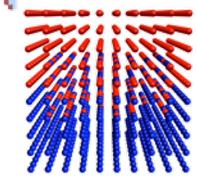
Courant Institute (NYU)

based on joint works with

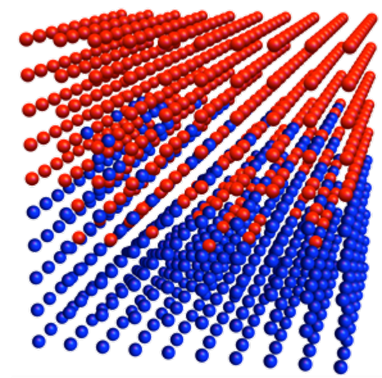
Reza Gheissari (UC Berkeley)

3D Ising interfaces

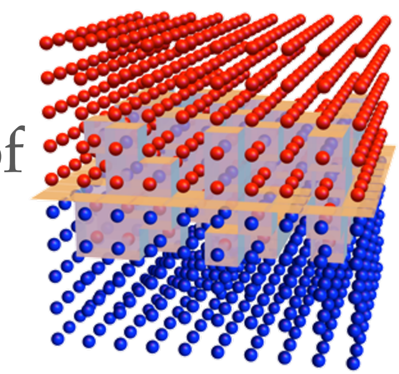


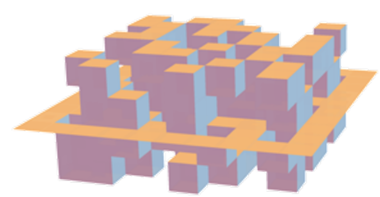


- Consider surfaces generated as follows:
  - 3D cylinder $\Lambda = [\![-n, n]\!]^2 \times (\mathbb{Z} + \frac{1}{2})$
  - σ is a 2-coloring of the vertices:
    - boundary vertices: $\begin{cases} - & \textit{upper half-space} \\ + & \textit{lower half-space} \end{cases}$
    - internal vertices: arbitrarily (*for now*).
  - Draw a **dual-face** $(u, v)^*$ if $\sigma_u \neq \sigma_v$.



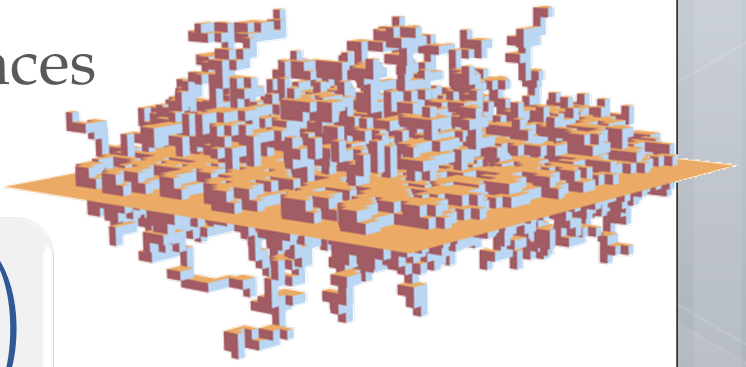
- **Interface**: (max) connected component $\mathcal{I}$ of dual-faces separating the boundary.





3D Ising interfaces (ctd.)

- Goal: understand random interfaces sampled via the distribution:



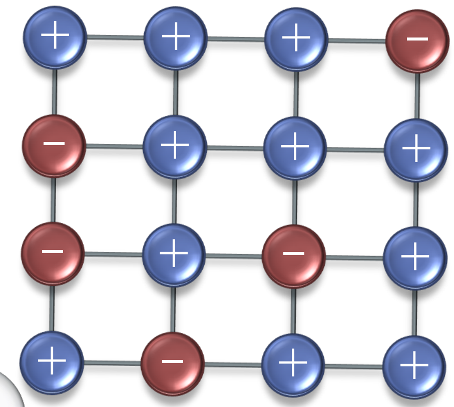
$$\mu(\mathcal{I}) \propto \exp\left(-\beta|\mathcal{I}| + \sum_{f\in\mathcal{I}} \mathbf{g}(f,\mathcal{I})\right)$$

  - $\beta > 0$: inverse temperature (large, fixed).
  - $\mathbf{g}(\cdot,\cdot)$: some complicated function, yet satisfying
    1) $\mathbf{g} \le K_0$
    2) $|\mathbf{g}(f,\mathcal{I}) - \mathbf{g}(f',\mathcal{I}')| \le e^{-c_0\,\mathbf{r}}$ if $B_{\mathbf{r}}(f,\mathcal{I}) \cong B_{\mathbf{r}}(f',\mathcal{I}')$

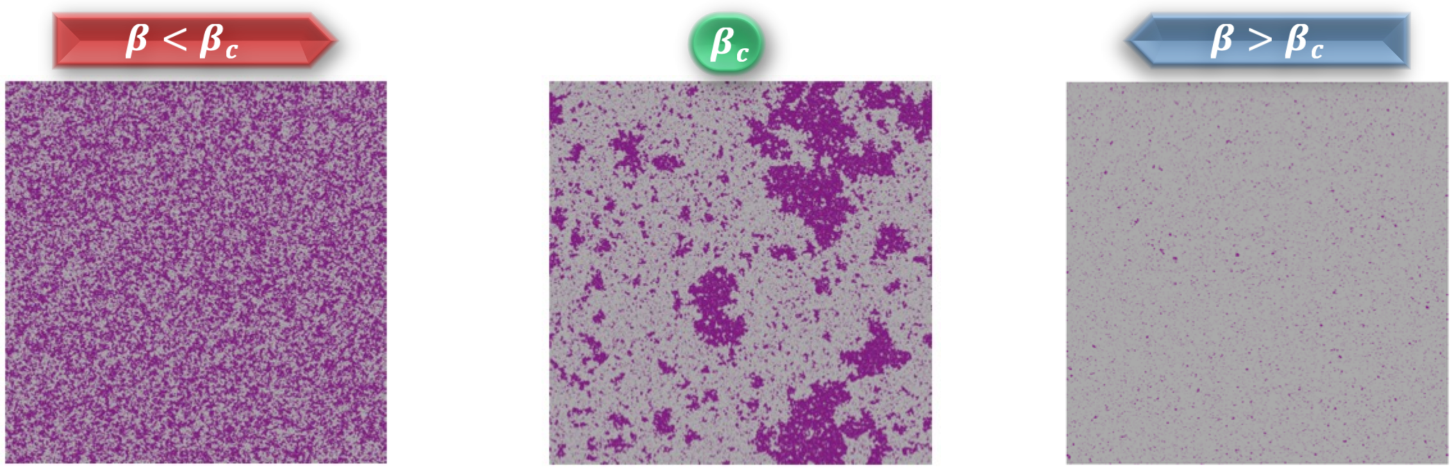
    for **absolute** constants c_0, K_0.

