Power law bounds for critical long-range percolation.

J. Hutchcroft

\[ J : \mathbb{Z}^d \rightarrow [0, \infty) \] integrable \[ \sum J(x) < \infty \]
\[ J(-x) = J(x) \]

Long-range percolation: Random graph with vertex set \[ \mathbb{Z}^d \], include a possible edge \[ \text{i} \rightarrow \text{j} \] with prob
\[ 1 - e^{-BJ(|\text{i} - \text{j}|)} \approx BJ(|\text{i} - \text{j}|) \] when \[ |\text{i} - \text{j}| \] large,
a parameter

Any pair \[ \text{i} \rightarrow \text{j} \] could be an edge.
No special role for nearest neighbours

Mostly interested in the case \[ J(x) \sim |x|^d - \alpha \], \( \alpha > 0 \)

No \( \infty \) clusters \( \Leftrightarrow \infty \) cluster

\[ 0 \quad B_c \quad \infty \]
$d = 1 : \beta_c < \infty \iff \alpha \leq 1$

Newman Shulman '86

$d > 1 : \beta_c < \infty \quad \forall \alpha > 0$.

Surprisingly, long-range percolation is better understood than nearest-neighbour!

Thm Aizenman & Newman '86

When $d = 1$ and $\alpha = 1$, the phase transition is discontinuous: there exists an $\infty$ cluster at $\beta_c$ a.s.

New proof on the arXiv this morning!

Thm Berger '02

If $\alpha < d$ then the phase transition is continuous: there are no $\infty$ clusters at $\beta_c$ a.s.
Why is $\alpha < d$ important?

Number of edges between two boxes

\[
\sim (n^d)^2 n^{-d-\alpha} = n^{d-\alpha}
\]

large when $\alpha < d$  
constant when $\alpha = d$  
zero whp when $\alpha > d$.

Beens's proof works by showing that the set

$\mathbb{P}(\beta > 0): \exists \text{ an infinite cluster at } \beta$'s

is open, and does not yield quantitative control of critical percolation.

Today:

\[
\text{.cluster of origin: } J(x) \geq C ||x||^{-d-\alpha}
\]

\[
\forall x \text{ with } ||x|| \geq r_0
\]

\[
\text{constant}
\]

\[
\text{Thm (H. 2020) If } \alpha < d \text{ then } \exists \text{C st.}
\]

\[
\mathbb{P}(\text{ } |K_0| = n) \leq C n^{-(d-\alpha)/(2d+\alpha)}
\]

\[
\frac{1}{r^d} \sum_{x \in T \leftrightarrow r^d} \mathbb{P}(0 \leftrightarrow x) \leq C r^{-2(d-\alpha)/3d}
\]
Based on a completely different method to Berger.

Most interesting when $d < 6$, $\alpha > \frac{d}{3}$ where the model is not expected to be mean-field, no previous power law upper bounds known.

Same proof gives similar results for groups other than $\mathbb{Z}^d$.

### Critical Exponents

**Conjecture:** For each $d \geq 1$ and $\alpha > 0$ there exist critical exponents $\delta$ and $\eta$ such that

\[
P_{\mathbb{Z}^d}(|K_\alpha| = n) \approx n^{-\frac{1}{\delta}}
\]

\[
P_{\mathbb{Z}^d}(x \leftrightarrow y) \approx \frac{1}{d} + \frac{1}{2} - \eta
\]

Intentionally vague!

