Erdős covering systems

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(Based on joint work with Paul Balister, Béla Bollobás, Julian Sahasrabudhe and Marius Tiba)

University of Oxford, March 31st, 2020

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Definition (Erdős, 1950)

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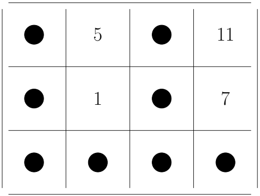

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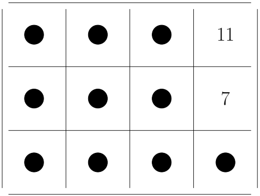

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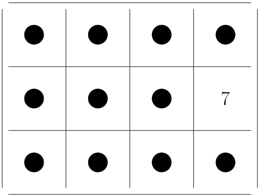

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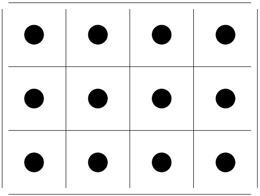

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Erdős used a similarly simple covering system to answer a question of Romanoff (and refute a conjecture of de Polignac), by showing that not all odd numbers are of the form $2^k + p$, where p is either 1 or prime.

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Conjecture (Erdős and Graham, 1980)

If the moduli of a system of arithmetic progressions are distinct and lie in the interval $[n, Cn]$, where $n \geqslant n_0(C)$ is sufficiently large, then the uncovered set has density at least δ for some $\delta = \delta(C) > 0$.

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It also implies there are only $2^{O(n^2)}$ minimal covering systems of size n.

Constructing many minimal covering systems

Recall that

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For each $1 \leqslant i \leqslant k$ and $1 \leqslant a \leqslant p_i - 1$, choose an arithmetic progression

$$\{a \cdot Q_{i-1} \pmod{d \cdot p_i}\} \quad \text{for some} \quad d \,\big|\, Q_{i-1}.$$

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A *frame* is a collection of arithmetic progressions as above.

An example of a frame

For each prime p we choose $p-1$ progressions of the form

$$\{a \cdot Q_{i-1} \text{ (mod } d \cdot p_i)\} \quad \text{for some} \quad d \,|\, Q_{i-1} = p_1 \cdots p_{i-1},$$

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- $p = 7$: we could take (for example):

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- and so on...

Counting the frames

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There are at least

$$\exp\left(\frac{\Omega(n^{3/2})}{(\log n)^{1/2}}\right)$$

minimal covering systems of $\mathbb{Z}$ of size n.

Proof.

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Theorem (Balister, Bollobás, M., Sahasrabudhe and Tiba, 2020+)

The number of minimal covering systems of $\mathbb{Z}$ of size n is

$$\exp\left(\left(\frac{4\sqrt{\tau}}{3} + o(1)\right)\frac{n^{3/2}}{(\log n)^{1/2}}\right)$$

as $n \to \infty$, where

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To prove this result, we needed to study the 'rough typical structure' of a minimal covering system of size n.

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Let $\mathcal{A}$ be a covering system with distinct moduli $d_1, \ldots, d_k \geqslant M$. Then

$$\sum_{i=1}^{k} \frac{1}{d_i} \geqslant \frac{\log M \log\log\log M}{4 \log\log M}.$$

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They also proved the conjecture of Erdős and Graham.

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Theorem (Hough, 2015)

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We bound the (distorted) measure of the set covered when revealing the prime p, and show that if the minimum modulus is sufficiently large, then the total (distorted) measure removed can be made arbitrarily small. □

A geometric setting

Let $S_1, \ldots, S_n$ be finite sets with at least two elements, and set

$$Q_k = S_1 \times \cdots \times S_k$$

for each $1 \leqslant k \leqslant n$.

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$$Q_k = S_1 \times \cdots \times S_k$$

for each $1 \leqslant k \leqslant n$.

A *hyperplane* in Q_n is a set

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Theorem (Balister, Bollobás, M., Sahasrabudhe and Tiba, 2020+)

If $|S_k| \geqslant 4k$ for all sufficiently large k, then there exists a constant C such that the following holds. Let $\mathcal{A}$ be a collection of hyperplanes that cover $Q_n = S_1 \times \cdots \times S_n$. Then either two of the hyperplanes are parallel, or there exists a hyperplane $A \in \mathcal{A}$ with $F(A) \subset \{1, \ldots, C\}$.

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(Proof: Set $S_k = \{1, \ldots, p_k\}$, where p_k is the kth prime, and use the Chinese Remainder Theorem to map progressions to hyperplanes.)

