Further Progress towards Hadwiger's Conjecture

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Oxford Discrete Mathematics and Probability Seminar

October 27, 2020

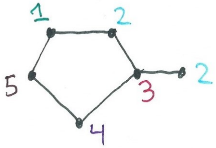
Part I

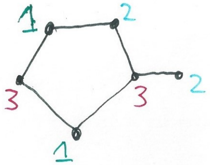
Hadwiger's: A History

Coloring

Definition (Coloring)

- A ***k*-coloring** of a graph G is an assignment of colors $1, 2, \ldots, k$ to vertices of G s.t. no two adjacent vertices receive the same color.
- We say G is ***k*-colorable** if G has a k-coloring.
- The **chromatic number** of G, denoted $\chi(G)$, is the smallest k such that G has k-coloring.





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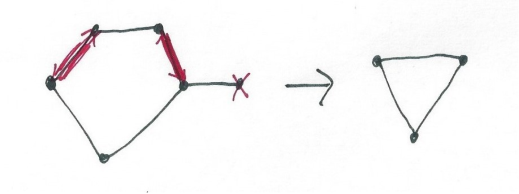

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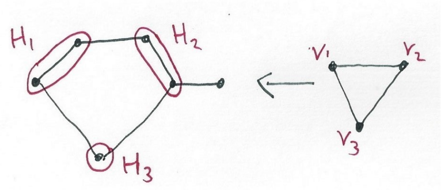

Definition (Models)

Let H be a graph with $V(H) = \{v_1, \ldots, v_t\}$. A **model of *H*** in a graph G is a collection of vertex-disjoint connected subgraphs $H_1, \ldots, H_t$ such that $\forall i \neq j \in [t]$ with $v_i v_j \in E(H)$, H_i is adjacent to H_j (i.e. $\exists$ an edge with one end in H_i and the other end in H_j).

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Do coloring and minors have any relation?

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- Open for $t \geq 7$.

General Bounds from Degeneracy

Linear Hadwiger's conjecture (e.g. Reed and Seymour 1998, Kawarabayashi and Mohar 2007)

$\exists C \geq 1$ *s.t.* $\forall t \geq 1$, *every graph with no* K_t *minor is* Ct*-colorable.*

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What is the degeneracy of graphs with no K_t minor?

Theorem (Kostochka 1982, Thomason 1984)

Every graph with no K_t minor is $O(t\sqrt{\log t})$-degenerate and hence $O(t\sqrt{\log t})$-colorable.

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Theorem (P. 2019+)

$\forall\ \beta > \frac{1}{4}$, *every graph with no* K_t *minor is* $O(t(\log t)^{\beta})$*-colorable.*

Part II

Variants of Hadwiger's

Weakening of Hadwiger's Conjecture: Independent Set and Fractional Coloring

Theorem (Duchet and Meyniel 1982)

$\forall t \geq 2$, every graph G with no K_t minor has an independent set of size at least $\frac{\mathrm{v}(G)}{2(t-1)}$.

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Theorem (Reed and Seymour 1998)

$\forall t \geq 2$, every graph G with no K_t minor satisfies $\chi_f(G) \leq 2(t-1)$.

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Improved to defect $t-2$ by **van den Heuvel** and **Wood** (2018).

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- $\lceil \frac{31t}{2} \rceil$ (**Kawarabayashi** and **Mohar** 2007)
- $\lceil \frac{7t-3}{2} \rceil$ (**Wood** 2010)
- $4t - 4$ (**Edwards** et al. 2015)
- $3t - 3$ (**Liu** and **Oum** 2015)
- $2t - 2$ (**van den Heuvel** and **Wood** 2018)
- $t - 1$ (announced by **Dvovrák** and **Norin**)

Generalizations of Hadwiger's Conjecture: Odd Minors

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Part III

Main Results

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Theorem (P., Oct 2020+)

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Theorem (P., Oct 2020+)

Every graph with no odd K_t minor is $O(t(\log \log t)^6)$-colorable.

Part IV

Proof Overview for Norin, P., Song Result

Coloring Small Graphs

We have the following corollary of Duchet and Meyniel result:

Corollary

If G is a graph with no K_t minor, then

$$\chi(G) \leq \left(\log_2\left(\frac{v(G)}{t}\right) + 2\right) t.$$

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$\forall s \geq 1$, $\exists g(s)$ s.t. if G is a graph with $\mathrm{d}(G) > 0$, and we let $D = s \cdot \mathrm{d}(G)$, then G contains at least one of the following:

(i) a minor J with $\mathrm{d}(J) \geq D$, or

(ii) a subgraph H with $\mathrm{v}(H) \leq g(s) \cdot \frac{D^2}{\mathrm{d}(G)}$ and $\mathrm{d}(H) \geq \frac{\mathrm{d}(G)}{g(s)}$.