$d > 6$ or $\alpha < \frac{d}{3} \Rightarrow \delta = 2$, $\eta = 0$

"Mean-field behaviour"    Hara & Slade 1990

Chen & Sakai 2015
Always have \( \delta \geq 2, \ 2 - \eta \leq d \)

Our results: If \( \delta, \eta \) are well-defined then

\[
\delta \leq \frac{2d + \alpha}{d - \alpha}, \quad 2 - \eta \leq \frac{d}{3} + \frac{2\alpha}{3}
\]

There is a surprisingly simple prediction for the true value of these exponents:

\( d = 1 \):
\[ 2 - \eta = \alpha, \quad \delta = \frac{1 + \alpha}{1 - \alpha} \sqrt{2} \]

\( d = 2 \):
\[ 2 - \eta = \alpha \left( \eta_{\text{NSR}} \right), \quad \delta = \frac{d + \alpha}{d - \alpha} \sqrt{2} \eta_{\text{SR}} \delta_{\text{SR}} \]

Interesting things can happen here (log corrections)
Our bounds are of reasonable order inside the conjectured "long-range dominant" regime.
Proof overview:
Two basic strategies for proving no percolation at $p_c/\beta_2$

"Supercritical strategy": Prove that if infinite clusters exist, then they must be "large" in some way, that guarantees they have $p_c < 1$.
This shows that $\mathbb{P}: \infty \text{ cluster exists is } p_c$ does not typically yield quantitative control of critical percolation.

E.g. Harrs $\mathbb{Z}^2$ (1960), Benjamini, Lyons, Peres, Schramm nonamenable groups (1999)
Slabs $\mathbb{Z}^2 \times [0,1]$, $\mathbb{Z}^{d-2}$ Damkold, Copin, Sidoravicius, Tasmin (2016)

"Subcritical strategy": Try to prove that $\mathbb{P}: \infty \text{ clusters do not exist is } p_c$ is closed
by proving that some non-trivial upper bound on the distribution of $K_0$ holds uniformly on $(0, p_c)$.

Often uses a bootstrapping argument
Prove that some non-trivial bound implies a strictly stronger version of itself

E.g., Hara–Slade lace expansion method roughly works by showing that in high-dimensions

\[
(\mathbb{P}_p (x \leftrightarrow y) \leq 3 G(x, y) \quad \forall x, y \in \mathbb{Z}^d)
\]

\[
(*) \Rightarrow (\mathbb{P}_p (x \leftrightarrow y) \leq 2 G(x, y) \quad \forall x, y \in \mathbb{Z}^d)
\]

Where $G(x, y) = \|y - x\|_{d+2}$ is the Greens function.

If (*) is established, a continuity argument yields that the strong form of the bound holds uniformly on $(0, p_c)$ and hence at $p_c$ also.
Our proof is also based on a bootstrapping argument.

- Builds on ideas originally used to analyze percolation on some large groups, some joint with Jonathan Herman.

- One key ingredient is the two-ghost inequality

\[
\sum_x \sqrt{J(x)(e^{\beta J(x)} - 1)} P^x_{\beta \mathcal{G}}(\sum_{v \in V} \Psi(v) = \phi) \leq C 
\]

Thin Normalize so that \( \sum J(x) = 1 \).

\[
P(\text{all distinct each with at least } n \text{ vertices, at least } 1 \text{ finite}) \leq C \frac{\log n}{n^2}
\]

Uses ideas from Aizenman-Kesten-Newman '86. We'll see stronger versions later.
Let's now prove that if $x < \frac{1}{4}$ then
\[ P_{\beta_c}(|k| = n) \leq C n^{-\frac{d-4x}{4d}} \]

Suffices to prove that $\exists C$ st. if $\frac{\beta}{2} \leq \beta < \beta_c$ then $\forall A < \infty$

\[ P_{\beta}(|k| = n) \leq A n^{-\theta} \quad \forall n \geq 1 \]

(*) \Rightarrow \left( P_{\beta}(|k| = n) \leq C(A+1) n^{-\theta} \quad \forall n \geq 1 \right)

Indeed, if we define $A(\beta)$ to be minimal st
\[ P_{\beta}(|k| = n) \leq A(\beta) n^{-\theta} \quad \forall n \geq 1 \text{ then} \]

Shapress of the phase transition $\Rightarrow A(\beta) < \infty \quad \forall \beta < \beta_c$

(*) \Rightarrow A(\beta) \leq C A(\beta) + 1 \quad A(\beta) \leq 4 C^2 \quad \forall \frac{\beta}{2} \leq \beta < \beta_c
Let's now prove $(\ast)$. Let 

$$ \Lambda_r = \mathbb{E}^{-r} \mathbb{R}^d $$

Fix \( P_c \frac{\beta_c}{2} \leq \beta < \beta_c \) and let \( A \) be such that \( P_{\beta}(|k_1| \geq n) \leq A n^{-\delta} \).