Definition: the classical Ising model

- Underlying geometry: finite $\Lambda \subset \mathbb{Z}^d$.
- Set of possible configurations: $\Omega = \{\pm 1\}^\Lambda$
- Probability of a configuration $\sigma \in \Omega$ given by the *Gibbs distribution*:



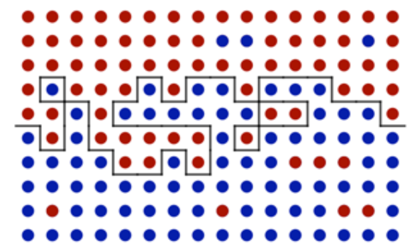
$$\mu_\Lambda(\sigma) \propto \exp\left(-\beta \sum_{x\sim y} \mathbf{1}_{\{\sigma_x \neq \sigma_y\}}\right)$$



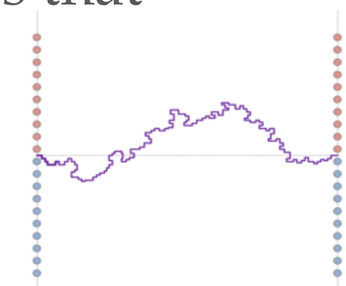
$\beta = 0.75$ $\beta = 0.88$ $\beta = 1$

2D Ising interfaces

- $\mu_\Lambda^{\mp}$: Ising model on 2D cylinder $\Lambda = [\![-n,n]\!] \times (\mathbb{Z}+\frac{1}{2})$
  - Boundary conditions: $\begin{cases} \ominus \ \textit{upper half}\text{–plane} \\ \oplus \ \textit{lower half}\text{–plane} \end{cases}$
  - Draw a dual-**edge** $(u,v)^*$ if $\sigma_u \neq \sigma_v$.



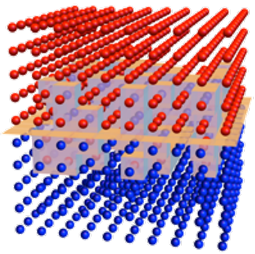
- **Interface**: connected component $\mathcal{I}$ of dual-edges that separates the the boundary components.



- Known [Higuchi '79], [Dobrushin, Hryniv '97], [Hryniv '98], [Dobrushin, Kotecký, Shlosman '92] :
  - Interface has a scaling limit: $\frac{\mathcal{I}(x/n)}{\sqrt{c_\beta n}} \to$ Brownian bridge
  - Maximum M_n is $O_P(\sqrt{n})$, and $M_n - \mathbb{E}[M_n]$ is also $O_P(\sqrt{n})$.

3D Ising interfaces

- $\mu_\Lambda^\mp$: Ising model on 3D cylinder $\Lambda = [\![-n,n]\!]^2 \times (\mathbb{Z}+\frac{1}{2})$
  - Boundary conditions: $\begin{cases} \ominus \textit{ upper half–}\text{plane} \\ \oplus \textit{ lower half–}\text{plane} \end{cases}$
  - Draw a dual-face $(u,v)^*$ if $\sigma_u \neq \sigma_v$.

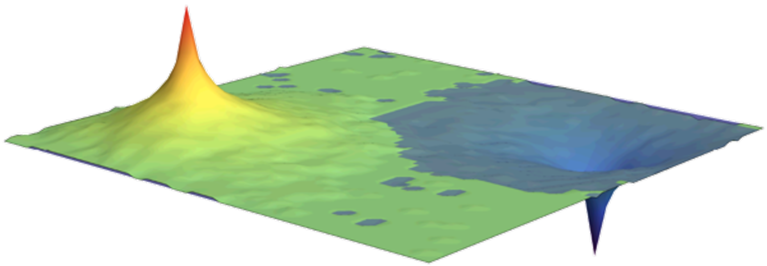


- **Interface**: maximal connected component $\mathcal{I}$ of dual-faces that separates the boundary components.
- [Minlos, Sinai '67],[Dobrushin '72]: $\mu_\Lambda^\mp(\mathcal{I}) \propto e^{-\beta|\mathcal{I}|+\sum_{f\in\mathcal{I}} \mathbf{g}(f,\mathcal{I})}$ (*cluster expansion;* valid for large β)
- THEOREM: [Dobrushin '72] (*rigidity of the interface*)

> There exists $\beta_0 > 0$ such that $\forall \beta > \beta_0$ and $\forall x_1, x_2, h$,
> $$\mu_\Lambda^\mp\big(\mathcal{I} \ni (x_1,x_2,h)\big) \le \exp\big(-\tfrac{1}{3}\beta h\big)$$

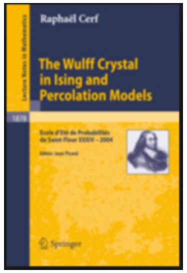
Plus/minus interface in 3D Ising

- M_n = maximum height of the interface $\mathcal{I}$ in 3D Ising with Dobrushin's boundary conditions.
  - [Dobrushin '72]: $\exists C_\beta$ s.t. $\mu_\Lambda^\mp(M_n \leq C_\beta \log n) \to 1$.
  - $\Rightarrow$ (via straightforward matching order lower bound) the maximum of the interface has **order** $\log n$.
- Asymptotics of the maximum (LLN)? Tightness?
- Structure of interface conditional on the rare event of reaching height $h \gg 1$ above some fixed point?



Related work on 3D Ising interfaces

- Alternative simpler argument by [van Beijeren '75] for [Dobrushin '72]'s result on the rigidity of the 3D Ising interface.
- Rigidity argument extended to
  - Widom–Rowlinson model [Bricmont, Lebowitz, Pfister, Olivieri '79a], [Bricmont, Lebowitz, Pfister '79b, '79c]
  - Super-critical percolation / random cluster model conditioned to have interfaces [Gielis, Grimmett '02]
- Tilted interfaces: [Cerf, Kenyon '01] (zero temperature, 111 interface), [Miracle Sole '95] (1-step interface), [Sheffield '03] ($|\nabla\phi|^p$ models), **many** works on the conjectured behavior, related to the (non-)existence of non-translational invariant Gibbs measures
- Wulff shape, large deviations for the magnetization, surface tension [Pisztora '96], [Bodineau '96], [Cerf, Pisztora '00], [Bodineau '05], [Cerf '06]
- Plus/minus phases away from the interface [Zhou '19]



LLN for the maximum

- Recall: M_n = maximum of the interface $\mathcal{I}$ in 3D Ising; [Dobrushin '72]: $M_n = O_{\mathrm{P}}(\log n)$.
- THEOREM: ([Gheissari, L. '19a])

There exists β_0 such that for all $\beta > \beta_0$,

$$\lim_{n\to\infty} \frac{M_n}{\log n} = \frac{2}{\alpha}\,, \qquad \textit{in probability,}$$

LLN

where

$$\alpha(\beta) = \lim_{h\to\infty} -\frac{1}{h} \log \mu^{\mp}_{\mathbb{Z}^3}\left((0,0,0) \overset{+}{\longleftrightarrow} (\mathbb{R}^2 \times \{h\}) \right)$$

*-connected in $\mathbb{Z}^2\times[0,h]$

and satisfies $\alpha(\beta)/\beta \to 4$ as $\beta \to \infty$.

> existence of the limit α nontrivial: sub-multiplicativity argument relying on new results on the interface shape.

Tightness and tails for the maximum

- THEOREM: ([Gheissari, L. '19b])

1. There exists β_0 such that for all $\beta > \beta_0$,
$$M_n - \mathbb{E}M_n = O_{\mathrm{P}}(1).$$
2. There exist $C, \bar{\alpha}, \underline{\alpha}$ such that $\forall\, r \geq 1$,
$$\begin{cases} e^{-(\bar{\alpha}r + C)} \leq \mu_n^{\mp}(M_n \geq \mathbb{E}[M_n] + r) \leq e^{-(\underline{\alpha}r - C)} \\ e^{-e^{\bar{\alpha}r+C}} \leq \mu_n^{\mp}(M_n \leq \mathbb{E}[M_n] - r) \leq e^{-e^{\underline{\alpha}r - C}} \end{cases}$$
where $\bar{\alpha}/\underline{\alpha} \to 1$ as $\beta \to \infty$.