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- **P.** 2020+: $g(s) = O((1 + \log s)^6)$.

Finding Many Small Dense Graphs

Let $f(t) := 3.2^2 \cdot g(3.2\sqrt{\log t}) = O((\log\log t)^6)$.

Corollary

$\forall k \geq t$, if G is a graph with $\mathrm{d}(G) \geq k \cdot f(t)$ and G contains no K_t minor, then G contains a subgraph H with $\mathrm{v}(H) \leq t \cdot f(t) \cdot \log t$ and $\mathrm{d}(H) \geq 2k$.

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Mader's Theorem (1972): if $\mathrm{d}(H) \geq 2k$, then H contains a k-connected subgraph.

Corollary

If G is a graph with no K_t minor and

$$\chi(G) \geq k \cdot f(t) + 2t \log f(t) + 6t \log r,$$

then G contains r vertex-disjoint k-connected subgraphs $H_1, \ldots, H_r$ with $\mathrm{v}(H_i) \leq t \cdot f(t) \cdot \log t$ for every $i \in [r]$.

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Definition (Linked)

A graph G is ***k*-linked** if for any set of vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$ of G, there exist internally vertex-disjoint paths $(P_i : i \in [k])$ from s_i to t_i.

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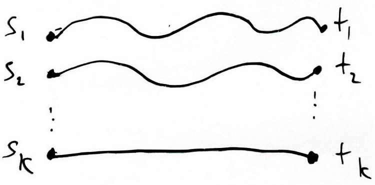

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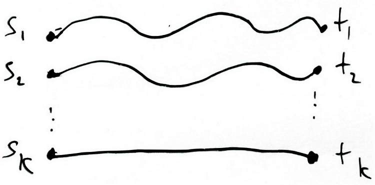

Theorem (Bollobás and Thomason 1996)

If a graph G is $\Omega(k)$-connected, then G is k-linked.

Woven

Definition (Woven)

A graph G is **(a, b)-woven** if for every three sets of vertices $R = \{r_1, \ldots, r_a\}$, $S = \{s_1, \ldots, s_b\}$, $T = \{t_1, \ldots, t_b\}$ in $V(G)$, there exists a K_a model in G rooted at R internally vertex-disjoint from a set of internally vertex-disjoint paths $(P_i : i \in [k])$ from s_i to t_i.

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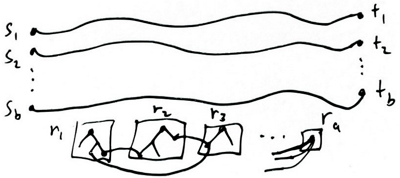

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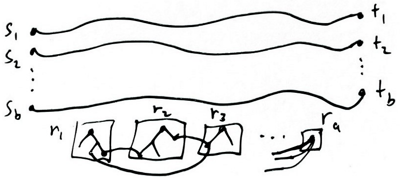
A graph G is (a,b)-**woven** if for every three sets of vertices $R=\{r_1,\ldots,r_a\}$, $S=\{s_1,\ldots,s_b\}$, $T=\{t_1,\ldots,t_b\}$ in $V(G)$, there exists a K_a model in G rooted at R internally vertex-disjoint from a set of internally vertex-disjoint paths $(P_i : i\in[k])$ from s_i to t_i.



Theorem (Norin and Song 2019+)

If a graph G is $\Omega(a\sqrt{\log a}+b)$-connected, then G is (a,b)-woven.

Building a Minor when $\chi = \Omega(t(\log t)^{1/4} \cdot f(t))$

Let $y = (\log t)^{1/4}$ and $x = \frac{t}{y} = \frac{t}{(\log t)^{1/4}}$.