Let \( S_{x,n} = \{ \mathbb{Z}_n \otimes \mathbb{Z}_n \} \) then

two-ghost inequality \( \Rightarrow \)

$$ \sum_{x \in \Lambda_r} P(S_{x,n}) \leq C r^{d+\alpha} \sum_{x \in \mathbb{Z}_d} \sqrt{\mathcal{J}(x) e^{\beta s_{k_0} - 1}} P(S_{x,n}) $$

$$ \leq C r^{d+\alpha} / n^{\frac{1}{2}}. $$

But we also have

$$ P(S_{x,n}) = P(|k_0|, |k_x| \geq n) - P(\delta \leq x) $$

$$ \geq P(|k_0| \geq n)^2 - P(\delta \leq x) $$

\( \uparrow \) FKG.
And $\sum_{x \in \Delta_r} P_B(o \leftrightarrow x) = E_B |k_o \cap \Delta_r|^d$

$\leq E_B |k_o| \times (2r+1)^d$

$= \sum_{i} P_B(1k_0|zn)$

$\leq CArd(1-\Theta)$

Summing over $x$ and rearranging gives

$P(1k_0|zn) \leq \frac{1}{|\Delta_r|} \sum_{x \in \Delta_r} P(o \leftrightarrow x)$

$+ \frac{1}{|\Delta_r|} \sum_{x \in \Delta_r} P(S_{x,n})$

$\leq CArd(1-\Theta) + C |r|^{d+\alpha}$

Optimize over $r$ by taking $r = n^{1-\frac{4\Theta}{2\alpha}}$

$\leq C(A+1)^{-2\Theta}$

$\Theta$ was chosen to make this step work. $\Box$
Two ingredients to improve this proof:

- Find a better way to convert volume-tail bounds into two-point function bounds.

- Improved two ghost inequality.

\[
\text{Thm } \sum J(x) = 1. \text{ Assume that } P(|k_0| \leq n) \leq A n^{-\theta} \text{ for some } A < \infty \text{ and } 0 \leq \theta < 1/2. \text{ Then }
\]

\[
\sum (e^{B^2(x)} - 1) P(S_{x,n})^2 \leq \frac{C A^2}{(1-\theta)^2} n^{1+2\theta}
\]

- Gets something out of the bootstrapping hypothesis.
- Handles large \( n \) better.
Universal tightness of the maximum cluster size.

\[ G = (U, E, J) \] weighted graph

\[ J : E \rightarrow [0, \infty) \] percolation defined as before.

Given \( \Delta \subseteq V \) finite, define

\[ |K_{\text{max}}(\Delta)| = \max \{ |K_v \cap \Delta| : v \in V \}. \]

\[ M_\beta(\Delta) = \min \{ n : P_\beta(|K_{\text{max}}(\Delta)| \geq n) \leq \frac{1}{e} \} \]

"Typical value", essentially the median.

We will prove that \( |K_{\text{max}}(\Delta)| \)

is always of order \( M_\beta(\Delta) \), with universal upper and lower tail bounds.
Theorem 2.2 (Universal tightness of the maximum cluster size). Let $G = (V, E, J)$ be a countable weighted graph and let $\Lambda \subseteq V$ be finite and non-empty. Then the inequalities

$$P_\beta\left(|K_{\text{max}}(\Lambda)| \geq \alpha M_\beta(\Lambda)\right) \leq \exp\left(-\frac{1}{9} \alpha\right)$$  \hspace{1cm} (2.5)