Tightness

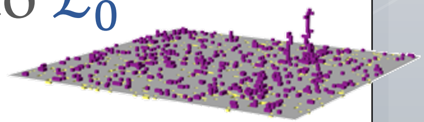
Gumbel tails

- PROPOSITION: ([Gheissari, L. '19b])

There *does not* exist a deterministic sequence (m_n) s.t. $(M_n - m_n)$ converges weakly to a nondegenerate law.

Steppingstone: Dobrushin's argument

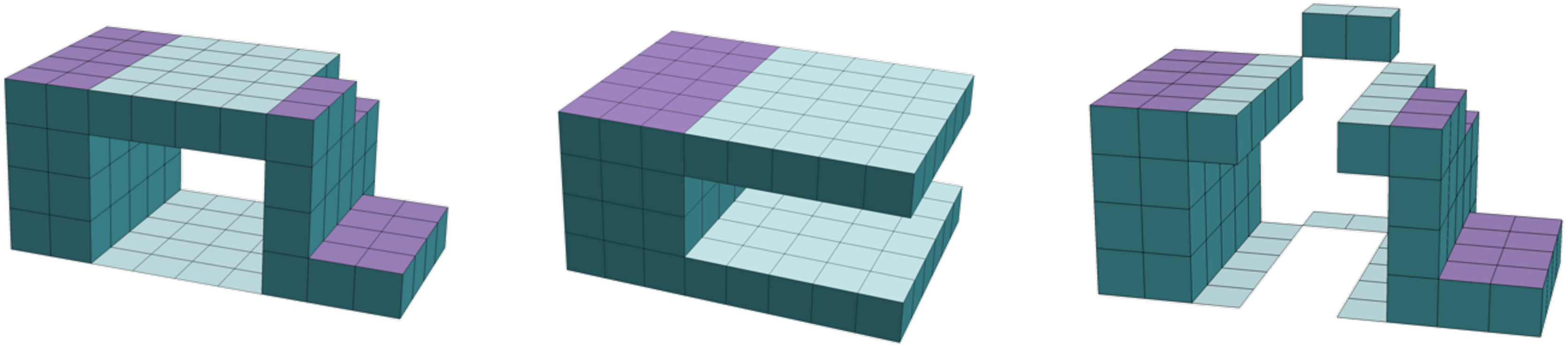
- Notation: $\mathcal{L}_0 = \mathbb{R}^2 \times \{0\}$; π = projection onto $\mathcal{L}_0$



- DEFINITION: [**ceiling** and **walls**]

1. *Ceiling face* : a horizontal face $f \in \mathcal{I}$ such that
$$\pi(f') \neq \pi(f) \quad \forall f' \neq f.$$
Ceiling C : connected component of ceiling faces.

2. *Wall face* : all other faces.
Wall W : connected component of wall faces.



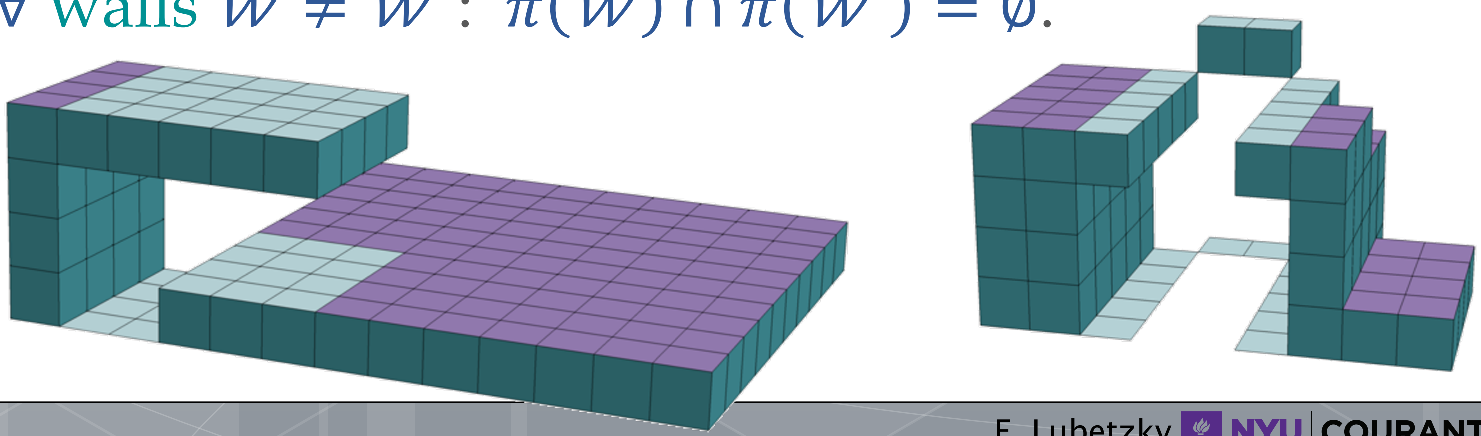
Steppingstone: Dobrushin's argument

- DEFINITION: [**ceiling** and **walls**]

1. *Ceiling face* : a horizontal face $f \in \mathcal{I}$ with $\pi(f') \neq \pi(f) \quad \forall f' \neq f$.
   Ceiling $\mathcal{C}$: connected component of ceiling faces.
2. *Wall face* : all other faces.
   Wall $\mathcal{W}$: connected component of wall faces.

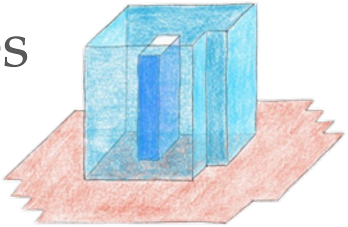
- FACTS:

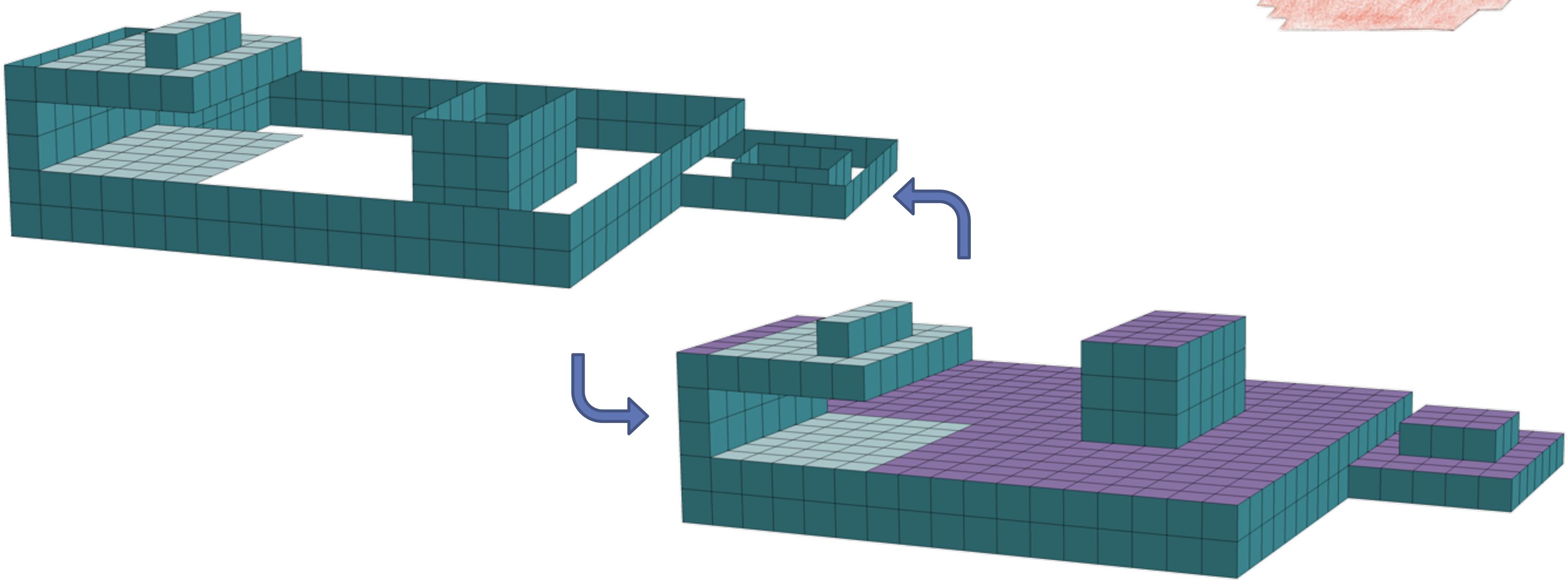
1. $\forall$ ceiling $\mathcal{C}$ has a single height.
2. $\forall$ wall $\mathcal{W}$: $\pi(\mathcal{W})$ is connected.
3. $\forall$ walls $\mathcal{W} \neq \mathcal{W}'$: $\pi(\mathcal{W}) \cap \pi(\mathcal{W}') = \emptyset$.



