Percolation on triangulations: A bijective path to Liouville quantum gravity

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Percolation on triangulations

CLE on Liouville Quantum Gravity
Percolation on triangulations

“Random curves on a random surface”

CLE on Liouville Quantum Gravity
Percolation on triangulations

CLE on Liouville Quantum Gravity

convergence
Percolation on triangulations

CLE on Liouville Quantum Gravity

Kreweras excursion

2D Brownian excursion

convergence
Percolation on triangulations
Percolation on a regular lattice

Triangular lattice
Percolation on a regular lattice

**Site percolation:** Color vertices *black* or *white* with probability $1/2$. 
Percolation on a regular lattice

\[ \text{An } \times \text{Bn box} \]

Site percolation: Color vertices \textit{black} or \textit{white} with probability \(1/2\).

Questions:

- Crossing probabilities?
Percolation on a regular lattice

Site percolation: Color vertices *black* or *white* with probability $1/2$.

Questions:
- Crossing probabilities?
- Law of interfaces?
Percolation on a regular lattice

Site percolation: Color vertices black or white with probability $\frac{1}{2}$.

Questions:
- Crossing probabilities?
- Law of interfaces?
- Mixing properties?
Triangulations (of the disk)

Def. A **triangulation** of the disk is a decomposition into triangles.
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(multiple edges allowed, loops forbidden)
Triangulations (of the disk)

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Def. A triangulation is rooted by marking an edge on the boundary.

Def. A triangulation is rooted by marking an edge on the boundary.
Percolation on triangulations

We can consider percolation on random triangulations of the disk. ($k$ exterior vertices, $n$ interior vertices; uniform probability)
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Same questions:
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We can consider percolation on random triangulations of the disk. 
\((k \text{ exterior vertices}, n \text{ interior vertices}; \text{uniform probability})\)

Goal 1: Answer these questions. (as \(n \to \infty\), \(k \sim \sqrt{n}\))
Percolation on triangulations

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Goal 1: Answer these questions. (as $n \to \infty$, $k \sim \sqrt{n}$)

We can alternatively consider infinite triangulations.

Uniform Infinite Planar Triangulation
[Angel, Schramm 04]
Regular lattices Vs random lattices

Is it interesting to study statistical mechanics on random lattices?
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Yes! New tools: random matrices, generating functions, bijections.
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Yes! Critically weighted random lattices \(\leadsto\) random surfaces.
Triangulations as a random surface

Uniformly random triangulation with $n$ triangles of side length $n^{-1/4}$.
Uniformly random triangulation with \( n \) triangles of side length \( n^{-1/4} \).

**Theorem** [LeGall 2013, Miermont 2013]*,**

Convergence in law as a metric space (Gromov-Hausdorff topology).
Limit is a random compact metric space homeomorphic to 2D sphere, of Hausdorff dimension 4.

(* for a different family of planar maps)  (** based on prior bijective results)
Triangulations as a random surface

Goal 2: Say something new about this random surface.
Conformal Loop Ensemble (CLE) on Liouville Quantum Gravity (LQG)
What is ... Liouville Quantum Gravity?
What is Liouville Quantum Gravity?

\textbf{LQG} is a random area measure $\mu$ on a $\mathbb{C}$-domain which is related to the Gaussian free field.
What is Liouville Quantum Gravity?

Brownian motion

\[ \mu = e^{\gamma h} dx \]
Random function chosen with probability proportional to
\[ e^{-\sum_{i=1}^{n} \frac{(h(i) - h(i-1))^2}{2}} \]

Brownian motion

\[ h = \lim_{n \to \infty} h_n \]

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What is . . . **Liouville Quantum Gravity**?

Random function chosen with probability proportional to
\[ e^{-\sum_{u \sim v} \frac{(h(u) - h(v))^2}{2}} \]

**Gaussian Free Field**
(a distribution)

**LQG**
(area measure)

\[ h_n : [n]^2 \to \mathbb{R} \]

\[ h = \lim h_n \]

\[ \mu = e^{\gamma h} dx dy \]
What is ... Liouville Quantum Gravity?

\[ h_n : [n]^2 \to \mathbb{R} \]

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\[ \gamma \in [0, 2] \] controls how wild LQG measure is.

Today: \[ \gamma = \sqrt{8/3} \]. “pure gravity”
What is... a **SLE** (Schramm–Loewner evolution)?
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\[
\text{SLE}_\kappa \text{ is a random (non-crossing, parametrized) curve in a } \mathbb{C}\text{-domain.}
\]
What is... a SLE (Schramm–Loewner evolution)?

\( \text{SLE}_\kappa \) is a random (non-crossing, parametrized) curve in a \( \mathbb{C} \)-domain.

The parameter \( \kappa \) determines how much the curve “wiggles”.

\( \text{SLE}_\kappa \) were introduced to describe the scaling limit of curves from statistical mechanics.
What is... a **SLE** (Schramm–Loewner evolution)?

SLE are **characterized** by:

- Conformal invariance property
- Markov domain property
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\[
\begin{align*}
\gamma(t) & \quad \tilde{\phi} \text{ conformal} \\
0 & \to 1
\end{align*}
\]
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**Theorem [Smirnov 01]:** Convergence.
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Theorem [Smirnov 01]: Convergence.

Theorem [Camia, Newman 09]: Convergence.
Conjectural relation (1990s)
LQG was introduced in physics as a model of random surface describing space-time evolution of strings.
Conjectural relation (1990s)

Riemann mapping

Random triangulations gives another natural model of random surfaces.

