On groups of finite upper rank

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Rank and upper rank

For a finite group G with Sylow p-subgroup P the *rank* and the *p-rank of* G are defined by

$$\mathrm{r}(G) = \sup\{\mathrm{d}(H) \mid H \leq G\},$$
$$\mathrm{r}_p(G) = \mathrm{r}(P),$$

where as usual $\mathrm{d}(H)$ denotes the minimal size of a generating set for H. When G is an arbitrary group, $\mathcal{F}(G)$ denotes the set of finite quotient groups of G, and we define the ('local' and 'global') *upper ranks* of G:

$$\mathrm{ur}_p(G) = \sup\{\mathrm{r}_p(Q) \mid Q \in \mathcal{F}(G)\}$$
$$\mathrm{ur}(G) = \sup\{\mathrm{r}(Q) \mid Q \in \mathcal{F}(G)\}.$$

A theorem of Lucchini [L], first proved for soluble groups by Kovács [K], asserts that for a finite group G,

$$\sup_p \mathrm{r}_p(G) \leq \mathrm{r}(G) \leq 1 + \sup_p \mathrm{r}_p(G),$$

so the analogue holds for the upper ranks of an infinite group; in particular, $\mathrm{ur}(G)$ is finite if and only if the local upper ranks $\mathrm{ur}_p(G)$ are bounded as p ranges over all primes.

Let us denote by $\mathcal{U}$ *the class of all groups* G *such that* $\mathrm{ur}_p(G)$ *is finite for every prime* p. One can describe $\mathcal{U}$ more colourfully as *the class of groups whose profinite completion has a p-adic analytic Sylow pro-p subgroup for every prime* p [DDMS].

Background

More than 20 years ago, Alex Lubotzky conjectured that there is a 'subgroup growth gap' for finitely generated soluble groups. We had recently established that a finitely generated (f.g.) residually finite group has polynomial subgroup growth if and only if it is virtually a soluble minimax group (see [LMS] or [LS], Chapter 5). I showed in [S3] that there exist f.g. groups of arbitrarily slow non-polynomial subgroup growth; the Lubotzky question amounts to: do there exist such groups that are *soluble*?

Now if a f.g. soluble group G has subgroup growth of type strictly less than $n^{\log n/(\log\log n)^2}$ then $\mathrm{ur}_p(G)$ is finite for every prime p ([MS], Prop. 2.6, [S2], Proposition C). On the other hand, it is known that a finitely generated residually finite group has finite upper rank if and only if it is virtually a soluble minimax group [MS1]. So Lubotzky's conjecture would follow from

Conjecture A [S2] *Let G be a f.g. soluble group. If $G \in \mathcal{U}$ then G has finite upper rank.*

Equivalently: if the upper p-ranks of G are all *finite*, then they are *bounded.* If G is assumed to be residually finite, this conclusion is equivalent to saying that G is a minimax group.

In fact, Conjecture A would imply that a f.g. soluble group canot have subgroup growth of type strictly between polynomial and $n^{\log n}$ ([S5], Proposition 5.1).

I am now doubtful about this conjecture, having spent over two decades failing to prove it. What follows is a survey of what is known on the topic.

Olshanski-Osin groups

In [MS] we raised the question: is Conjecture A true even without the solubility hypothesis? If G is a group with $\mathrm{ur}_2(G)$ finite then G has a subgroup H of finite index such that every finite quotient of H is soluble ([LS], Theorem 5.5.1). This (at first sight surprising) consequence of the Odd Order Theorem suggests that the solubility hypothesis in Conjecture A may be redundant. Without that hypothesis, however, the conjecture is *false*, as was recently pointed out to me by Denis Osin. I am very grateful to him for allowing me to reproduce his argument here. It depends on

Theorem 1 ([OO] Theorem 1.2) *Let $P = (p_i)$ be an infinite sequence of primes. There exists an infinite 2-generator periodic group $G(P) = G_0$ having a descending chain of normal subgroups $(G_i)_{i\geq 0}$ with $\bigcap G_i = 1$ such that G_{i-1}/G_i is abelian of exponent dividing p_i for each $i \geq 1$.*

