Ellipticity in profinite groups

If f_w is a homomorphism (e.g. when G is an abelian group) then

$$w(G) = G_w = G_w^{*1}.$$

In general,

$$w(G) = \bigcup_{n=1}^{\infty} G_w^{*n}, \tag{1}$$

w has *width* m in G if $w(G) = G_w^{*m}$

$m_G(w)$ denotes the least finite m such that w has width m in G;
$m_G(w) = \infty$ if there is no such m

G is w-*elliptic* if $m_G(w)$ is finite
G is *verbally elliptic* if it is w-*elliptic* for every word w.

Theorem 1 (P. Stroud) *Every finitely generated abelian-by-nilpotent group is verbally elliptic.*

Theorem 2 (DS) *Every virtually soluble minimax group is verbally elliptic.*

Theorem 3 (P. Stroud) *The free two-generator* $\mathfrak{N}_2$*-by-abelian group is* not *verbally elliptic.*

Theorem 4 (A. Rhemtulla) *Non-abelian free groups are* not w*-elliptic for* any *non-trivial, non-universal word* w.

Important earlier contributions were made by *V. A. Romankov* and *K. George*. I will not discuss these results here; they are expounded at length in my book. They exemplify the general principle that verbal ellipticity is some sort of weak commutativity condition.

G profinite: then

$$G = \varprojlim \mathcal{F}(G)$$

where

$$\mathcal{F}(G) = \{G/N \mid N \lhd_o G\}$$

$N \lhd_o G$ means 'N is an open normal subgroup of G'.

Properties of the *topological* group G reflect *uniform* properties of the family $\mathcal{F}(G)$ of finite groups. In particular:

1. G is (topologically) finitely generated iff $\mathrm{d}(G)$ is finite, and

$$\mathrm{d}(G) = \sup\{\mathrm{d}(Q) \mid Q \in \mathcal{F}(G)\}.$$

$\mathrm{d}(G)$: the minimal size of a *topological* generating set for G

2. Let w be a word. Then G is w-elliptic iff $m_G(w)$ is finite, and

$$m_G(w) = \sup\{m_Q(w) \mid Q \in \mathcal{F}(G)\}.$$

($m_G(w)$ and $w(G)$ are understood *algebraically*)

Proposition 1 (B. Hartley) *Let G be a profinite group and w a word. Then G is w-elliptic if and only if $w(G)$ is closed.*

Proof. Suppose $m_G(w) = m$ is finite. Then

$$\begin{aligned} w(G) &= G_w^{*m} \\ &= \bigcup G_{+w}^{\varepsilon(1)} \ldots G_{+w}^{\varepsilon(m)} \end{aligned}$$

where $(\varepsilon(1), \ldots, \varepsilon(m))$ ranges over $(\pm 1)^{(m)}$. For each such tuple, the set $G_{+w}^{\varepsilon(1)} \ldots G_{+w}^{\varepsilon(m)}$ is the image of

$$f_w^{\varepsilon} : G^{(k)} \to G$$
$$(\mathbf{g}_1, \ldots, \mathbf{g}_m) \mapsto w(\mathbf{g}_1)^{\varepsilon(1)} \ldots w(\mathbf{g}_m)^{\varepsilon(m)}.$$

This map is continuous, so its image is compact, hence closed in G. Therefore $w(G)$ is closed.

The converse is a little more subtle: it is an application of the Baire category theorem to the expression (1).

Definition. The word w is *elliptic* in a *family* of groups $\mathcal{C}$ if

$$\exists m \in \mathbb{N} \text{ such that } m_G(w) \leq m \quad \forall G \in \mathcal{C}.$$

A profinite group G is a *pro-$\mathcal{C}$ group* if $\mathcal{F}(G) \subseteq \mathcal{C}$. Now combining **1.**, **2.** and Proposition 1 we get

Proposition 2 *A word w is elliptic in a family of finite groups $\mathcal{C}$ if and only if $w(G)$ is closed in G for every pro-$\mathcal{C}$ group G.*

A little history.

