I Introduction

$G(n,p)$

vertices: $1, 2, \ldots, n$

edges: keep each of the $n(n-1)$ possible directed edges independently with probability $p$. 
We're interested in strongly connected components (SCC).

\[ x \text{ and } y \text{ are in the same SCC if there are two paths: } x \rightarrow y \rightarrow x \]

b) Phase Transition For the SCC

Same as the Erdős–Rényi graph:

\[ p = \frac{c}{n} \]

\[ c > 1 \rightarrow \text{unique giant component} \]

\[ c < 1 \rightarrow \text{only small components} \]

In fact, something more precise is known

**Theorem: Łuczak–Sciersta (1985)**

Write \[ p = \frac{1}{n} + \frac{d_n}{n^{3/2}} \]

Suppose \[ d_n = o(n^{3/2}) \]

- If \[ d_n \rightarrow \infty \] then:
  - Largest SCC has size \( \sim 4n^{2/3}d_n \)
  - 2nd largest has size \( O(n^{2/3}/d_n) \)
- If \[ d_n \rightarrow \infty \] then:
  - Largest SCC has size \( O(n^{2/3}/d_n) \)
Q: What happens if $dn$ stays bounded on converges?

**Case of the Erdős–Rényi graph**

$$G(n, p) \quad p = \frac{1}{n} + \frac{d}{n^{2/3}} + o\left(\frac{d}{n^{2/3}}\right) \quad d \in \mathbb{R}$$

**Theorems:**

- **Aldous (1977):** let $Z_n \geq Z_n \geq \cdots$
  
  be the sizes of the connected components of $G(n, p)$. Then
  
  \[
  \frac{Z_i}{n^{2/3}} \xrightarrow{	ext{in } l^2, \text{ as sequences}} \alpha_i
  \]

- **Addario-Berry, Broutin, Goldschmidt (2012):**

  Let $(C_i^h)_i$ be the cc of $G(n, p)$ with
\[ \# C_i^n = Z_i^n \cdot \text{Then} \]

\[
\frac{C_i^n}{n^{2/3}} \xrightarrow{\text{oll}} \mathbb{C}_i
\]

(in Gromov-Hausdorff, as a sequence)

The main takeaway is that the distance between two typical points is of order \( n^{2/3} \).

(side note: both the \( C_i^n \) and \( \mathbb{C}_i \) have descriptions as "binary trees with a few additional edges")
Our results

\[ p = \frac{2}{n} + \frac{a}{n^{\log_2 3}} + \Theta \left( \frac{1}{n^{\log_2 3}} \right) \]

\section{Informally}

Within the SCC:

- w.h.p. no vertices with degree \( \geq 4 \)
- number of vertices with degree 3 is \( \approx n^3 \)
- number of vertices with degree 2 is \( \approx n^3 \)

\[ \sim a \cdot n^{1.5} \]
\[ \sim b \cdot n^{2.3} \]

\section{Metric directed multigraphs (MDM)}

\textbf{Definition:} \( \mathcal{G} \) is the set of finite multigraphs where each edge
has a direction and a length

- An isomorphism is a pair of bijections
  \( f: V(X) \to V(Y) \)
  \( g: E(X) \to E(Y) \)

which preserve the structure

**Examples:** How many isomorphisms?

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
  \draw (0,0) -- (0,1);
  \draw (1,0) -- (1,1);
  \draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} & \rightarrow & \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
  \draw (0,0) -- (0,1);
  \draw (1,0) -- (1,1);
  \draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} & \circ \quad 0
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
  \draw (0,0) -- (0,1);
  \draw (1,0) -- (1,1);
  \draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} & \rightarrow & \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
  \draw (0,0) -- (0,1);
  \draw (1,0) -- (1,1);
  \draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} & ?
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
  \draw (0,0) -- (0,1);
  \draw (1,0) -- (1,1);
  \draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} & \rightarrow & \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
  \draw (0,0) -- (0,1);
  \draw (1,0) -- (1,1);
  \draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} \quad 2
\end{align*}
\]

distance: \( d_{\Sigma} (X,Y) = \inf_{\{f \in \mathcal{F} \}} \sum_{e \in E(X)} |f_e - g_{f(e)}| \)
\[
\frac{1}{\delta^2} \left( \frac{1}{\epsilon} + \frac{1}{2} + \frac{1}{2} \right) \lesssim 3 + 3 = 6
\]

Remark: this metric is very rigid:

\[
\frac{1}{\delta^2} \left( \frac{1}{\epsilon} \right) \leq 2 \Rightarrow \delta \geq 2
\]

c) Main Theorem:

Let \( (c_i, n) \), \( n \in \mathbb{N} \) be the SCC of \( \overline{G}^2(n, p) \), viewed as MDMs by removing the

\( \delta^0 = 2 \) vertices. Then:

\[
\left( \frac{c_i}{n^{3/2}}, i \in \mathbb{N} \right) \xrightarrow{d} (c_i, i \in \mathbb{N})
\]

where \( c \in \mathbb{R}^\infty \) s.t.
- Finitely many terms are 3-regular
- the rest are cycles.

\[ \text{topology for sequences}: \quad d(A_i, B) = \sum_{i \geq 1} d_{\mathcal{F}}(A_i, B_i) \]

IV Exploration and structure
A version of depth-first search
b) Edge classification

- tree edge (forward)
- surplus edge (forward)
- back edge (backward)

Interaction between forward and backward
→ SCC.

\( \square \) Limit behaviour of forest and surplus

prop: "Forward edges without arrows"

\[
\frac{1}{n!} = \mathcal{O}(n, p) \quad \text{Erős-Rényi}
\]

So we know a lot about the trees and surplus edges. In particular:

- number of vertices in *Tree ~ n^{2/3}
- typical distances are ~ n^{1/3}
- surplus edges are $O(n)$ in number.

It turns out, surplus edges don’t count!

**prop:** $\Pr(\text{a SCC of } G_{n,p} \text{ has a surplus edge}) \rightarrow 0$

```
"p<":
```

```
\Pr(\text{"p": counts})
```

\[\text{VI Limit of the back edges}\]

\[\text{a1 problem?}\]

In a single tree, there are $k(k-1)$ possible back edges, appearing independently with probability $p \approx \frac{1}{n}$.

But $k \approx n^{3/2}$ ...
\[ E[\text{# of back edges}] \approx \frac{1}{2} n^4 \frac{\lambda}{\mu} \rightarrow \infty \]

b) solution: Most of the BE don't matter!

How to find those which do matter:

- go around the tree
- keep the first ancestral BE
- keep any BE which is either ancestral or links into something we've already selected.

This turns out to converge to a PPP on the limiting continuum tree.
What we end up with:
Other works:

- Coulson (’79): estimates for size of largest component
- De Panafieu, Douris, Diatchkou, Rasheed, Husaini
- Wagner (’20): asymptotics for probability of acyclic $\sim \frac{c}{m}$
- only singleton and cyclic SCCs.

$C$ and $C'$ explicit