Now let $G = G(P)$ where P consists of distinct primes. Each quotient G/G_n is finite. Given $m \in \mathbb{N}$ there exists n such that $p_i \nmid m$ for all $i \geq n$. It is easy to see that each element of G_n has order coprime to m, whence $G_n \leq G^m$. It follows that for each prime p,

$$\begin{aligned}\mathrm{ur}_p(G) &= \sup\{\mathrm{ur}_p(G/G^m) \mid m \in \mathbb{N}\} \\ &= \sup\{\mathrm{r}_p(G/G_n) \mid n \in \mathbb{N}\} = \left\{ \begin{array}{c} \mathrm{r}_p(G/G_k) \text{ if } p = p_k \\ 0 \text{ if } p \neq p_i\ \forall i \end{array} \right\} < \infty.\end{aligned}$$

Thus $G \in \mathcal{U}$. On the other hand, G is residually finite and not virtually soluble (as it is infinite, f.g. and periodic), and so G has infinite upper rank by the theorem from [MS1] quoted above.

Whether Conjecture A holds with 'soluble' replaced by 'torsion-free' is still an open problem.

The groups of slow subgroup growth constructed in [S3] are built out of finite simple groups. The groups $G(P)$, in contrast, have all their finite quotients soluble: I call such groups of *prosoluble type* (because their profinite completions are prosoluble). As far as I know, these provide the first such examples with arbitrarily slow non-polynomial subgroup growth; they show that Lubotzky's conjecture becomes false if 'soluble' is replaced by 'of prosoluble type':

Proposition 2 *let $f : \mathbb{N} \to \mathbb{R}_{>0}$ be an unbounded non-decreasing function. Then there exists a sequence P of primes such that the group $G = G(P)$ satisfies*

$$s_n(G) \leq n^{f(n)}$$

for all large n, but G does not have polynomial subgroup growth.

Here, $s_n(G)$ denotes the number of subgroups of index at most n in G.

Proof. Suppose that $P = (p_i)$ is a strictly increasing sequence of primes. Let H be a proper subgroup of index $\leq n$ in G. Then $G_0 > H \geq G^{n!} \geq G_k$ for some k. Let k be minimal such. Then $G_k \leq H \cap G_{k-1} < G_{k-1}$, so

$$p_k \mid |G_{k-1} : H \cap G_{k-1}| \leq n.$$

It follows that

$$s_n(G) = s_n(G/G_{k(n)})$$

where $k(n)$ is the largest k such that $p_k \leq n$.

Put $Q_n = G/G_{k(n)}$. According to [LS], Corollary 1.7.2,

$$s_n(Q_n) \leq n^{2+r(n)}$$

where $r(n) = \max_p \mathrm{r}_p(Q_n)$. Write $m_j = |G : G_{j-1}|$ for each $j \geq 1$. Since G is a 2-generator group, G_{j-1} can be generated by $1 + m_j$ elements, and so $\mathrm{r}_{p_j}(Q_n) \leq 1 + m_j$ for $j \leq k(n)$, while $\mathrm{r}_p(Q_n) = 0$ if $p \notin \{p_1, \ldots, p_{k(n)}\}$.

Now we can choose the sequence P recursively as follows: p_1 is arbitrary. Set $\mu_1 = 1$. Given p_i and μ_i for $i \leq t$, set

$$\mu_{t+1} = \mu_t \cdot p_t^{1+\mu_t}$$

and let $p_{t+1} > p_t$ be a prime so large that

$$f(p_{t+1}) \geq 3 + \mu_{t+1}.$$

Note that $|G_{j-1} : G_j| \leq p_j^{1+m_j}$ for each j, so $m_{j+1} \leq m_j \cdot p_j^{1+m_j}$. It follows that $m_j \leq \mu_j$ for all j. Then

$$\begin{aligned} r(n) \leq \max\{1 + m_j \mid j \leq k(n)\} &\leq \max\{1 + \mu_j \mid j \leq k(n)\} \\ &= 1 + \mu_{k(n)} \\ &\leq f(p_{k(n)}) - 2 \leq f(n) - 2. \end{aligned}$$