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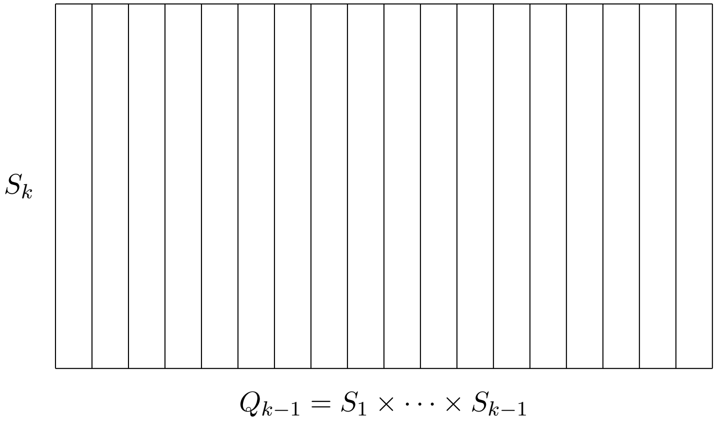

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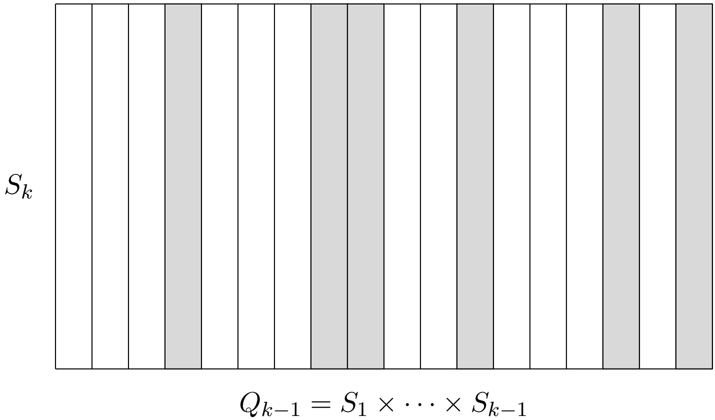

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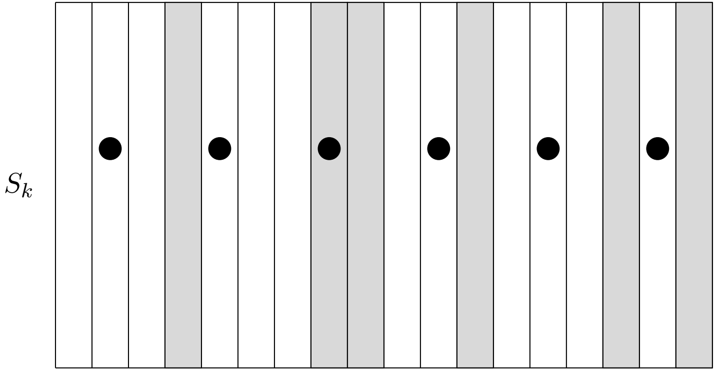

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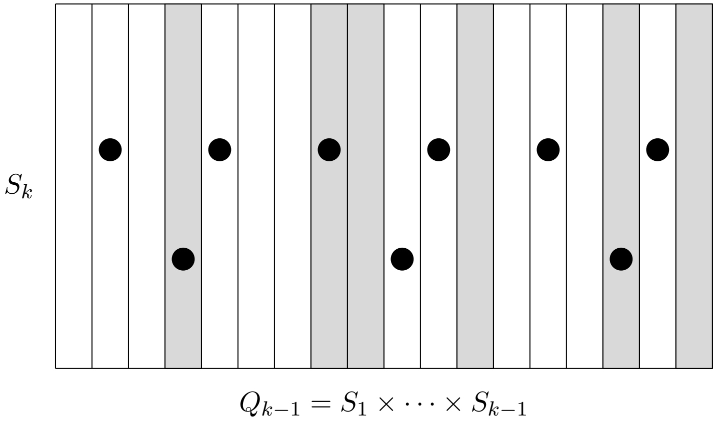

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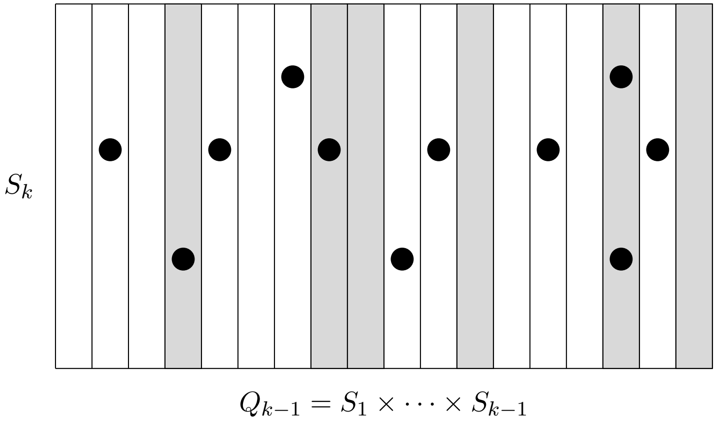

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However, it turns out to be simpler to do something more complicated!

The distortion method (BBMST version)

Recall that $Q_k = Q_{k-1} \times S_k$, and for each $x \in Q_{k-1}$ define

$$\alpha_k(x) := \frac{\big|\{y \in S_k : (x,y) \in B_k\}\big|}{|S_k|},$$

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- If $\alpha_k(x) > \delta$, then we 'cap' the distortion by increasing the measure at each point of $F_x \setminus B_k$ by a factor of $1/(1-\delta)$, and decreasing the measure on points of $F_x \cap B_k$ by a corresponding factor.