Lemma 1 *If* $G = \langle g_1, \ldots, g_d \rangle$ *is nilpotent then the derived group* G' *satisfies*

$$G' = [G, g_1] \ldots [G, g_d].$$

This shows that the word $[x, y]$ is elliptic in the class $\mathfrak{N} \cap \mathfrak{G}_d$ of d-generator nilpotent groups. It also implies that the word

$$v(x, y, z) = [x, y]z^p$$

(here p is any prime) is elliptic in $\mathfrak{N} \cap \mathfrak{G}_d$ since

$$v(G) = G'G^p = G' \cdot \{g^p \mid g \in G\}.$$

So Proposition 2 gives

Corollary 1 *If* G *is a finitely generated pro-*p *group then the subgroups* G' *and* $G'G^p$ *are closed.*

Any d-generator group H with $v(H) = 1$ satisfies $|H| \leq p^d$.

Definition. The word w is *d-locally finite* if

$$\beta(w) := |F_d/w(F_d)| < \infty.$$

Lemma 2 *Let G be a d-generator profinite group and w a d-locally finite word. If $w(G)$ is closed then $w(G)$ is open.*

Proof. If $w(G) \leq N \lhd_o G$ then G/N is a finite d-generator group killed by w, so $|G/N| \leq \beta(w)$. Hence there is a smallest such N, call it M, and as $w(G)$ is closed we have

$$w(G) = \bigcap_{w(G) \leq N \lhd_o G} N = M.$$

Remark (for later use): it's enough to assume that there is a finite upper bound $\overline{\beta}(d, w)$ for the size of each *finite* quotient of $F_d/w(F_d)$.

Anyway, we can now deduce:

- *if G is a finitely generated pro-p group then $G'G^p$ is open in G.*

This implies that $G'G^p$ has finite index in G, hence $G'G^p$ is again a finitely generated pro-p group.
Now let K be any normal subgroup of finite index in G.
Exercise: G/K is necessarily a p-group.

Let $\Phi(G) = G'G^p$. Then

$$\Phi^n(G) \leq K$$

for some n. As $\Phi^n(G)$ is open in G, K is also open. As every subgroup of finite index contains a normal subgroup of finite index, we have proved

Theorem 5 (Serre) *In a finitely generated pro-p group every subgroup of finite index is open.*

Serre's question (1975): *is the same true for finitely generated profinite groups in general*?

Theorem 6 (Nikolov/Segal) *In a finitely generated profinite group every subgroup of finite index is open.*

Hence

- $\{\text{finite-index subgroups}\} = \{\text{open subgroups}\}$
  $=$ base for the neighbourhoods of 1

whence

Corollary 2 *In a finitely generated profinite group the topology is uniquely determined by the group structure.*

More generally:

- *Every group homomorphism from a finitely generated profinite group to any profinite group is continuous.*

Lemma 3 *Let $d \in \mathbb{N}$ and let H be a finite group. Then there exists a d-locally finite word w such that $w(H) = 1$.*

An algebraic theorem:

Theorem 7 (Nik/Seg) *Every d-locally finite word is elliptic in the class of d-generator finite groups.*

Proof of Theorem 6. Let G be a d-generator profinite group and K a normal subgroup of finite index in G. There exists a d-locally finite word w such that $w(G/K) = 1$. Theorem 7 implies that $w(G)$ is closed, and then Lemma 2 shows that $w(G)$ is open. As $K \geq w(G)$ it follows that K is open.

In the same work we established a companion result:

Theorem 8 (Nik/Seg) *Let G be a profinite group and H a closed normal subgroup of G. Then the subgroup*

$$[G, H] = \langle [g, h] \mid g \in G,\ h \in H \rangle$$

is closed in G.