Thus

$$s_n(G) = s_n(Q_n) \leq n^{2+r(n)} \leq n^{f(n)}.$$

Of course, G does not have polynomial subgroup growth because it has infinite upper rank, as observed above. ■

Minimax groups: a reminder

Let us denote by $\mathcal{S}$ the class of all residually finite virtually soluble minimax groups. The following known results will be used without special mention:

- If $G \in \mathcal{S}$ then G is virtually nilpotent-by-abelian.
- If $G \in \mathcal{S}$ then G is virtually residually (finite nilpotent).
- A minimax group is in $\mathcal{S}$ if and only if it is virtually torsion-free.
- The class $\mathcal{S}$ is extension-closed.
- If G is f.g. and virtually residually nilpotent then G is residually finite.
- If G has a nilpotent normal subgroup N such that G/N' is (a) minimax *resp.* (b) of finite upper rank, then G is (a) minimax, *resp.* (b) of finite upper rank.
- Let G be f.g. and residually finite. Then $\mathrm{ur}(G)$ is finite if and only if $G \in \mathcal{S}$.

For most of these, see [LR], Chapter 5 and Chapter 1. The penultimate claim is an easy consequence of [LR], **1.2.11**. The final claim is [MS1], Theorem A.

Some known cases

Proposition 3 *Let G be a f.g. nilpotent-by-polycyclic group. If $G \in \mathcal{U}$ then G is a minimax group.*

Proof. Let N be a nilpotent normal subgroup of G with G/N polycyclic. It will suffice to show that G/N' is minimax, so replacing G by this quotient we may assume that N is abelian. Then N is Noetherian as a G/N-module, so the torsion subgroup T of N has finite exponent, e say. Let σ be the set of prime divisors of e.

By P Hall's 'generic freeness lemma' (cf. [LR], 7.1.6) N/T has a free abelian subgroup F_1/T such that N/F_1 is a π-group for some finite set of primes π. Then $F_1 = T \times F$ where F is free abelian, and N/F is a $\pi \cup \sigma$-group.

Let $p \notin \pi \cup \sigma$ be a prime. Then $N = FN^p$ and $N^p \cap F = F^p$, so $F/F^p \cong N/N^p$. Now G/N^p is residually finite and the image of N/N^p in any finite quotient of G/N^p has rank at most $\mathrm{ur}_p(G) = r_p$; it follows that $|F/F^p| = |N/N^p| \leq p^{r_p}$. Therefore F has rank at most r_p. Hence for each prime $q \notin \pi \cup \sigma$ we have

$$\mathrm{ur}_q(G) \leq \mathrm{ur}(G/N) + \mathrm{ur}(F) \leq \mathrm{ur}(G/N) + r_p.$$

It follows that $\mathrm{ur}(G)$ is finite, since G/N is polycyclic and $\pi \cup \sigma$ is finite.

As G is residually finite it follows that G is a minimax group. (For the quoted properties of f.g. abelian-by-polycyclic groups, see for example [LR], Chapters 4 and 7.) ■

The upper p-rank of a group G can equivalently be defined as *the rank of a Sylow pro-p subgroup P of $\widehat{G}$*, the profinite completion of G, where for a profinite group P, the rank of P is

$$\mathrm{r}(P) = \sup\{\mathrm{r}(P/N) \mid N \lhd P,\ N \text{ open}\}.$$

The pro-p groups of finite rank are well understood (see [DDMS]); in particular, they are linear groups in characteristic 0.

Proposition 4 ([LS], Window 8, Lemma 9) *Let K be a f.g. residually nilpotent group. Suppose that the pro-p completion $\widehat{K}_p$ of K has finite rank for some prime p. Then there exists a finite set of primes π such that the natural map*

$$K \to \prod\nolimits_{q\in\pi} \widehat{K}_q$$

is injective.