Building a Minor when $\chi = \Omega(t(\log t)^{1/4} \cdot f(t))$

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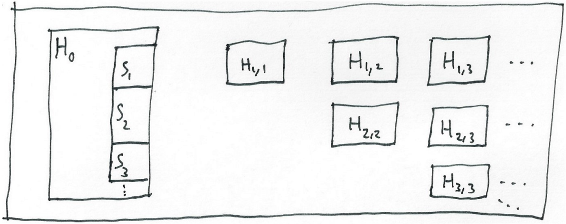

- In H_0: choose vertices $S_i = \{s_{i,j,k} : j \in [y], k \in [x]\}$ $\forall i \in [y]$.
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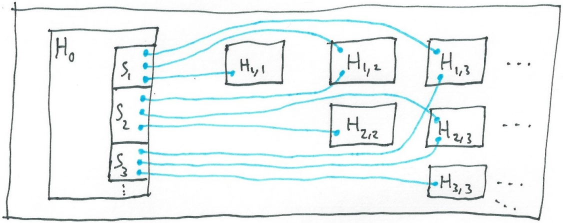

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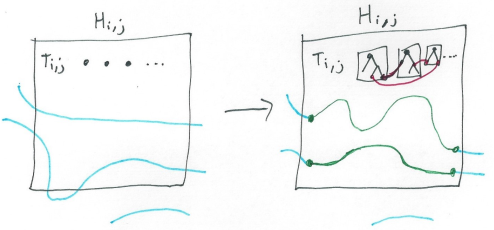
- Since G is xy^2-linked, there exists paths $\mathscr{P}$ from $s_{i,j,k}$ to $t_{i,j,k}$.

Building a Minor Continued

- $\forall i, j$: weave in $H_{i,j}$ a K_{2x} model rooted at $T_{i,j}$ while "preserving" $\mathscr{P}$.



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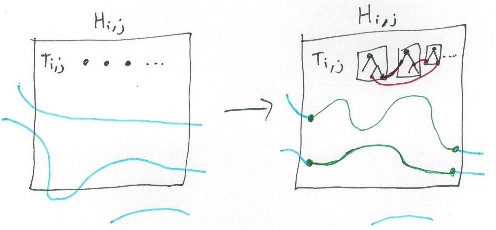
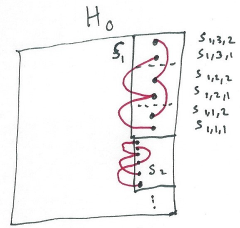


- $\forall i,k$, link all vertices in $\{s_{i,j,k} : j \in [y]\}$ in H_0.



Part V

Further Progress: Breaking into Two Cases

How to Build a Complete Minor?

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Theorem (Girão and Narayanan 2020+)

*For every positive integer k, if G is a graph with $\chi(G) \geq 7k$, then G contains a k-**connected subgraph** H with $\chi(H) \geq \chi(G) - 6k$.*

Key Concept: Chromatic Separability

Definition (Chromatic Separable)

Let $s \geq 0$. A graph G is ***s*-chromatic-separable** if there exist two vertex-disjoint subgraphs H_1, H_2 of G s.t. $\forall i \in \{1,2\}$

$$\chi(H_i) \geq \chi(G) - s,$$

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Let $s \geq t$. If G is a graph with $\chi(G) \geq \Omega(s \log\log t)$ and every subgraph H of G with $\chi(H) \geq \frac{\chi(G)}{2}$ is s-chromatic-separable, then G contains a K_t minor.

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Lemma (Inseparable Case)

Let $s = \Omega(t \log\log t)$. If G is a s-chromatic-inseparable graph with $\chi(G) \geq \Omega(t \cdot (f(t) + \log\log t))$, then G contains a K_t minor.

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Since $\chi(H) \geq \Omega(t(\log\log t)^6) \geq \Omega(t \cdot (f(t) + \log\log t))$, we have by the Inseparable Case Lemma that H contains a K_t minor. □

Part VI

Always Separable Case

Recursive Weaving: Finding Subgraphs

Let $a = t$ and $b = 0$, we will show that G is $(a, 0)$-woven.

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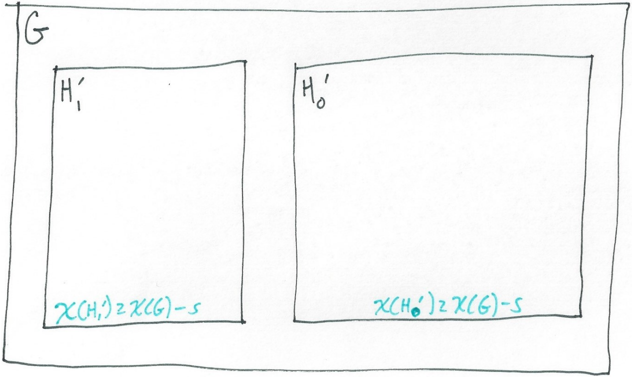
$$\chi(H_1'), \chi(H_0') \geq \chi(G) - s.$$



