Minimum Weight Disk
Triangulations & Fillings

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**Abstract:**
Downward-closed $Y \subseteq 2^V$

\[
\emptyset, 1, 2, 3, 4, 5, \\
12, 13, 23, 14, 34, 45, \\
123
\]

**Example:**

\[
\{123, 234, 345, 145, 125\}
\]

**Geometric Realization:**
Clump of simplices

Diagram of simplicial complex.
Simplicial complexes:

Abstract:
Downward-Closed $Y \subseteq 2^V$

\[ \emptyset, 1, 2, 3, 4, 5, \]
\[ 12, 13, 23, 14, 15, 1 \]
\[ 23 \]

Geometric Realization:
Gluing of simplices

Random Simplicial 2-Complexes
Generalizing $G(n,p)$

$Y_2(n,p)$ [Linial-Meshulam]

(*) $n$ vertices $\{1, \ldots, n\}$

(*) All $\binom{n}{2}$ edges

(*) Each of $\binom{n}{3}$ 2-dimensional faces appears ind. with probability $p = p(n)$. 

\[ n = 4 \]
Homological Connectivity

Theorem [Linial–Meshulam]: \[ \forall \varepsilon > 0, \quad \mathbb{P}(\overline{H_1(Y_2(n,p) \cup F_2)} = 0) \xrightarrow{n \to \infty} \begin{cases} 0 & p < (1 - \varepsilon) \cdot \frac{2 \log n}{n} \\ 1 & p > (1 + \varepsilon) \frac{2 \log n}{n} \end{cases}. \]

(*) Every cycle is a linear combination of boundaries of the 2-faces of \( Y_2(n,p) \).

- Linial–Meshulam Theorem was extended to:
  - Other rings of coefficients
  - Higher dimensional complexes
  - Hitting–time results

Simple Connectivity

**Theorem [Babson-Hoffman-Kahle]:** \( \forall \epsilon > 0 \)

\[
P\left( Y_2(n,p) \text{ is simply-connected} \right) \to 0 \quad \text{if} \quad p < n\frac{1}{2} - \epsilon
\]

(*) *Every cycle is null-homotopic:*

\( \boxempty \) **\( \Delta \)-ion of the disk:**

(-) Vertices labelled in \( \{1, \ldots, n\} \)
(-) Boundary is given cycle.
(-) All faces from \( Y_2(n,p) \)

**Open:** Improve this bound
**Simple Connectivity**

**Theorem [Babson - Hoffman - Kahle]:** \( \forall e > 0 \) 
\[
P\left( Y_2(n, p) \text{ is simply-connected} \right) \to 0 \quad \text{if} \quad p < n^{-\frac{1}{2} - e},
\]

(*) Every cycle is null-homotopic:

- A region of the disk:
  - Vertices labelled in \( \{1, \ldots, n\} \)
  - Boundary is given cycle.
  - All faces from \( Y_2(n, p) \)

Vertex labels in a null-homotopy may repeat.

**Diagrams:**

- Dunce Hat
- Tweak of \( RP^2 \)
Simple Connectivity: Upper Bounds

Upper Bounds: BHTK, Günter-Wagner, Komárd-P-Sudakov

**Theorem [Luria - P.]:** Let \( \delta = \frac{4y}{\beta^2} \), \( \forall \varepsilon > 0 \)

\[ p = (1 + \varepsilon)(\delta n)^{-\frac{1}{2}} \Rightarrow \text{w.h.p. every cycle of length 3 is null-homotopic in } Y_2(n, p) \]

by a proper \( \Delta \)-ion

\( \Rightarrow Y_2(n, p) \) is w.h.p. simply-connected

**Def:** A labeled \( \Delta \)-ion is called proper if the vertex labels are distinct.

**Theorem [Tutte]:** # rooted triangulations with labeled \( K \) internal

\[ \text{planar} \]

\[ \text{unlabeled} \]

\[ \text{vertices} \]

\[ = (C + o(1)) K^{-\frac{5}{2}} \gamma^K \]

**Conjecture:** \((\delta n)^{\frac{1}{2}}\) is the sharp threshold prob. for simple-connectivity.
\[ \frac{2 \log n}{n} \quad \frac{\gamma_2(n, p)}{Y_2(n, p)} \quad \frac{n^{-\frac{1}{2}} = \varepsilon}{(\sigma n)^{-\frac{1}{2}}} \quad p \]

- Homological Connectivity
- Simple Connectivity
- ? proper triangulations
Homological Connectivity

Simple Connectivity = proper triangulations

Minimum Weight Triangulations

Assign independent $\text{Exp}(1)$ weights to the $\binom{n}{3}$ 2-dimensional simplices on $\{1, \ldots, n\}$.

What is the total minimum weight of a triangulation whose boundary is a fixed cycle $\Delta_3$?

If $F_2$-homological filling? Null-homotopy? Proper?

$F_n \leq W_n \leq D_n$

Bounds from $Y_2(n,p)$: $n^{-\frac{1}{2}} \leq W_n \leq D_n = O\left(\frac{\log n}{n}\right)$
Minimum Weight Triangulations

What is the total minimum weight of a triangulation whose boundary is fixed cycle \( \begin{array}{c} \hline \end{array} \)?

\( F_2 \)-homological filling? Null-homotopy? Proper?

\[ F_n \leq W_n \leq D_n \]

**Theorem [Benjamini - Lubetzky - P.]:**

1. \( D_n = (\gamma n)^{\frac{1}{2}} \left( \frac{\log(n)}{2} + \frac{5 \log \log(n)}{2} + O_p(1) \right) \).

2. WHP \( F_n = W_n = D_n \)
Ideas from the Proof

1) To find $D_n$: 1st + 2nd moment:
(1st):
$$E\left[\#(\text{proper d-ions of total weight})\leq (\frac{\log n}{2})^2 + \frac{3\log\log n + A}{2}\right] \rightarrow c' \cdot e^{\frac{A}{n^2}}$$

(2nd): Consider $D$-ions with $kn = \Theta(n)$ internal vertices and throw out:
- Triangulation with $P = \Theta(1)$ vertices
- Inner triangulation of a face with $k-n$ vertices

2) To show $F_n = D_n$:
- Every $F_2$-homological filling contains a surface
- 1st moment on # surfaces below given weight $=$ genus expansion $=$