This is the key to

Theorem 5 ([S5], Theorem 5) *Let G be a f.g. group that is virtually residually nilpotent. If $G \in \mathcal{U}$ then G has finite upper rank.*

Proof. It follows from Proposition 4 that G has a subgroup K of finite index such that K is residually (finite nilpotent of rank at most r); here $r = \max_{q\in\pi} \mathrm{ur}_q(G)$. By a result mentioned above, we may also take it that every finite quotient of K is soluble. The main result of [S1] now shows that K is virtually nilpotent-by-abelian (see also [LS], Window 8, Corollary 5), and the result follows by Proposition 3. ■

Let $\mathcal{H}$ denote the class of all groups G with the property: *every virtually residually nilpotent quotient of G is a minimax group.*

Theorem 5 shows that finitely generated groups in $\mathcal{U}$ belong to $\mathcal{H}$. It is *not* true that every f.g. soluble residually finite group in $\mathcal{H}$ has finite upper rank:

Proposition 6 ([PS], Proposition 10.1) *Let p be a prime, let*

$$H = \langle x_n \ (n \in \mathbb{Z}); x_n^p = x_{n-1}\rangle$$

be the additive group of $\mathbb{Z}[1/p]$ written multiplicatively, and let τ be the automorphism of H sending x_n to x_{n+1} for each n. Extend τ to an automorphism of the group algebra $\mathbb{F}_pH$ and then to an automorphism of $W = \mathbb{F}_pH \rtimes H = C_p \wr H$. Set $G = W \rtimes \langle\tau\rangle$. Then

- *G is a 3-generator residually finite abelian-by-minimax group*
- $G \in \mathcal{H}$
- $\mathrm{ur}_q(G) = 2$ *for every prime* $q \neq p$
- $\mathrm{ur}_p(G)$ *is infinite.*

This shows, also, that the hypothesis of Conjecture A can't be weakened by omitting finitely many primes.

Still, the strongest result so far obtained towards Conjecture A rests on a consideration of certain groups in $\mathcal{H}$. It seems clear that the trouble with the last example is due to the presence of 'bad' torsion; if we exclude this we obtain the following:

Theorem 7 ([PS], Theorem 3.2) *Let $G \in \mathcal{H}$ be f.g. and residually finite. Suppose that G has a metabelian normal subgroup N with G/N polycyclic. Then G/N' is minimax. If N' has no π-torsion where $\pi = \mathrm{spec}(G/N')$ then G is minimax.*

(For a minimax group H, $\mathrm{spec}(H)$ denotes the (finite) set of primes p such C_{p^∞} is a section of H.)

From this, it is relatively straightforward to deduce

Theorem 8 (cf. [PS], Corollary 3.3) *Let G be a finitely generated group that is nilpotent-by-abelian-by-polycyclic. If $G \in \mathcal{U}$ then G has finite upper rank.*

Proof. We may assume that G satisfies the hypotheses of Theorem 7. Keeping the notation there, put $A = N'$, an abelian normal subgroup of G. For a prime p and $K \lhd_f G$ let $D_p(K)/(A \cap K)$ be the p'-component of the finite abelian group $A/(A \cap K)$. Then

$$\mathrm{r}_p(G/K) = \mathrm{r}_p(G/KD_p(K)).$$

So if we set

$$D = \bigcap_{\substack{p \notin \pi \\ K \lhd_f G}} KD_p(K),$$

we have $\mathrm{ur}_p(G) = \mathrm{ur}_p(G/D)$ for all $p \notin \pi$.

Now $AD/D \cong A/(A \cap D)$ has no π-torsion, since each $A/(A \cap KD_p(K))$ is a p-group. Clearly G/D is residually finite, so Theorem 7 applies to show that G/D is a minimax group. Hence

$$\mathrm{ur}_p(G) = \mathrm{ur}_p(G/D) \leq \mathrm{ur}(G/D) < \infty$$

for every $p \notin \pi$, and as π is finite it follows that $\mathrm{ur}_p(G)$ is bounded over all primes p. ■

The hypotheses in Theorem 8 seem rather restrictive. However, if we could only replace 'nilpotent-by-abelian' with 'abelian-by-nilpotent' then we could deduce the full force of Conjecture A; this is explained below.

