Sections of high rank varieties and applications

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Setting : Let $\mathbf{k}$ be a field. Consider $R = \mathbf{k}[x_1, \ldots, x_n]$ the ring of polynomials in n variables. We are interested in *stability* properties of ideals $I \subset R$, namely properties that are independent of the number of variables n.

Example 1:

Let $Q_1, \dots, Q_c \in \mathbf{k}[x_1, \dots, x_n]$ be homogeneous polynomials.

Hilbert Syzygy Theorem

The ideal $I = (Q_1, \dots, Q_c)$ is of projective dimension at most n.

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Stillman Conjecture

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Ananyan-Hochster Theorem (2016): YES !

Example 2:

Easy algebraic fact: $V|_{\mathbf{k}}$ a vector space, $W \subset V$ a subspace. If $f : W \to \mathbf{k}$ is a polynomial, $\deg f = a$, then $\exists P : V \to \mathbf{k}$ a polynomial, $\deg P = a$, such that $P|_W = f$.

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Question: Let $X \subset V$ be a degree d hypersurface.
For example, when $d = 2$

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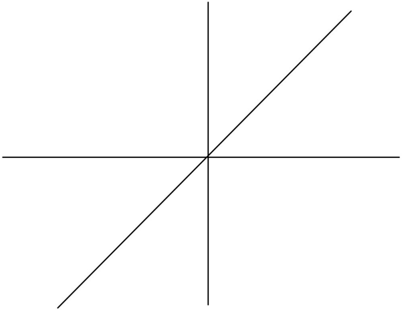
$\exists P : V \to \mathbf{k}$ a polynomial, $\deg P = a$, such that $P|_X = f$?

No !

Example: $Q(x,y) = xy(x-y)$, $X_Q = \{x \in V : Q(x) = 0\}$.

$$f(x,0) = f(0,y) = 0, \quad f(x,x) = x$$



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Polynomial Extension Theorem [Kazhdan-Z]

Let $a, d > 0$, $\mathbf{k}$ be admissible. $\exists r = r(a, d, c)$ such that:
For any $\bar{Q} = \{Q_i\}_{i=1}^c \subset \mathbf{k}[x_1, \dots, x_n]$, $\deg Q_i \leq d$ and nc-rank $> r$, $X_{\bar{Q}}$ has property $*_a$.

Rank

$Q \in \mathbf{k}[x_1, \ldots, x_n]$, $\deg Q = d > 1$. The rank of Q is the minimal r such that $\exists S_i, R_i$ of degrees $< d$ such that

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$\tilde{Q} : V^d \to \mathbf{k}$ is the symmetric d-tensor:

$$\tilde{Q}(h_1, \ldots, h_d) = \Delta_{h_1} \ldots \Delta_{h_d} Q$$

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If char $\mathbf{k} > d$ then $Q(x) = \tilde{Q}(x, \ldots, x)/d!$, and $r(Q) \leq r_{nc}(Q)$.

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Theorem [Kazhdan - Z]

$\exists \gamma = \gamma(d, c)$ such that for any $\mathbf{k}$, any $\bar{Q}$ defined over $\mathbf{k}$. Then

$$r_{nc}(\bar{Q}(\bar{\mathbf{k}})) \leq \gamma s(\bar{Q})^{\gamma}.$$

A key property of high rank varieties (stated for a hypersurface):

V a $\mathbf{k}$-vector space, $Q : V \to \mathbf{k}$, $\deg Q = d$.
Call Q m-universal if for any $R \in \mathbf{k}[x_1, \ldots, x_m]$, $\deg R = d$, there exists an affine map $\phi : \mathbf{k}^m \to V$ such that $R = Q \circ \phi$.

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Theorem [Kazhdan - Z]

$\exists r = r(d)$ such that for any $\mathbf{k}$ which is either finite or algebraically closed, any Q with $r_{nc}(Q) > r$ is m universal.