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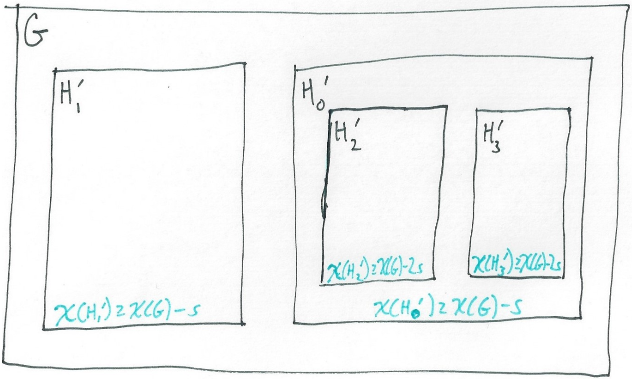
- Since $\chi(H'_0) \geq \frac{\chi(G)}{2}$, H'_0 is s-chromatic-separable. So there exists vertex-disjoint subgraph H'_2, H'_3 of H'_0 with

$$\chi(H'_2), \chi(H'_3) \geq \chi(G) - 2s.$$



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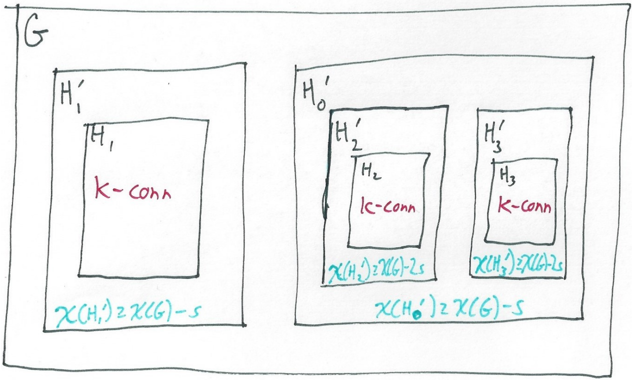
- By the **Girão-Narayanan** Theorem, $\forall i \in [3]$, there exists a ***k*-connected** subgraph H_i of H'_i with

$$\chi(H_i) \geq \chi(H'_i) - 6k \geq \chi(G) - 2s - 6k.$$



Recursive Weaving: Building a Minor

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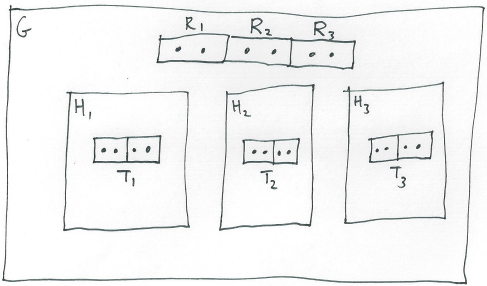
- $R_i = \{r_{a(i-1)/3+1}, \ldots, r_{ai/3}\}$, $\forall i \in [3]$.
- $s_i = s_{t+i} = r_i$, $\forall i \in [t]$.
- $T_i = \{t_{2a(i-1)/3+1}, \ldots, t_{2ai/3}\} \subseteq V(H_i)$, $\forall i \in [3]$.



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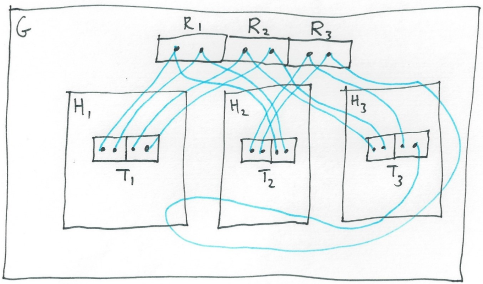

Since G is $2a$-linked, there exists paths $\mathscr{P}$ from s_i to t_i $\forall i \in [2a]$.

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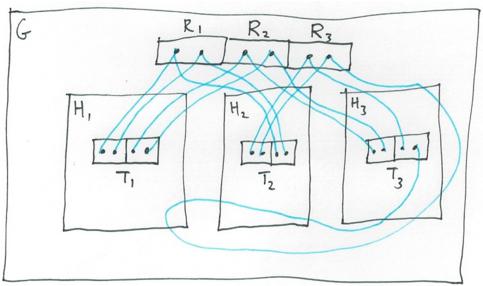

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Checking the parameters

Inductively show (a, b)-woven:

i	a_i	b_i
0	t	0
1	$\frac{2}{3} \cdot t$	$2a_0$
2	$\left(\frac{2}{3}\right)^2 \cdot t$	$2a_0 + 2a_1$
i	$\left(\frac{2}{3}\right)^i \cdot t$	$2 \cdot \sum_{j=0}^{i} a_j$
$\ldots$	$\ldots$	$\leq 2t \cdot \sum_{j=0}^{\infty} \left(\frac{2}{3}\right)^j$
$\Omega(\log \log t)$	$\leq \frac{t}{\log t}$	$\leq 6t$

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$$\chi(G) - (2s + 6k) \cdot \Omega(\log \log t) \geq \chi(G)/2$$

where $k = \Omega(t)$.