Modules of finite upper rank

Let G be a counterexample to Conjecture A of least possible derived length, l; we may assume that G is residually finite. Let A be maximal among abelian normal subgroups of G that contain $G^{(l-1)}$. Then G/A is residually finite (by an elementary lemma) and has finite upper rank, so G/A is a minimax group. In particular, G/A is virtually nilpotent-by-abelian and so G is abelian-by-nilpotent-by-polycyclic: this is the point of the final remark in the preceding section.

Putting $\Gamma = G/A$ we consider A as a Γ-module, written additively as A_Γ.

If B is a submodule of finite index in A_Γ then G/B is residually finite (because $\mathcal{S}$ is extension-closed), whence

$$\mathrm{r}_p(A/B) \leq \mathrm{ur}_p(G)$$

for each prime p; and it is clear that

$$\mathrm{ur}(G/B) \leq \mathrm{r}(A/B) + \mathrm{ur}(G/A).$$

Let us define the upper rank of a Γ-module M by $\mathrm{ur}(M) = \sup\{\mathrm{r}(M/B) \mid B \leq_\Gamma M,\ M/B \text{ finite}\}$, and set

$$\begin{aligned}\mathrm{ur}_p(M) &= \sup\{\mathrm{r}(M/B) \mid pM \leq B \leq_\Gamma M,\ M/B \text{ finite}\} \\ &= \mathrm{ur}(M/pM).\end{aligned}$$

I will say that M is a *quasi-f.g.* Γ-module if there exists a f.g. group G that is an extension of M by Γ. The preceding observations now show that A_Γ is a counterexample to

Conjecture B *Let Γ be a f.g. residually finite soluble minimax group and let M be a quasi-f.g. Γ-module. If $\mathrm{ur}_p(M)$ is finite for every prime p then M has finite upper rank.*

Conversely, it is easy to see that if M is a counterexample to Conjecture B then the corresponding extension G is a counterexample to Conjecture A. So the two conjectures are equivalent.

Theorem 8 establishes Conjecture B for the special case where Γ is abelian-by-polycyclic. A reduction step in the proof is Proposition 5.2 of [PS], which shows that M contains a finitely generated Γ-submodule B such that the finite module quotients of B are 'nearly all' isomorphic to finite quotients of M, and conversely. The hypothesis that Γ is abelian-by-polycyclic is used in the proof of this reduction, but can be dispensed with; this is explained in the next section. The main part of the proof, however, does depend on Γ having an abelian normal subgroup A such that Γ/A is polycyclic. Following a strategy devised by P. Hall

[H] and further developed by Roseblade [R], one examines the structure of B as a module for the group ring $\mathbb{Z}A$, with Γ/A as a group of operators. The necessary module theory is developed in [S4] and [S2].

For the general case of Conjecture B, it would seem necessary to generalize this machinery in one of two directions: either allow A to be nilpotent (rather than abelian), or allow Γ/A to be minimax (rather than polycyclic – while still assuming $\Gamma/C_\Gamma(A)$ to be polycyclic, if one takes A inside the centre of the Fitting subgroup of Γ). Whether either of these approaches is feasible remains unclear. Machinery relevant to the first approach has been developed by Tushev [T]. A major difficulty with the second approach is the fact that the 'generic freeness' property mentioned above definitely fails when Γ/A is not polycyclic, as observed by Kropholler and Lorensen in [KL], Cor. 5.6. Other aspects of the Hall-Roseblade theory have been usefully generalized by Brookes [B].

On the other hand, if one is seeking a counterexample to conjecture B, the simplest candidate would seem to be the following group: Let K be the Heisenberg group over $\mathbb{Z}[1/2]$ and take $\Gamma = K \rtimes \langle t \rangle$ where t acts on a matrix by doubling the off-diagonal entries (and multiplying the top right corner entry by 4). Then M could be the quotient $\mathbb{Z}\Gamma/J$ where J is a carefully constructed right ideal: generators of J should be chosen to ensure that $\mathbb{Z}\Gamma/J$ has finite upper p-rank for each prime p, but in such a way that these ranks are unbounded.