and

$$P_\beta\left(|K_{\text{max}}(\Lambda)| < \varepsilon M_\beta(\Lambda)\right) \leq 27 \varepsilon$$  \hspace{1cm} (2.6)

hold for every $\beta \geq 0$, $\alpha \geq 1$, and $0 < \varepsilon \leq 1$. Moreover, the inequality

$$P_\beta\left(|K_u \cap \Lambda| \geq \alpha M_\beta(\Lambda)\right) \leq \varepsilon P_\beta\left(|K_u \cap \Lambda| \geq M_\beta(\Lambda)\right) \exp\left(-\frac{1}{9} \alpha\right)$$  \hspace{1cm} (2.7)

holds for every $\beta \geq 0$, $\alpha \geq 1$, and $u \in V$.

We will deduce this theorem as a corollary of the following more general inequality.

Theorem 2.3. Let $G = (V, E, J)$ be a countable weighted graph and let $\Lambda \subseteq V$ be finite and non-empty. Then the inequalities

$$P_\beta\left(|K_{\text{max}}(\Lambda)| \geq 3^k \lambda\right) \leq P_\beta\left(|K_{\text{max}}(\Lambda)| \geq \lambda\right)^{3^{k-1}+1}$$  \hspace{1cm} (2.8)

and

$$P_\beta\left(|K_u \cap \Lambda| \geq 3^k \lambda\right) \leq P_\beta\left(|K_{\text{max}}(\Lambda)| \geq \lambda\right)^{3^{k-1}} P_\beta\left(|K_u \cap \Lambda| \geq \lambda\right)$$  \hspace{1cm} (2.9)

hold for every $\beta \geq 0$, $\lambda \geq 1$, $k \geq 0$, and $u \in V$.

This has the following consequence:

If $G_0$ is such that $P(1|K_u| \geq n) \leq \text{An}^{-\theta}$ \hspace{1cm} $\forall n \geq 1$

then in fact

$$P(1|K_u \cap \Lambda| \geq n) \leq C_0 \text{An}^{-\theta} e^{-\frac{n}{18M_\beta(\Lambda)}}$$ \hspace{1cm} $\forall n \geq 1$.

(Proof is just calculus.)

$\text{constant} = e^{(18)^0}$
This inequality is extremely useful!

**Corollary** If $A < \infty$ and $0 \leq \theta < 1$ are such that $P_B(1_{k \in \mathbb{Z}_n}) \leq A n^{-\theta}$ \(\forall n \geq 1\) and $u \in V$ then

$$M_B(\Delta) \geq C_6 A^{1/4+\theta} |\Delta|^{1/4+\theta}$$

and

$$\frac{1}{|\Delta|} \sum_{v \in \Delta} P_B(u \leftrightarrow v) \leq C_6 A^{2/4+\theta} |\Delta|^{-2/4+\theta}$$

\(\forall \Delta \subseteq V\) finite and $u \in V$.

- Can be used to give slick new proofs of some classical things. E.g.,

  max cluster size of critical Erdős-Rényi $\leq n^{2/3}$ whp.

  Bollobás & Łuczak 1980s

  **Why?** Stochastic domination by branching process gives

  $P_B(1_{k \in \mathbb{Z}_n}) \leq A n^{-\theta} \forall n \geq 1$.
Similarly, let us get upper bound for high-dim tar at $p_c$ given $L^d$ results of Hook & Slade.