The distortion method (key lemma)

Lemma

Let $\mathcal{A}$ be a collection of hyperplanes in $Q_n = S_1 \times \cdots \times S_n$. If

$$\frac{1}{4\delta(1-\delta)} \sum_{k=1}^{n} \mathbb{E}_{k-1}\big[\alpha_k(x)^2\big] < 1,$$

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Proof.

A simple calculation (using the inequality $\max\{a-b,0\} \leqslant a^2/4b$) gives

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Hence $\sum_k \mathbb{P}_n(B_k) = \sum_k \mathbb{P}_k(B_k) < 1$, and so $\mathcal{A}$ does not cover Q_n. □

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A straightforward induction shows that if $F(A) \subset [k]$, then

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The lemma now follows from a simple union bound. □

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Theorem (Balister, Bollobás, M., Sahasrabudhe and Tiba, 2020+)

If $|S_k| \geqslant (3+\varepsilon)k$ *for all* $k \geqslant k_0$*, then there exists* $C = C(\varepsilon, k_0)$ *such that the following holds. Let* $\mathcal{A}$ *be a collection of hyperplanes that cover* $Q_n = S_1 \times \cdots \times S_n$*. Then either two of the hyperplanes are parallel, or there exists a hyperplane* $A \in \mathcal{A}$ *with* $F(A) \subset \{1, \ldots, C\}$*.*

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The previous lemma gives (via a straightforward calculation)

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By the Key Lemma, it follows that $\mathcal{A}$ does not cover Q_n, as required. □

The density of the uncovered set

Theorem (Filaseta, Ford, Konyagin, Pomerance and Yu, 2007)

If $n \gg \exp\big(\log C \log\log C\big)$*, then for any system of arithmetic progressions with distinct moduli* $d_1, \ldots, d_k \subset [n, Cn]$*, the uncovered set has density at least*

$$(1 + o(1)) \prod_{i=1}^{k} \left(1 - \frac{1}{d_i}\right).$$

The density of the uncovered set

Question (Filaseta, Ford, Konyagin, Pomerance and Yu, 2007)

If a covering system has distinct moduli $d_1, \ldots, d_k$ *satisfying*

$$d_1, \ldots, d_k \geqslant M \qquad \textit{and} \qquad \sum_{i=1}^{k} \frac{1}{d_i} < C,$$

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Theorem (Balister, Bollobás, M., Sahasrabudhe and Tiba, 2020+)

For every $M > 0$ and $\delta > 0$, there exists a finite collection of arithmetic progressions with distinct moduli $d_1, \ldots, d_k \geqslant M$, such that

$$\sum_{i=1}^{k} \frac{1}{d_i} < 1$$

and the density of the uncovered set is less than δ.

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Theorem (Balister, Bollobás, M., Sahasrabudhe and Tiba, 2020+)

Let χ *be the multiplicative function defined by*

$$\chi(p^i) = 1 + \frac{(\log p)^4}{p}$$

for all primes p *and integers* $i \geqslant 1$. *There exists* $M > 0$ *so that for any system of arithmetic progressions with distinct moduli* $d_1, \ldots, d_k \geqslant M$, *if*

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The function χ cannot be replaced by one of the form $\chi(p^i) = 1 + O(1/p)$.

Some additional consequences

Conjecture (Schinzel, 1967)

In any covering system, one of the moduli divides another.

Some additional consequences

Theorem (Balister, Bollobás, M., Sahasrabudhe and Tiba, 2020+)

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Does there exist a covering system with all moduli distinct and odd?

Some additional consequences

Theorem (Balister, Bollobás, M., Sahasrabudhe and Tiba, 2020+)

In any covering system, one of the moduli divides another.

The Erdős–Selfridge problem

Does there exist a covering system with all moduli distinct and odd?

Theorem (Hough and Nielsen, 2019)

In any covering system with distinct odd moduli, one of the moduli is divisible by 3.

Some additional consequences

Theorem (Balister, Bollobás, M., Sahasrabudhe and Tiba, 2020+)

In any covering system, one of the moduli divides another.

The Erdős–Selfridge problem

Does there exist a covering system with all moduli distinct and odd?

Theorem (Balister, Bollobás, M., Sahasrabudhe and Tiba, 2020+)

In any covering system with distinct odd moduli, the least common multiple of the moduli is divisible by either 9 *or* 15.

Some additional consequences

Theorem (Balister, Bollobás, M., Sahasrabudhe and Tiba, 2020+)

In any covering system, one of the moduli divides another.

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Does there exist a covering system with all moduli distinct and odd?

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In any covering system with distinct odd moduli, the least common multiple of the moduli is divisible by either 9 or 15.

Theorem (Balister, Bollobás, M., Sahasrabudhe and Tiba, 2020+)

No covering system exists with distinct odd square-free moduli.

Thank you!