A possible reduction: quasi-f.g. modules.

Let Γ be a f.g. residually finite soluble minimax group. Then Γ has a nilpotent normal subgroup K such that Γ/K is virtually abelian. We fix a normal subgroup Z of Γ with $Z \leq \mathrm{Z}(K)$, and let $R = \mathbb{Z}Z$ denote its group ring. For a multiplicatively closed subset Λ of R, an R-module M is said to be Λ-*torsion* if every element of M is annihilated by some element of Λ.

Proposition 9 *Let A be a quasi-f.g. Γ-module. Then A has a finitely generated Γ-submodule B such that A/B is Λ-torsion for each Λ of the form $R \smallsetminus L$ where L is a maximal ideal of finite index in R not containing the augmentation ideal $(Z-1)R$.*

Before giving the proof we note a corollary. For a Γ-module M, let $\mathcal{F}(M)$ denote the set of isomorphism types of finite quotient Γ-modules of M.

Corollary 10 *For A and B as above, we have*

$$\mathcal{F}(A) \smallsetminus \mathcal{S} = \mathcal{F}(B) \smallsetminus \mathcal{S}$$

where $\mathcal{S}$ consists of the finite Γ-modules that have a composition factor on which Z acts trivially.

This is essentially a formal consequence of the stated condition on Λ-torsion, which implies that

$$AJ + B = A, \quad AJ \cap B = BJ$$

whenever J is the annihilator in R of some finite Γ-module not in $\mathcal{S}$. Thus questions about the upper ranks of A might be reduced to questions about the upper ranks of the finitely generated module B, if - by some subsidiary argument - one could leave aside the quotients lying in $\mathcal{S}$ (this is in principle the approach taken in [PS], §§5, 6).

To establish the Proposition, we consider a f.g. group E with an abelian normal subgroup A such that $E/A = \Gamma$. In E there is a series of normal subgroups

$$E \rhd K_1 \geq Z_1 \geq A \geq \gamma_{c+1}(K_1)[Z_1, K_1]$$

where $K_1/A = K$ is nilpotent of class c, say, and $Z_1/A = Z$. Now Z is an abelian minimax group, hence contains a finite subset Y_1 such that $Z/\langle Y_1 \rangle$ is divisible. Since $E/K_1 \cong \Gamma/K$ is virtually abelian and E is f.g., K_1 is finitely generated as a normal subgroup of E; we choose a finite set $X = X^{-1}$ of normal generators for K_1 and assume that X contains a set Y of representatives for the elements of Y_1. Finally, let $S = S^{-1}$ be a finite set of generators for E.

Lemma 11 *Let L be a maximal ideal of finite index in $R = \mathbb{Z}Z$ not containing $Z - 1$. Then $\Lambda = R \setminus L$ satisfies*

$$(\Lambda^g + 1) \cap Y_1 \neq \varnothing \tag{1}$$

for every $g \in E$.

Proof. Write $D = Z \cap (L + 1)$. If (1) fails for g then $D^g \supseteq Y_1$ which implies $D^g = Z$ since $Z/\langle Y_1 \rangle$ is divisible while $|Z : D^g|$ is finite. Hence $D = Z$ and so $L \supseteq Z - 1$. ∎

Now we define B to be the E-submodule of A generated by the finite set

$$\{[x, y],\ [x^s, y] \mid x \in X,\ y \in Y,\ s \in S\}.$$

We aim to show that if Λ is a multiplicatively closed subset of R satisfying (1) for every $g \in E$, then the R-module A/B is Λ-torsion; with Lemma 11 this will complete the proof of Proposition 9.