Starting with $H = G$ and arguing by induction we infer that each term $\gamma_n(G)$ of the lower central series is closed, and hence get

Corollary 3 *For each d and n the word $\gamma_n = [x_1, \ldots, x_n]$ is elliptic in the class of d-generator finite groups.*

These results suggest the

Definition. The word w is *uniformly elliptic* in a class $\mathcal{C}$ if for each $d \in \mathbb{N}$ there exists $f(w,d) \in \mathbb{N}$ such that

$$m_G(w) \leq f(w, (\mathrm{d}(G))$$

for every finite group $G \in \mathcal{C}$.

As we have seen,

- *w is uniformly elliptic in $\mathcal{C}$ iff $w(G)$ is closed in G for every finitely generated pro-$\mathcal{C}$ group G.*

Are all words are uniformly elliptic? No!

V. A. Romankov: G'' is *not closed* in the free 3-generator pro-p group.

So the word $\delta_2 = [[x_1, x_2], [x_3, x_4]]$ is *not* uniformly elliptic in p-groups.

Identify words with elements of the free group

$$F = F(x_1, x_2, \ldots)$$

on countably many generators.

Theorem 9 (Jaikin-Zapirain) *Let p be a prime and let $1 \neq w \in F$. The following are equivalent:*

(a) $w(G)$ *is closed in G for every finitely generated pro-p group G;*

(b) $w \notin F''(F')^p$;

(c) $G/\overline{w(G)}$ *is virtually nilpotent for every finitely generated pro-p group G.*

Definition. $w \in F$ is a *J-word* if $w \notin F''(F')^p$ for every prime p, or if $w = 1$.

This is a *necessary* condition for w to be uniformly elliptic in all finite groups. Is it *sufficient*? I don't know.

Definition. $\mathcal{Y}$ denotes the *smallest locally-closed and residually-closed class of groups that contains all virtually soluble groups*.

Theorem 10 (i) *Let G be a finitely generated profinite group and w a J-word. If $G/w(G) \in \mathcal{Y}$ then $w(G)$ is closed in G.*

(ii) *Let E be the free profinite group on $d \geq 2$ generators. If $w(E)$ is closed in E then w is a J-word, and $G/w(G)$ is virtually nilpotent for every d-generator profinite group G.*

(iii) *Let w be a word such that $F/w(F) \in \mathcal{Y}$. Then w is uniformly elliptic in all finite groups if and only if w is a J-word.*

Note: $F/w(F) \in \mathcal{Y}$ if and only if the variety defined by w is generated by $\mathcal{Y}$-groups, for example by finite groups or by soluble groups.

Proof also uses results of Burns, Macedońska and Medvedev about group varieties.

Problem. (Well known!) Let $w = [x, y, \ldots, y]$ be an Engel word. Is $F/w(F)$ residually finite?

If the answer is 'yes', then w is uniformly elliptic.

If it is 'no', we have an example of a J-word that is not uniformly elliptic.

Burnside words

x^q is not locally finite if q is large – negative solution of the Burnside Problem.

Positive solution of the Restricted Burnside Problem (RBP): this fact is invisible in the universe of finite groups!

But: there is a subtle way in which the infinitude of a Burnside group might manifest itself within finite group theory. See next lecture.

Theorem 11 *For each $q \in \mathbb{N}$ the Burnside word x^q is uniformly elliptic in all finite groups.*

This is the same as saying that the power subgroup G^q is closed in every finitely generated profinite group G. With the positive solution of RBP and a remark above, this in turn is equivalent to

Corollary 4 *In a finitely generated profinite group G the power subgroups G^q are open.*

In turn, this now implies

Theorem 12 *Every non-commutator word is uniformly elliptic in all finite groups.*

Proof. A non-commutator word takes the form

$$w = x_1^{e_1} \dots x_k^{e_k} u$$

where $u \in F'$ and at least one of the e_i is non-zero. Say $|e_i| = q > 0$. Let G be a finitely generated profinite group. Then $w(G) \geq G^q$ which is open, so $w(G)$ is open, hence closed. ∎