Steppingstone: Dobrushin's argument

- A wall $\mathcal{W}$ is **standard** if $\exists\, \mathcal{I}$ whose only wall is $\mathcal{W}$.
- <u>FACT</u>**:** $1{:}1$ correspondence between interfaces and *admissible** collections of standard walls.

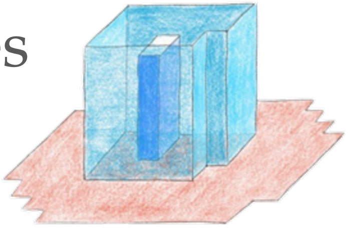




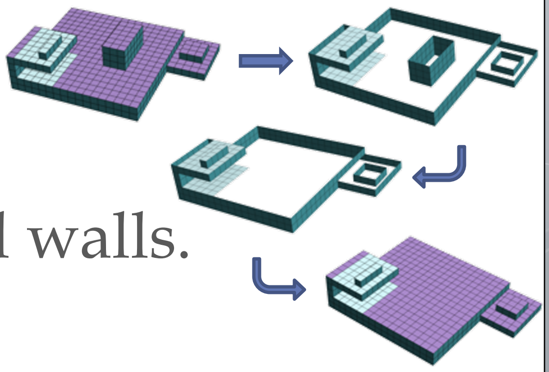
* *admissible: walls are disjoint components and so are their projections*

Steppingstone: Dobrushin's argument

- A wall $\mathcal{W}$ is **standard** if $\exists\, \mathcal{I}$ whose only wall is $\mathcal{W}$.
- <u>FACT</u>: 1: 1 correspondence between interfaces and *admissible* collections of standard walls.



- Basic idea: given $x \in \mathcal{L}_0$, construct a map Φ:
  - "*standardize*" every wall $\mathcal{W}$ in $\mathcal{I}$;
  - delete the wall $\mathcal{W}_x$ of x;
  - "*reconstruct*" $\mathcal{I}'$ from other standard walls.



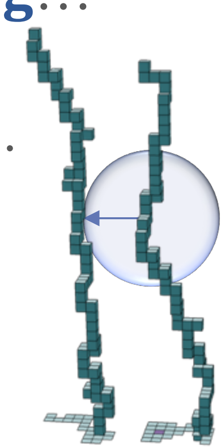
- Goal: establish for this map Φ:

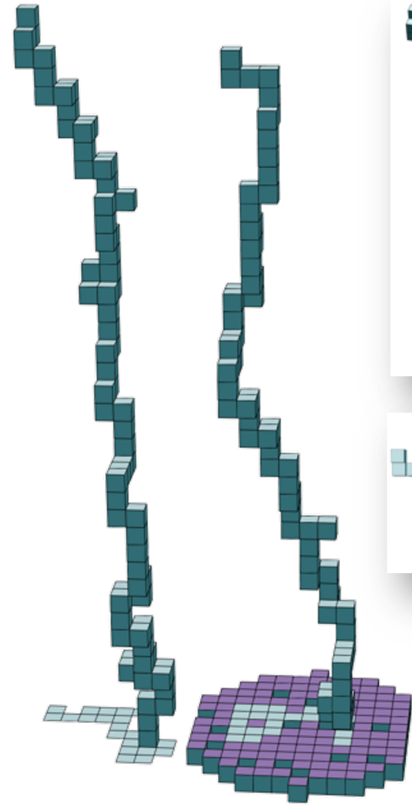
1. (*Energy bound*) $$\frac{\mu(\mathcal{I})}{\mu(\Phi(\mathcal{I}))} \leq e^{-c\beta|\mathcal{W}_x|}$$
2. (*Multiplicity bound*) $\#\{\mathcal{I} \in \Phi^{-1}(\mathcal{I}') : |\mathcal{W}_x| = \ell\} \leq e^{c\ell}$

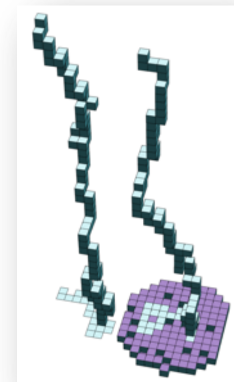
Steppingstone: Dobrushin's argument

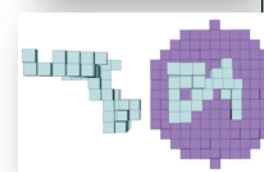
recall $\mu_\Lambda^\mp(\mathcal{I}) \propto e^{-\beta|\mathcal{I}|+\sum_{f\in\mathcal{I}}\mathbf{g}(f,\mathcal{I})}$

- Basic idea: delete the wall $\mathcal{W}_x$ of x.
- *Energy bound* ($\frac{\mu(\mathcal{I})}{\mu(\Phi(\mathcal{I}))} \leq e^{-c\beta|\mathcal{W}_x|}$) :
  - Gain $\beta|\mathcal{W}_x|$ from $\beta(|\mathcal{I}| - |\Phi(\mathcal{I})|)$
  - Problem: effect on **non-deleted** faces that moved due to $\mathbf{g}$…
    - The effect of $\mathbf{g}$ is **local** (decays exp. in distance).
    - BUT: **tall** nearby walls can pick up a cost that cancels our $\beta|\mathcal{W}_x|$ gain.







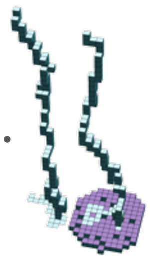


- Solution: also delete **tall** walls that are **close** to $\mathcal{W}_x$.

Steppingstone: Dobrushin's argument

recall $\mu_\Lambda^{\mp}(\mathcal{I}) \propto e^{-\beta|\mathcal{I}|+\sum_{f\in\mathcal{I}} \mathbf{g}(f,\mathcal{I})}$

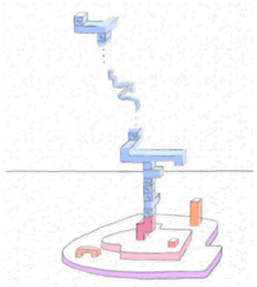
- *Energy bound* ($\frac{\mu(\mathcal{I})}{\mu(\Phi(\mathcal{I}))} \leq e^{-c\beta|\mathcal{W}_x|}$) :
  - Gain $\beta|\mathcal{W}_x|$ from $\beta(|\mathcal{I}| - |\Phi(\mathcal{I})|)$, but must handle $\mathbf{g}$…
  - … must also **delete tall** walls that are **close**.