Note that $\gamma_{i+1}(K_1)$ is generated by the elements $v_i(\mathbf{x}, \mathbf{w})^g$ for $g \in E$ and

$$v_i(\mathbf{x}, \mathbf{w}) = [x_0, x_1^{w_1}, \ldots, x_i^{w_i}],$$

$x_j \in X,\ w_j \in E$. Put

$$A_i = \langle [v_i(\mathbf{x}, \mathbf{w}), z]^g \mid x_j \in X,\ z \in Y,\ g, w_j \in E \rangle,$$
$$B_i = \langle [x^v, y]^g \mid x \in X,\ y \in Y,\ g, v \in E,\ l(v) \leq i \rangle$$

where $l(v)$ denotes the least n such that $v = s_1 \ldots s_n$ ($s_j \in S$). Note that $B_1 = B$.

Claim: A/A_c is Λ-torsion.

To see this, choose $y \in Y$ with $\overline{y} - 1 \in \Lambda$ where $\overline{y} = Ay$. Then (mixing additive and multiplicative notation)

$$A(\overline{y} - 1)^c = [A,_c y] \subseteq \gamma_{c+1}(K_1) \subseteq A.$$

Given a generator $v_c(\mathbf{x}, \mathbf{w})^g$ of $\gamma_{c+1}(K_1)$, choose $z \in Y$ such that $\overline{z}^g - 1 \in \Lambda$. Then

$$v_c(\mathbf{x}, \mathbf{w})^g(\overline{z}^g - 1) = [v_c(\mathbf{x}, \mathbf{w}), z]^g \in A_c.$$

Claim: For $i > 1$, B_i/B_{i-1} is Λ-torsion.

To see this, say $b = [x^{\gamma u}, y]^g$ is a generator of B_i where $l(u) \leq i - 1$. Choose $z \in Y$ such that $\overline{z}^{ug} - 1 \in \Lambda$. Then

$$\begin{aligned}(b(\overline{z}^{ug} - 1))^{-g^{-1}y} &= [x^{\gamma u}, y, z^u]^{-y^{-1}} \\ &= [z^u, x^{-\gamma u}, y^{-1}]^{x^{\gamma u}} + [y^{-1}, z^{-u}, x^{\gamma u}]^{z^u}.\end{aligned}$$

The first term lies in $B_1 \leq B_{i-1}$ and the second term lies in B_{i-1}. The claim follows since B_{i-1} is E-invariant.

Claim: Write $B_\infty = \bigcup_j B_j$. Then for $i > 1$, $A_i \subseteq B_\infty + A_{i-1}$.

To see this, let $x = (\mathbf{x}', x)$ and $\mathbf{w} = (1, w_1, \ldots) = (\mathbf{w}', w)$ be $(i+1)$-tuples in X, E respectively, and let $z \in Y$. Then

$$\begin{aligned}[v_i(\mathbf{x}, \mathbf{w}), z]^{-x^{-w}} &= [v_{i-1}(\mathbf{x}', \mathbf{w}'), x^w, z]^{-x^{-w}} \\ &= [x^w, z^{-1}, v_{i-1}(\mathbf{x}', \mathbf{w}')]^z + [z, v_{i-1}(\mathbf{x}', \mathbf{w}')^{-1}, x^w]^{v_{i-1}(\mathbf{x}', \mathbf{w}')}.\end{aligned}$$

The first term lies in B_∞ and the second term lies in A_{i-1}. The claim follows since each of these modules is E-invariant.

The three claims together now imply that A/B is Λ-torsion, and the proof is complete.

Further reductions

Suppose that the pair (Γ, M) furnishes a counterexample to Conjecture B, where M is finitely generated as a Γ-module. With quite a lot of extra work, generalizing some ideas from [S4], one can establish

Proposition 12 *The module M has a torsion-free residually finite quotient $\widetilde{M}$ of infinite upper rank such that every proper, π-torsion-free residually finite quotient of $\widetilde{M}$ has finite rank, where $\pi = \mathrm{spec}(\Gamma)$.*

(Here $\mathrm{spec}(\Gamma)$ denotes the (finite) set of primes p such that Γ has a section C_{p^∞}.)

This reduces the problem to consideration of a 'minimal counterexample', in a rather weak sense. Whether this is any help is not clear, and there seems little point in including the proof here.

Further results that may be relevant are obtained in [KL1]; these can be used to show that a module like our putative counterexample has many finite-rank quotients that split as direct sums.

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