Part VII

Inseparable Case

Building a Minor Sequentially

The Plan: build a K_t model in $y = \sqrt{\log t}$ stages, where in each stage we ensure that a set of $x = \frac{t}{y}$ new parts are adjacent to every other part.

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A model $\mathscr{H} = \{H_1, \ldots, H_{\mathsf{v}(H)}\}$ of H in a graph G is **tangent** to a subgraph G' of G if $\forall i \in [\mathsf{v}(H)]$, $|V(H_i)| \cap |V(G')| = 1$.

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Goal: $\forall i \in [y]$

1. **Extend a K_{ix} model $\mathscr{H}_i$ to a $K_{(i+1)x}$ model $\mathscr{H}_{i+1}$ in G_i** (i.e. add $H_{ix+1}, \ldots, H_{(i+1)x}$ **adjacent to all H_j** while preserving previous adjacencies).

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A model $\mathscr{H} = \{H_1, \ldots, H_{\mathrm{v}(H)}\}$ of H in a graph G is **tangent** to a subgraph G' of G if $\forall i \in [\mathrm{v}(H)]$, $|V(H_i)| \cap |V(G')| = 1$.

Goal: $\forall i \in [y]$

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Building a Minor Sequentially

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3. **Make $\mathscr{H}_{i+1}$ tangent to G_{i+1}**.

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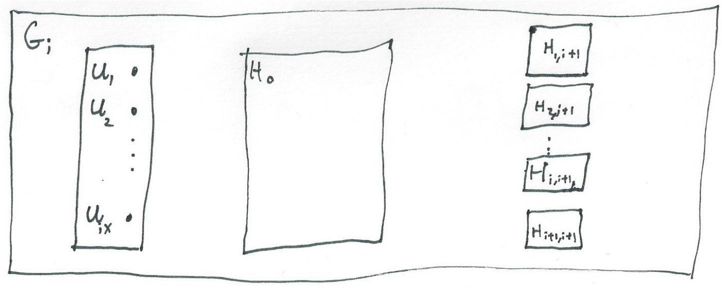
In $H_{j,i+1}$: choose vertices $T_{j,i+1} = \{t_{j,i+1,k}, t_{i+1,j,k} : k \in [x]\}$.



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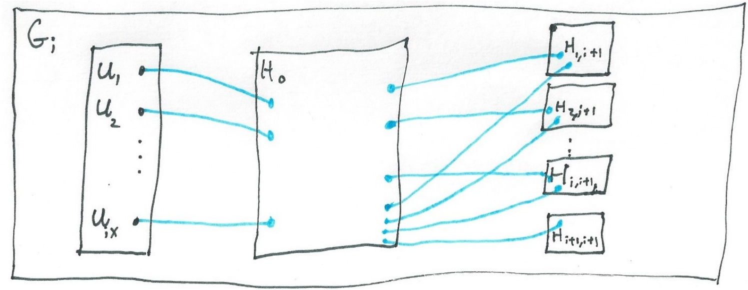

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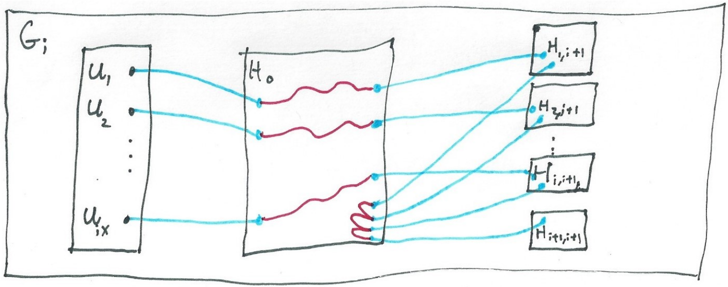