- *Multiplicity bound* ($\#\{\mathcal{I} \in \Phi^{-1}(\mathcal{I}') : |\mathcal{W}_x| = \ell\} \leq e^{c\ell}$) :
  - Problem: accounting for the **extra** walls we deleted…

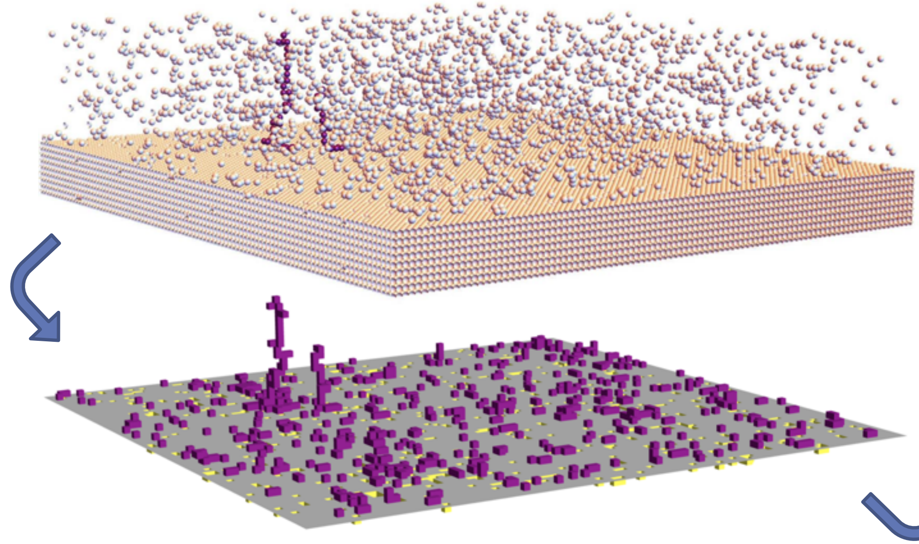
- Dobrushin's criterion: **groups of walls**: for $x, y \in \mathcal{L}_0$,
  $\mathcal{W}_x \sim \mathcal{W}_y \quad \Longleftrightarrow \quad d(x,y)^2 \leq \max\{|\pi^{-1}(x)|, |\pi^{-1}(y)|\}$.
  (a "tall" $\mathcal{W}_x$ (many faces above x) is easier to group with)
- The map Φ deletes the entire **group of walls** of $\mathcal{W}_x$: analysis becomes 2D (but too crude for detailed questions).

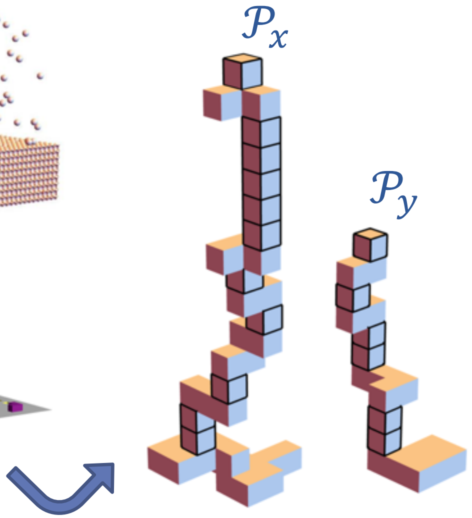
New approach: pillars in the interface



<u>DEFINITION</u>: [$\mathcal{P}_x$, the **pillar** at $x \in \mathbb{R}^2 \times \{0\}$]

1. Take the interface $\mathcal{I}$ (filling in $\forall$ bubble)
2. Discard $\mathbb{R}^2 \times (-\infty, 0)$ from the sites below $\mathcal{I}$
3. The pillar $\mathcal{P}_x$ is the remaining ⊕ *-connected component of x

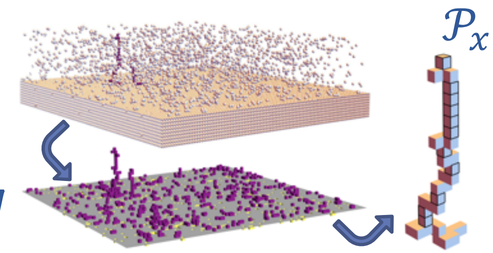




Goal: second moment argument for $M_n = \max_x \mathrm{ht}(\mathcal{P}_x)$

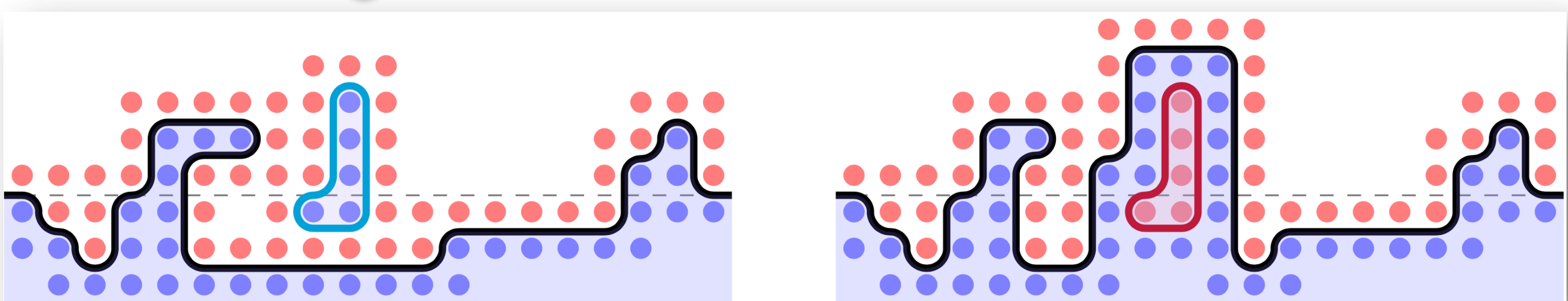
Pillars vs. connected + components

DEFINITION: [$\mathcal{P}_x$, the **pillar** at $x \in \mathbb{R}^2 \times \{0\}$]



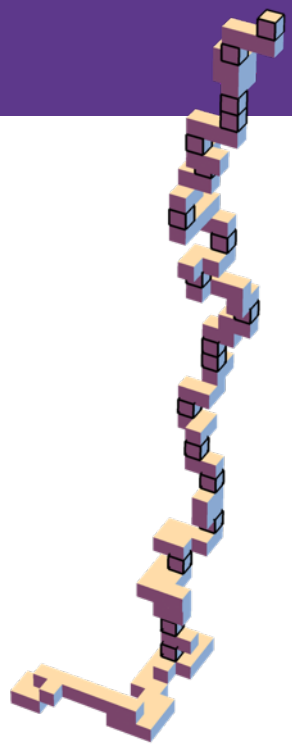
1. Take the interface $\mathcal{I}$ (filling in $\forall$ bubble)
2. Discard $\mathbb{R}^2 \times (-\infty, 0)$ from the sites below $\mathcal{I}$
3. The pillar $\mathcal{P}_x$ is the remaining + *-connected component of x

REMARK: No monotonicity the height of the pillar $\mathcal{P}_x$ and the height of the + component of x (in either direction)



Goal: second moment argument for $M_n = \max_x \mathrm{ht}(\mathcal{P}_x)$

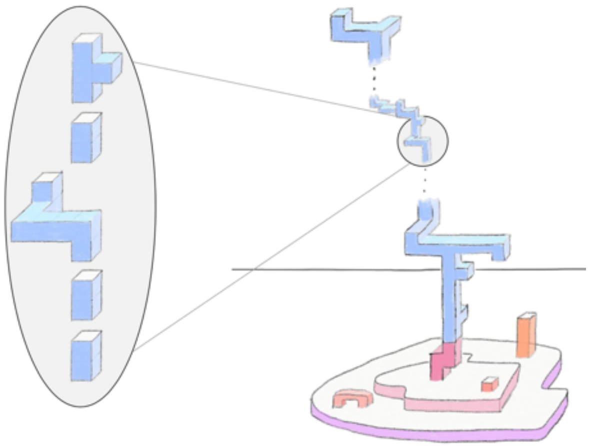
Decomposition of pillars



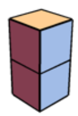
- DEFINITION: [**cutpoint** of the pillar]
  a cell v_i which is the only intersection of the pillar $\mathcal{P}_x$ with a horizontal slab.