* Yields the "hyperscaling inequality"

\[ 2 - n \leq d \frac{\sigma - 1}{\sigma + 1} \]

\[ P_{bc}(x \leftrightarrow y) \sim \|y - x\|^{d+2-n} \]

\[ P_{bc}(\|k_0\| = n) \sim n^{-1/\sigma} \]

Believed to be an equality in low dimensions.
Deduction of the corollary from the theorem: Write $M = M_B(\Delta)$. As before,

$$\sum_{u \in A} P(u \leftrightarrow v) = E[|k_{u \otimes D}|] = \sum_{n \geq 1} P(|k_{u \otimes D}| = n)$$

\[
\leq C_0 A \sum n^{-\theta} e^{-\frac{n}{18M}}
\leq C_0 A M^{-\theta} \quad \text{calculus}
\]

On the other hand,

$$\sum_{u,v \in A} P(u \leftrightarrow v) = E[\sum_{u \in A} 1(u \leftrightarrow v)] \geq E[|k_{\max}(D)|]$$

\[
\geq C M^2
\]

Comparing these two inequalities and rearranging yields the claim! \[\square\]
Let's now prove the universal tightness theorem.

Key combinatorial lemma:
Let $G = (V,E)$ be a connected, locally finite graph, and let $A \subseteq V$ be finite, $|A| \leq B$. Then $\exists E_1, E_2 \subseteq E$ disjoint such that
- $E_1, E_2$ both span connected subgraphs of $G$.
- The sets $V(E_i)$ and $V(E_2)$ of vertices incident to the two edge sets satisfy
  \[ \frac{1}{3}|A| \leq |V(E_i) \cap A| \leq \frac{2}{3}|A| \quad i=1,2. \]
Proof. Suffices to consider the case $G$ is a tree, taking a spanning tree otherwise.

We will take $E_1, E_2$ to be a partition of the edge set.

Root $T$ at an element of $A$.

Recursively construct

$\emptyset = E^0 \subseteq E^1 \subseteq \ldots \subseteq E^N$

so that $E^i$ and $E \setminus E^i$ span connected subgraphs $1 \leq i \leq N$.

$V^0 = S$, $V^i =$ unique vertex incident to $E^i$ and $E \setminus E^i$.

- If $V^i$ has exactly one child not incident to $E^i$

$E^{i+1} = E^i \cup \{\text{edge connecting } V^i \text{ to its child}\}$
- Otherwise $v^i$ has at least 2 children not incident to $E_i$.

Pick the child that has the fewest elements of $A$ descended from it.

$E_i^{hi} = E_i^i \cup \{ \text{all edges in subtree with this child} \}$

Stop when every vertex of $A$ touches $E_i^i$.

$V_i^i = V(E_i^i)^{i \downarrow i} \setminus V_o = \emptyset^3$

$\{ 0 \leq n \leq N : |V_n| > \frac{|A|}{3} \}$ contains $N$ but not $0$. 

{$V$}
\[ m = \min \{ n \mid 0 \leq n \leq N : |V_m| > \frac{|A|}{3} \} \]

\[ \frac{|A|}{3} < |V_m \cap A| \leq |V_{m-1} \cap A| + \max \left\{ 1, \frac{1}{2} |A \setminus V_m| \right\} \leq \frac{2}{3} |A| \]

as required \( \square \)

Applying this lemma iteratively, we get:

**Lemma** If \(|A| \geq 3^k\) then there exists \(m = 3^{k-1} + 1\) and a collection of disjoint sets \(E_1, \ldots, E_m\) such that

- \(E_i\) spans a connected subgraph of \(G\) \(\forall 1 \leq i \leq m\),
- \[ \frac{|A|}{3^k} \leq |A \cap V(E_i)| \leq \frac{|A|}{3^{k-1}} \]
  \(\forall 1 \leq i \leq m\)
Given this lemma, the universal tightness theorem follows from the BK inequality:

\[ |k_{\max}(\Delta)| \geq 3^{k} \lambda^{3} \]

\[ \leq \underbrace{|k_{\max}(\Delta)| \geq \lambda^{3} \circ \cdots \circ |k_{\max}(\Delta)| \geq \lambda^{3}}_{3^{k-1} \text{ times}} \]

\[ |k_{\max}(\Delta)| \geq 3^{k} \lambda^{3} \]

\[ \leq \underbrace{|k_{u} \cap \Delta| \geq \lambda^{3} \circ \cdots \circ |k_{\max}(\Delta)| \geq \lambda^{3}}_{3^{k-1} \text{ times}} \]

BK inequality: \( \mathbb{P}(A_{1} \circ \cdots \circ A_{n}) \leq \mathbb{P}(A_{1}) \)

\( A_{i} \) increasing.
The Aizenman-Kesten-Newman method. (If we have extra time)

$W_p$: Bernoulli $p$ bond percolation


On $\mathbb{Z}^d$, $W_p$ has at most one $\infty$ cluster as.