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Key result from model theory

Let T be the theory of algebraically closed fields. Then any first order property in T true for algebraic closure of finite fields is true for all algebraically closed fields.

Key technical tool: rank can be measured analytically.

Bias-Rank (Bhowmik-Lovett, Milicévic)

Let $s > 0$. $\exists r = r(d,s)$, such that for any $\mathbf{k}$, $|\mathbf{k}| = q$, any $\mathbf{k}$-vector space V, and $Q : V \to \mathbf{k}$ of degree d, if $r_{nc}(Q) > r$ then for any non trivial additive character $\chi : V \to \mathbb{C}^*$

$$|V|^{-1} \sum_{v \in V} \chi(Q(v)) < q^{-s}$$

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Question: can we have $D(d) = 1$?

Proof of polynomial extension theorem for a hypersurface, **k** finite.

X_Q has property $*_a$ if for any $f : X_Q \to \mathbf{k}$ weakly polynomial of degree $\leq a$, $\exists P : V \to \mathbf{k}$ a polynomial, $\deg P = a$, $P|_{X_Q} = f$.

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- Step 3: Show that if Y is of rank $> r(d)$, $Y = X \cap W \subset W$, W a hyperplane, then if $*_a$ holds for Y then $*_a$ hold for X. Now induct via a flag. (use $r(Q|_W) \leq r(Q)$).

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  Key: If $W_t = \{l(x) = t\}$ then almost any line in $W_t \cap X$ has a parallel line in $W \cap X$.

Example of proof scheme.

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Example: $Q(x,y) = xy + 1$. $\mathbb{F}_t(Q)$ is irred $t \neq 1$, but $\mathbb{F}_1(\bar{Q})$ not.

- $\mathbb{A}$ the affine line, $\mathbf{k} = \mathbb{A}(\mathbf{k})$.
- $V|_{\mathbf{k}}$ vector space $\rightsquigarrow \mathbb{V}(\mathbf{k}) = V$.
- $Q : V \to \mathbf{k} \rightsquigarrow \mathbb{Q} : \mathbb{V} \to \mathbb{A}$
- $\mathbb{F}_t(Q) := \mathbb{Q}^{-1}(t) \subset \mathbb{V}$, $t \in \mathbb{A}$

IFT holds for finite fields $\mathbf{k} = \mathbb{F}_q$.

- $\mathbf{k}_l := \mathbb{F}_{q^l}$, $\mathbb{X}$ an m-dimensional algebraic variety defined over $\mathbf{k}$.
- $c(\mathbb{X})$ - number of irreducible components of $\mathbb{X}$ of dimension m (considered as a variety over the algebraic closure $\mathbf{K} = \bar{\mathbf{k}}$).
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Theorem (Weil)

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Bias-rank $\implies \exists r = r(d)$ such that if $r_{nc}(Q) > r$, then for all $t \in \mathbf{k}_l$, $|a(t) - q^{-l}| \leq q^{-3l}$.

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We immediately deduce IFT holds for $\mathbf{K} = \bar{\mathbf{k}}$ the algebraic closure of $\mathbf{k} = \mathbb{F}_q$: $\mathbf{K} = \bigcup \mathbb{F}_{q^n}$ so we may assume that $t \in \mathbb{F}_{q^n}$.

IFT hold for any algebraically closed field:

Reformulation of IFT

Let $d \geq 1$ and $n \geq 1$. A field $\mathbf{k}$ has the property $\star(n,d,r)$ if for any polynomial $Q \in \mathbf{k}[x_1,\dots,x_n]$ of degrees d and nc-rank $\geq r$ all varieties $\mathbb{F}_t,\ t \in \mathbb{A}$ are irreducible of dimension $n-1$.

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IFT is equivalent to the following statement.

Claim: For any $d \geq 1$ there exists $r = r(d)$ such that any algebraically closed field has $\star(n,d,r)$ for all $n \geq 1$.

Now $\star(n,d,r)$ is a first order property in T.

Thank you !