- DEFINITION: [pillar **increment**]
  $\mathcal{X}_i$ = segment of $\mathcal{P}_x$ bounded between the cutpoints v_i, v_{i+1} (inclusively).

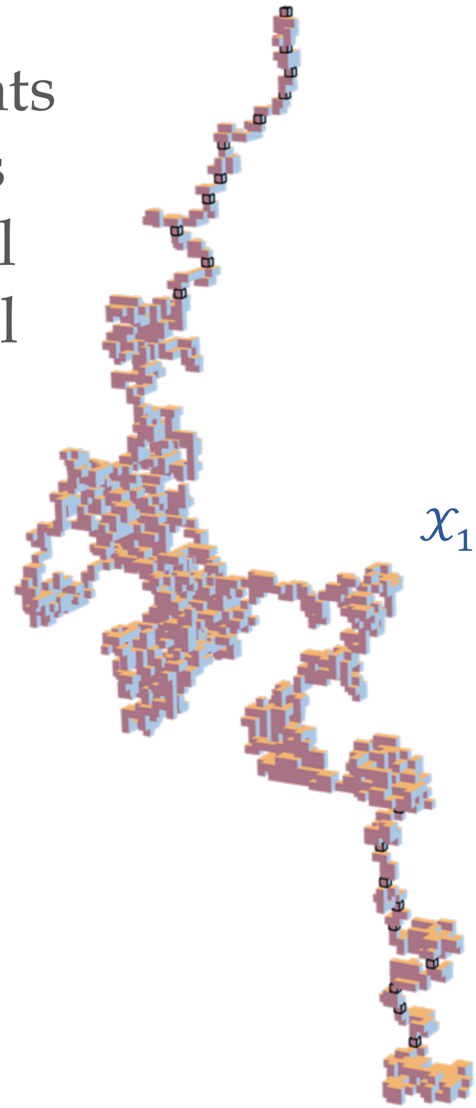
- Decompose $\mathcal{P}_x$ into:
  1. *increments* $(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_T)$
  2. *base* $\mathfrak{B}_x = \mathcal{P}_x \cap (\mathbb{R}^2 \times [0, \mathrm{ht}(v_1)])$



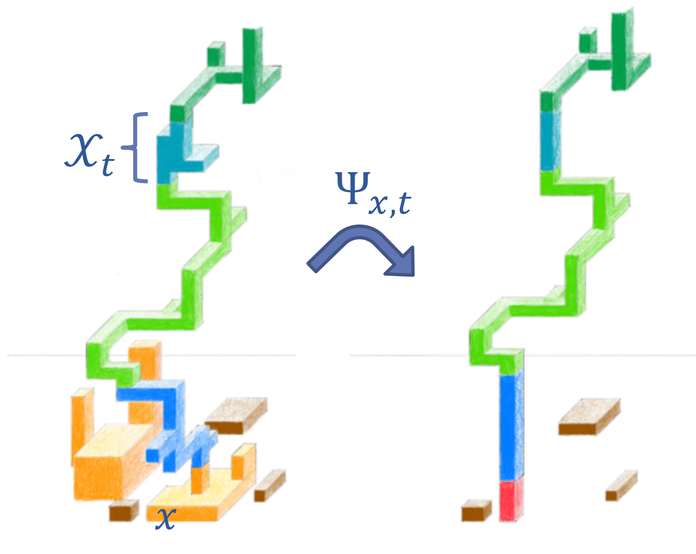
Decomposition of pillars

- Typical increments are perturbations (with exponential tails) of the trivial increment 

- But: (rarely) they can be quite complex…



The interface map $\Psi_{x,t}$

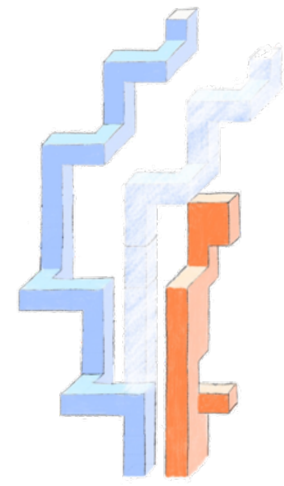


$\Psi_{x,t}$: $\{\mathcal{I} : \mathrm{ht}(\mathcal{P}_x) \geq h, |\mathfrak{B}_x| \vee |\mathcal{X}_t| \geq r\} \to \{\mathcal{I} : \mathrm{ht}(\mathcal{P}_x) \geq h\}$ s.t.

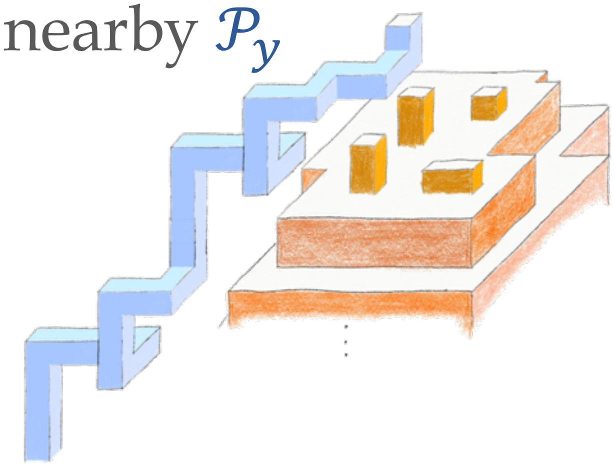
1. (*Energy bound*) $$\frac{\mu(\mathcal{I})}{\mu\big(\Psi_{x,t}(\mathcal{I})\big)} \leq e^{-c\beta\left(|\mathcal{I}|-|\Psi_{x,t}(\mathcal{I})|\right)}$$
2. (*Multiplicity bound*) $\#\{\mathcal{I} \in \Psi_{x,t}^{-1}(\mathcal{I}') : |\mathcal{I}| - |\mathcal{I}'| = \ell\} \leq e^{c\ell}$

Challenges due to interacting pillars

- The map $\Psi_{x,t}$ induces
  1. horizontal shifts
  2. vertical shifts (down & up)



- The pillar $\mathcal{P}_x$ to hit a nearby $\mathcal{P}_y$ (possibly making the map not well-defined)
- The pillar may get very close to a nearby $\mathcal{P}_y$ and heavily interact with it (destroying the energy control).