Moreover

$$P_p\left(\bigcap_{x \in \mathbb{Z}^d} \uparrow x \right) \leq C_p, \frac{\log n}{n^d}$$
Simplified presentation by Gandolfi, Grimmett & Russo
Only 4 pages!

\[ \Pr_c(\text{diam } \geq n) \leq n^{-\frac{1}{2}-\varepsilon_3} \]

H. 2018: New version that works for all Cayley graphs, works directly with volumes
\[ S_{\text{e,n}} = \frac{\varepsilon}{2} \text{ e closed, endpoints} \]
are in distinct clusters both of which touch at least \( n \) edges, at least one finite.

Then let \( G \) be a transitive unimodular graph of degree \( d \). Then

\[
P_p(S_{e,n}) \leq 42 d \left[ \frac{1-p}{p^n} \right]^{1/2}
\]

\( \forall e \in E, \; n \geq 1, \; p \in [0,1] \).

\( G \) has ghost field independent of \( \omega_p \), which contains each edge w/ prob \( 1-e^{-h} \).

\( T_e = \{ e \in \text{closed, endpoints of } e \in \text{distinct clusters} \} \).
(each of which touches a green edge and at least one of which is finite.

$n$ uniform random edge incident to $0$

\[
\text{Thm'}: \quad P_{\bar{P}} (T_2) \leq 21 \left[ \frac{1-P}{P} \right]^{1/2}
\]

Sketch of proof:

$F_e$: both clusters touching $e$ are finite

\[
P(F_e \cap T_e) = E \# \{ \text{finite clusters touching } e \text{ & the ghost} \} 1(\text{e closed})
\]
\[ -1 ( \exists \text{ finite cluster touching } e \text{ } \& \text{ the ghost } ) \]

\[ \text{IP}( \exists \text{ finite cluster touching } e \text{ } \& \text{ the ghost } ) \cap (e \text{ closed}) \]

\[ \text{IP}(\exists \text{ finite cluster touching } e \text{ } \& \text{ the ghost } ) \cap (e \text{ closed}) \]

\[ = \text{IP}(T_e \setminus F_e) \]

\[ + \text{IP}(F_e \cap \exists \text{ at least one cluster touching } e \text{ touches ghost } ) \cap (e \text{ closed}) \]

\[ \text{independent!} \]

\[ \text{Algebra} \implies \]

\[ \text{IP}(T_e) = \]

\[ - \]
\[ \mathbb{E} \left[ (1 - \text{closed}) - \frac{1}{p} \cdot 1 - \text{open} \right] \]

\# \{ \text{finite clusters touching} \} \\
\text{e} \& \text{ghost}

\text{Mass-transport} \quad \rightarrow

\mathbb{P}(T_n) = \mathbb{E} \left[ \sum_{k \in \mathbb{K}} \frac{h_p(k)}{|K|} \right]

\kappa \text{ a finite cluster touching } \mathbb{N} \text{ and ghost.}

\sum 1 - \text{closed} \\
- \frac{1}{p} \cdot 1 - \text{open}

\mathbb{P}(T_n) \leq 2 \mathbb{E} \left[ \frac{h_p(k_0)}{|K|} \right] (1 - e^{-\frac{c}{|k_0|}})
edge volume.

\[ = 2 \mathbb{E} \left| Z_{T-1} \right| \frac{1 - e^{-hT}}{T} 1(T < \infty) \]

for some martingale \( Z \) and stopping time \( T \).

Finish w/ martingale analysis. \( \square \)