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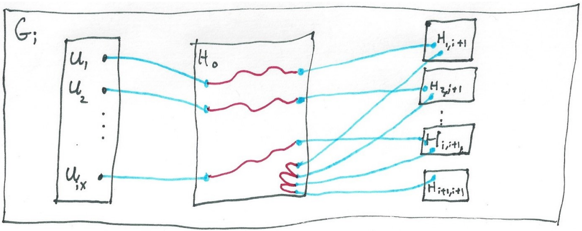
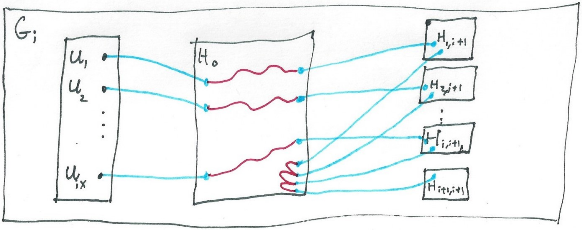


- **The Small Graphs**: Let $J = G[\{u_m : m \in [i]\} \cup V(H_0) \cup \bigcup_{j \in [i+1]} V(H_{i,j})]$.

Step Two: Large χ



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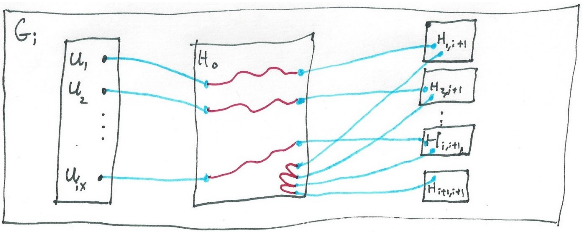

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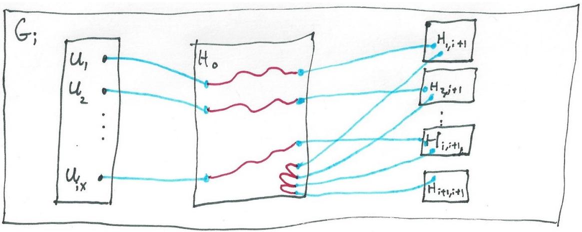

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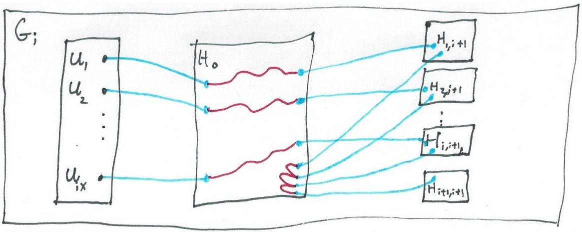

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- **Remaining:**

  $\chi(G_i - \mathscr{H}_{i+1}) \geq \chi(G_i) - \chi(J) - \chi(L) \geq \chi(G_i) - O(t \log\log t)$.

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  Works!

  - Make double the paths for $\mathscr{P}$.
  - Then using **Menger**'s make double the paths from G_{i+1} to H_0.
  - **Menger**'s then gives single copy paths for both.

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Apply lemma to each H_j in $\mathscr{H}_{i+1}$ with $|S_j| \leq t \log^3 t$. Hence $\exists$ $\mathscr{H}'_{i+1}$ tangent to G_{i+1} with

$$\chi(G_i - \mathscr{H}'_{i+1}) \geq \chi(G) - O(t \log\log t).$$

Back to Step Three: Still Tangent?

Hence by the **Girão-Narayanan** Theorem, there exists a subgraph G'_{i+1} disjoint from $\mathscr{H}'_{i+1}$ with

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- **Case 2** - otherwise:

  G_{i+1} and $G'_{i+1} \setminus V(G_{i+1})$ are vertex-disjoint and have

  $$\chi \geq \chi(G) - O(t \log\log t),$$

  contradicting that G is $O(t \log\log t)$-inseparable!

Summary

We proved:

Theorem (P. 2020+)

$\forall\ \beta > 0$, every graph with no K_t minor is $O(t(\log t)^{\beta})$-colorable.

The key was dividing into cases:

- Always Separable
- Inseparable

With the better density increment theorem, we get:

Theorem (P. 2020+)

Every graph with no K_t minor is $O(t(\log \log t)^6)$-colorable.

Future Directions

- **Improve the density increment theorem**?
  Say to $(1 + \log s)$?
- Avoid dividing into cases?
- Color small graphs better?
- Use just connectivity instead of chromatic number?

Theorem (Norin and P. 2020+)

$\forall \beta > \frac{1}{4}$, if G is $\Omega(t(\log t)^{\beta})$-connected and has no K_t minor, then $\mathrm{v}(G) \leq t(\log t)^{7/4}$.