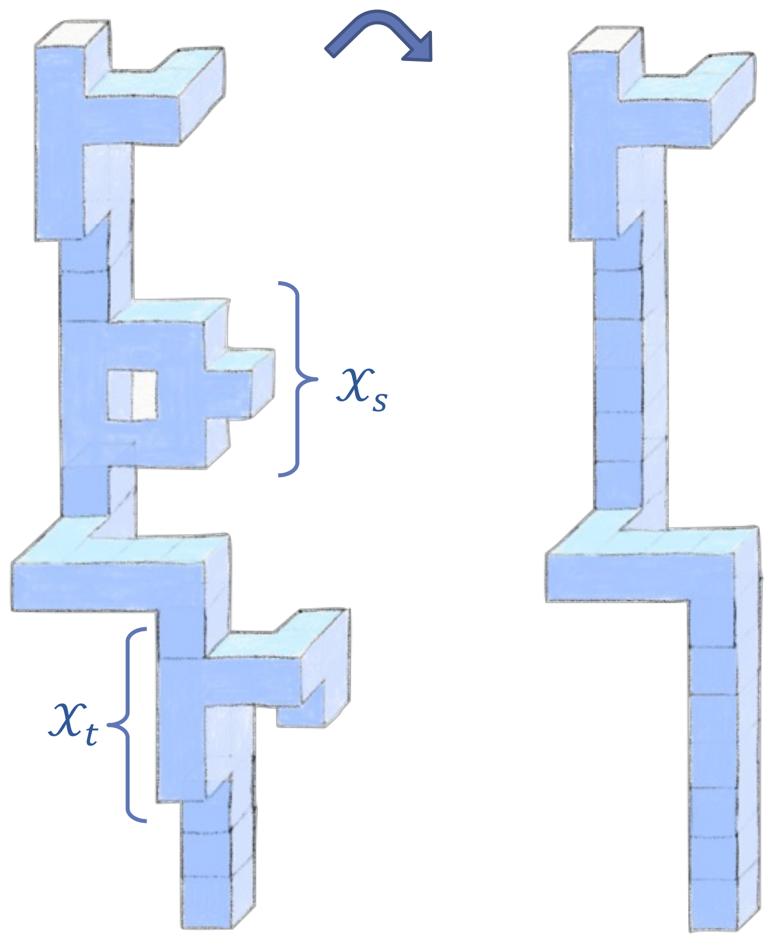


Basic map $\Psi_{x,t}$ to control increments

- Target the structure of the increment $\mathcal{X}_t$ by:
  - straightening $\mathcal{X}_t$ if its size is too large.
  - straightening any other increment $\mathcal{X}_s$ for $s \geq t$ whose size is at least

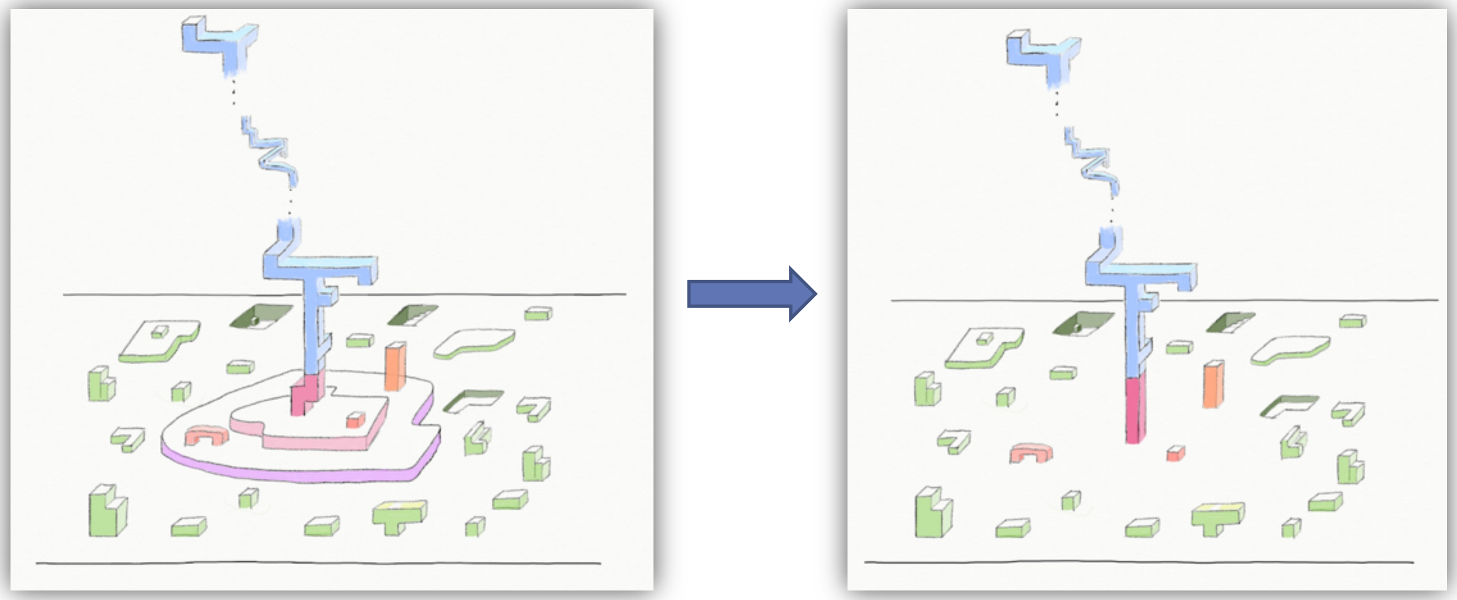
    $$e^{c|s-t|}$$

    (*too large w.r.t.* $\mathcal{X}_t$).



A basic $\Psi_{x,t}$ for controlling increments

- Base is delicate: incorporates interaction with other nearby pillars in the interface…
- Trying to relax the definition of the base to rule out such interactions gives an $O(\log h)$ error on its size: sufficient for LLN but *not for tightness*.



Algorithm for the refined map $\Psi_{x,t}$

- Defining $\Psi_{x,t}$:
  - $\forall\, j \geq 1$, determine whether to straighten $\mathcal{P}_x$ at the increment $\mathcal{X}_j$. If so:
    - $\forall y \neq x$, determine whether this action may cause $\mathcal{P}_x$ to draw to closely to $\mathcal{P}_y$. If so, delete $\mathcal{P}_y$ as well.
- Delicate balance between deleting too little (energy control) and deleting too much (multiplicity control).

Algorithm 1: The map $\Psi_{x,t}$

1 Let $\{\tilde W_y : y \in \mathcal{L}_{0,n}\}$ be the standard wall representation of the interface $\mathcal{I} \setminus \mathcal{S}_x$. Also let $\mathcal{O}_{v_1}$ be the nested sequence of walls of v_1, so that $\theta_{\mathrm{ST}}\mathcal{O}_{v_1} = \tilde{\mathfrak{W}}_{v_1}$.

`// Base modification`

2 Mark $[x] = \{x\} \cup \partial_0 x$ and $\rho(v_1)$ for deletion (where $\partial_0 x$ denotes the four faces in $\mathcal{L}_0$ adjacent to x).

3 **if** *the interface with standard wall representation* $\tilde{\mathfrak{W}}_{v_1}$ *has a cut-height* **then**
  - Let $h^\dagger$ be the height of the highest such cut-height.
  - Let $y^\dagger$ be the index of a wall that intersects $(\mathcal{P}_x \setminus \mathcal{O}_{v_1}) \cap \mathcal{L}_{h^\dagger}$ and mark $y^\dagger$ for deletion.