Conjectural relation (1990s)
It was conjectured that the two models were in fact exactly related.
Conjectural relation (1990s)

Riemann mapping  \rightarrow\text{Nice embedding}

Thm [Miller, Sheffield 2016]: Equality as metric spaces.
Goal 2’: Establish a relation between LQG and “embedded” random triangulations.
Goal 3: Establish a relation between percolation interfaces on random triangulations and CLE$_6$. 
Convergence results

Percolation on random triangulations

under nice embedding

CLE on Liouville Quantum Gravity
Convergence results

Percolation on random triangulations under nice embedding

CLE on Liouville Quantum Gravity
Thm [Bernardi, Holden, Sun 18]:
Let \((M_n, \sigma_n)\) uniformly random percolated triangulation of size \(n\) (\(n\) interior vertices, \(\sqrt{n}\) exterior vertices).

There exist embeddings \(\phi_n : M_n \rightarrow \mathbb{D}\) (and coupling) such that the following converge jointly in probability:

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- **Pivotal measures**: \(\forall \epsilon, i, j, \nu_{i,n}^\epsilon \to \nu_i^\epsilon\), and \(\nu_{i,j,n}^\epsilon \to \nu_{i,j}^\epsilon\).
- **Crossing events**: For random vertex \(v_n, E_b(v_n) \to E_b(v)\).
Strategy of proof:

$(M_n, \sigma_n)$

bijection

[Bernardi 2007]
[Bernardi, Holden, Sun 2018]

Convergence of walk

$LQG_{\sqrt{8/3}} + CLE_6$

measure preserving correspondence

[Duplantier, Miller, Sheffield 2014]
“mating of trees”

Convergence of walk

++++

++++
Strategy of proof:

Embedding \( \phi_n \)
defined using “space filling exploration”

measure preserving correspondence

Convergence of walk ++++

LQG \( \sqrt{\frac{8}{3}} \) + CLE_6

“mating of trees”
The bijection
Kreweras walks

Def. A **Kreweras walk** is a lattice walk on $\mathbb{Z}^2$ using the steps $a = (1, 0)$, $b = (0, 1)$ and $c = (-1, -1)$. 
Thm [Bernardi 07/ Bernardi, Holden, Sun 18]:
There is a **bijection** between:

- $\mathcal{K} =$ set of Kreweras walks starting and ending at $(0, 0)$ and staying in $\mathbb{N}^2$.
- $\mathcal{T} =$ set of percolated triangulations of the disk with 2 exterior vertices: one white and one black.
Example: $w = baabbcacc$
Example: $w = baabbcacc$
Example: \( w = \textbf{baabbcacc} \)

Definition:
Example: \( w = \textcolor{red}{baabbbeacc} \)

Definition:
**Thm:** This is a \textbf{bijection}.
Variants of the bijection

Spherical case

Disk case

UIPT case
<table>
<thead>
<tr>
<th>Dictionary</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>percolated triangulation</td>
<td>walk</td>
</tr>
<tr>
<td>edges</td>
<td>steps</td>
</tr>
<tr>
<td>left-boundary length</td>
<td>x-coordinate of walk</td>
</tr>
<tr>
<td>vertices</td>
<td>c-steps</td>
</tr>
<tr>
<td>black vertices</td>
<td>c steps of type abc</td>
</tr>
<tr>
<td>perco-interface toward $t$</td>
<td>walk of excursions</td>
</tr>
<tr>
<td>clusters</td>
<td>envelope intervals</td>
</tr>
<tr>
<td>cluster’s bubbles</td>
<td>cone intervals</td>
</tr>
</tbody>
</table>
Dictionary: percolation-interface to $v \longleftrightarrow$ walk of excursions
Dictionary: percolation-interface to $\nu \longleftrightarrow$ walk of excursions

- Flatten each sub-excursion into a single step
- Empty the bubbles
- Shuffle of two looptrees
- Flattened walk
discrete dictionary

[Bernardi, Holden, Sun]

continuum dictionary

[Duplantier, Miller, Sheffield]
Perfect correspondence!

Bernardi, Holden, Sun

Duplantier, Miller, Sheffield

Discrete dictionary

Continuum dictionary
Strengthening the convergence results

Holden, Sun + Albenque, Garban, Gwynne, Lawler, Li, Sepulveda + Miller, Sheffield

corvergence under **nice** embedding
Cardy embedding of triangulations

where $p_\bullet = P_{\text{perco}}(p_\bullet, p_\bullet, p_\bullet)$
Thm [Holden, Sun]: Convergence holds for the Cardy embedding. (because $\phi_n \approx \text{Cardy embedding}$)
Key ingredient used: “convergence componentwise”

Same triangulation

$k$ independent percolations

Same LQG

$k$ independent CLE

$k$ Kreweras walks

$k$ Brownian motions
Why useful?

To upgrade the “crossing event result” from an annealed result to a quenched result.
This implies (...) that $\phi_n \approx \text{Cardy embedding}$. 
Why useful?

To upgrade the “crossing event result” from an \textit{annealed result} to a \textit{quenched result}.
This implies (...) that \( \phi_n \approx \text{Cardy embedding} \).

How is it proved?

- \textbf{LQG stay the same}: prove the previous convergence is joint with convergence in Gromov-Hausdorff-Prokhorov topology.
- \textbf{CLE are independent}: prove CLE mixes fast (using pivotal point result).
Thanks.