`// Spine modification (A): the 1st increment`

4 Set $\mathfrak{s}_1 \leftarrow 0$ and $y^*_A \leftarrow \emptyset$.
for $j = 1$ **to** $\mathscr{T} + 1$ **do**
  - Let $s \leftarrow \mathfrak{s}_j$ and $\mathfrak{s}_{j+1} \leftarrow \mathfrak{s}_j$.
  - **if** $\mathfrak{m}(\mathscr{X}_j) \geq j - 1$ **then** `// (A1)`
    - Let $\mathfrak{s}_{j+1} \leftarrow j$.
  - **if** $\mathfrak{D}_x(\tilde W_y, j, -v_{s+1}, 0) \leq \mathfrak{m}(\tilde W_y)$ *for some y* **then** `// (A2)`
    - Let $\mathfrak{s}_{j+1} \leftarrow j$ and mark for deletion every y for which (A2) holds.
  - **if** $\mathfrak{D}_x(\tilde W_y, j, -v_{s+1}, 0) \leq (j-1)/2$ *for some y* **then** `// (A3)`
    - Let $\mathfrak{s}_{j+1} \leftarrow j$ and let y^*_A be the minimal index y for which (A3) holds.

Let $j^* \leftarrow \mathfrak{s}_{\mathscr{T}+2}$ and mark y^*_A for deletion.

`// Spine modification (B): the t-th increment`

5 **if** $t > j^*$ **then**
  - Set $\mathfrak{s}_t \leftarrow t - 1$ and $y^*_B \leftarrow \emptyset$.
  - **for** $k = t$ **to** $\mathscr{T} + 1$ **do**
    - Let $s \leftarrow \mathfrak{s}_k$ and $\mathfrak{s}_{k+1} \leftarrow \mathfrak{s}_k$.
    - **if** $\mathfrak{m}(\mathscr{X}_k) \geq k - t$ **then** `// (B1)`
      - Let $\mathfrak{s}_{k+1} \leftarrow k$.
    - **if** $\mathfrak{D}_x(\tilde W_y, j, -v_{s+1}, v_t - v_{j^*+1}) \leq \mathfrak{m}(\tilde W_y)$ *for some y* **then** `// (B2)`
      - Let $\mathfrak{s}_{k+1} \leftarrow k$ and mark for deletion every y for which (B2) holds.
    - **if** $\mathfrak{D}_x(\tilde W_y, j, -v_{s+1}, v_t - v_{j^*+1}) \leq (k-t)/2$ *for some y* **then** `// (B3)`
      - Let $\mathfrak{s}_{k+1} \leftarrow k$ and let y^*_B be the minimal index y for which (B3) holds.
  - Let $k^* \leftarrow \mathfrak{s}_{\mathscr{T}+2}$ and mark y^*_B for deletion.

else
  - Let $k^* \leftarrow j^*$.

6 **foreach** *index* $y \in \mathcal{L}_{0,n}$ *marked for deletion* **do** delete $\tilde{\mathfrak{F}}_y$ from the standard wall representation $(\tilde W_y)$.

7 Add a standard wall $W_x^{\mathcal{J}}$ consisting of $\mathrm{ht}(v_1) - \frac{1}{2}$ trivial increments above x.

8 Let $\mathcal{K}$ be the (unique) interface with the resulting standard wall representation.

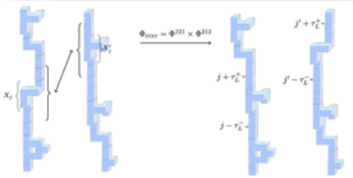
9 Denoting by $(\mathscr{X}_i)_{i\geq 1}$ the increment sequence of $\mathcal{S}_x$, set

$$\mathcal{S} \leftarrow \begin{cases} (\underbrace{X_\varnothing, X_\varnothing, \dots, X_\varnothing}_{\mathrm{ht}(v_{j^*+1})-\mathrm{ht}(v_1)}, \mathscr{X}_{j^*+1}, \dots, \mathscr{X}_{t-1}, \underbrace{X_\varnothing, X_\varnothing, \dots, X_\varnothing}_{\mathrm{ht}(v_{k^*+1})-\mathrm{ht}(v_t)}, \mathscr{X}_{k^*+1}, \dots) & \text{if } t > j^*, \\ (\underbrace{X_\varnothing, X_\varnothing, \dots, X_\varnothing}_{\mathrm{ht}(v_{j^*+1})-\mathrm{ht}(v_1)}, \mathscr{X}_{j^*+1}, \dots) & \text{if } t \leq j^*. \end{cases}$$

10 Obtain $\Psi_{x,t}(\mathcal{I})$ by appending the spine with increment sequence $\mathcal{S}$ to $\mathcal{K}$ at $x + (0, 0, \mathrm{ht}(v_1))$.

CLT for location of tip, volume, surface area

- Via additional maps ($2 \to 2$): tall pillars are $\approx$ stationary sequences of increments.



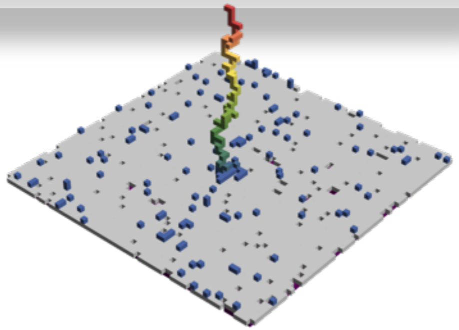
- THEOREM: ([Gheissari, L. '19a])

Let $(Y_1, Y_2, \mathrm{ht}(\mathcal{P}_x))$ be the location of the tip of the pillar $\mathcal{P}_x$. Conditional on $\mathcal{P}_x$ having at least $1 \ll T_n \ll n$ increments,

$$\frac{(Y_1, Y_2, \mathrm{ht}(\mathcal{P}_x)) - (x_1, x_2, \lambda T_n)}{\sqrt{T_n}} \xrightarrow{\mathrm{d}} \mathcal{N}\left(0, \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & (\sigma')^2 \end{pmatrix}\right)$$

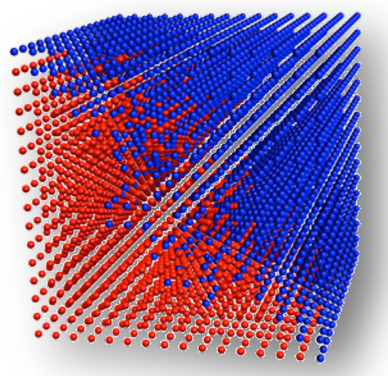
for some $\sigma, \sigma' > 0$.

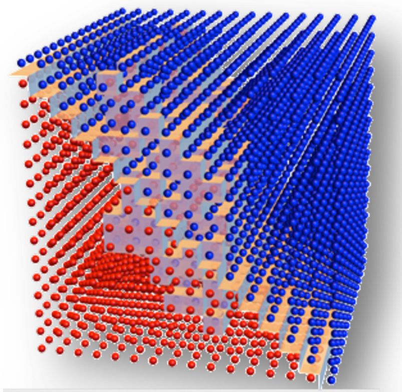
- CLT also holds, e.g., for the surface area and volume of $\mathcal{P}_x$.



Open: tilted interfaces

- Major open problem: roughness of **tilted** interfaces of the 3D Ising model at low temperature (β fixed, large).
  - Conjecture: $\mathrm{Var}(\mathrm{ht}_x(\mathcal{I})) \asymp \log n$.
  - Verified only for $\beta = \infty$ ([Cerf, Kenyon '01]).
  - For finite large β, unknown that $\mathrm{Var}(\mathrm{ht}_x(\mathcal{I})) \to \infty$…





Thank you!