Ellipticity in finite groups

Joint work with **Nikolay Nikolov**.

Given a word w and a finite d-generator group G, we want to prove that

$$m_G(w) \leq f = f(w,d).$$

How does this work for $w = [x,y]$ when $G = \langle g_1, \ldots, g_d \rangle$ is nilpotent? Recall Lemma 1 of Lecture 2:

$$G' = [G, g_1] \ldots [G, g_d].$$

Induction on the nilpotency class c.

Let $h \in G'$. Inductively, assume that

$$h = b \cdot [g_1, x_1] \ldots [g_d, x_d],$$

$$x_1, \ldots, x_d \in G$$

$$b \in \gamma_c(G) := K.$$

To kill b: seek $y_i \in G$ such that

$$[g_1, y_1 x_1] \dots [g_d, y_d x_d] = b[g_1, x_1] \dots [g_d, x_d]. \quad (1)$$

Assume $y_i \in \gamma_{c-1}(G) = N$. Then the equation reduces to

$$[g_1, y_1] \dots [g_d, y_d] = b. \quad (2)$$

This can be solved because the mapping

$$\begin{aligned} [\mathbf{g}, -] : N^{(d)} &\to K \\ \mathbf{y} &\mapsto [\mathbf{g}, \mathbf{y}] = [g_1, y_1] \dots [g_d, y_d] \end{aligned}$$

is *surjective*.

Note: $[\mathbf{g}, \mathbf{x}] := v(\mathbf{x})$ is a generalized word over G. Solve $v(\mathbf{x}) = h$ by successive approximation.

Key ingredient (cf. Hensel's Lemma): the 'derived mapping', in this case

$$f_v : N^{(k)} \to K$$

is surjective.

The hypothesis that ensures this: $\{g_1, \dots, g_d\}$ generates G.

In general? **Issues**:

1. How does the induction start?

2. If G is not nilpotent, what should K and N be?

3. If w is some complicated word, the derived mapping may be arbitrarily horrible.

Issue 3. Suppose

$$w(G) = \langle r_1, \ldots, r_\delta \rangle \quad \text{with } r_1, \ldots, r_\delta \in G_w. \quad (3)$$

Let H be a normal subgroup of $w(G)$ such that $w(G)/H$ is nilpotent. Then

$$\begin{aligned} w(G)' &= [w(G), r_1] \ldots [w(G), r_\delta] H \\ &\subseteq G_w^{*2\delta} H \end{aligned}$$

Suppose also that w *is not a commutator word*. Then

$$q = q(w) := |\mathbb{Z}/w(\mathbb{Z})| < \infty, \tag{4}$$

which implies that

$$g^q \in G_w \quad \forall g \in G.$$

Then for each $r \in G_w$ we have $\langle r \rangle \subseteq G_w^{*q}$, whence

$$\begin{aligned} w(G) &= \langle r_1 \rangle \cdots \langle r_\delta \rangle \, w(G)' \\ &\subseteq G_w^{*(\delta q + 2\delta)} H. \end{aligned} \tag{5}$$

Definition. The word w is d-*restricted* if both (4) and (3) hold for every finite d-generator group G, where δ depends only on w and d.

Parenthesis. Let $F = F_d = \langle x_1, \ldots, x_d \rangle$. Suppose that $B(d, q) = F/F^q$ is finite.

Then F^q is finitely generated, say

$$F^q = \langle u_1(\mathbf{x})^q, \ldots, u_\delta(\mathbf{x})^q \rangle .$$

Now if $G = \langle g_1, \ldots, g_d \rangle$ is any finite group and $\pi : F \to G$ sends x_i to g_i then

$$G^q = F^q \pi = \langle u_1(\mathbf{g})^q, \ldots, u_\delta(\mathbf{g})^q \rangle .$$

Conclusion: *if for each $\delta \in \mathbb{N}$ there exists a finite d-generator group G such that G^q is* not *generated by δ q*th *powers then $B(d, q)$ is infinite.*

i.e. if x^q is not d-restricted – detectable in *finite groups* – then $B(d, q)$ is infinite.
This *won't* lead to an alternative to Novikov-Adian: because x^q *is* in fact d-restricted, for every d.

Theorem 1 *Every d-restricted word is elliptic in d-generator finite groups.*

d-locally finite $\Rightarrow$ d-restricted; hence Theorem 7 of lecture 2.

It is *not* strong enough to show that the Burnside words are uniformly elliptic.

Now fix a d-restricted word w and let G be a finite d-generator group. Then

$$W = w(G) = \langle r_1, \ldots, r_\delta \rangle$$

where $r_1, \ldots, r_\delta \in G_w$. Let

$$H = \gamma_c(W) = [H, W]$$

be the nilpotent residual of W.

Henceforth: *forget* about G and w, and concentrate on W and H: need only prove:

- every element of H is a product of boundedly many commutators $[h, r_j]$ with $h \in H$, together with boundedly many qth powers.

Issue 1: *Starting the induction*. What if G happens to be a simple group?

Theorem 2 (Martinez/Zelmanov, Saxl/Wilson) *There exists $m = m(q)$ such that x^q has width m in every finite simple group.*

Using Theorem 2 and the ideas discussed above, we can reduce Theorem 1 to

Theorem 3 *Let* $W = \langle r_1, \ldots, r_\delta \rangle$ *be a finite group and* $H = [H, W]$ *an acceptable normal subgroup of* W*. Let* $q \in \mathbb{N}$*. Then*

$$H = \left(\prod_{i=1}^{\delta} [H, r_i] \right)^{*f_1(\delta,q)} \cdot H_q^{*f_2(q)} \qquad (6)$$

Acceptable is a small technical condition we'll ignore. $H_q = \{h^q \mid h \in H\}$.

Both for the reduction argument mention above, and for the special application to Burnside words, we also need a variant:

Theorem 4 *Let* W *be a finite group and* $H = [H, W]$ *an acceptable normal subgroup of* W*. Suppose that* $\mathrm{d}(W) = \delta$ *and that* $\mathrm{Alt}(n)$ *is not involved in* W*. If* $W = H \langle s_1, \ldots, s_t \rangle$ *then*

$$H = \left(\prod_{i=1}^{t} [H, s_i] \right)^{*f_3(n,\delta)} .$$

The difference in the hypothesis is that we are not given generators for W, only generators for W modulo H. In applications, the s_i will be w-values where w is a word that isn't known to be d-restricted, so that we can't *a priori* find a bounded set of w-values that generate W.

The difference this makes in the proof: W acts on a minimal normal (non-central) subgroup A. When $W = \langle r_1, \ldots, r_\delta \rangle$ it follows that at least one of the the sets $[A, r_j]$ must be quite big. In the other case, we need to know that one of the sets $[A, s_j]$ is quite big; this is proved using special representation-theoretic properties of groups that don't involve $\mathrm{Alt}(n)$.

In particular, we can use this for a Burnside word, and deduce

Proposition 1 *For each $n \in \mathbb{N}$, the word x^q is uniformly elliptic in the class of finite groups that do not involve* $\mathrm{Alt}(n)$.

Issue 2. The proof of Theorem 3 is by induction on $|H|$. Since $H = [H, W]$, we can choose a normal subgroup N of W minimal subject to

$$H \geq N = [N, W].$$

C*ase 1:* N is nilpotent of class at most 2 and exponent prime or 4;

Case 2: N is quasi-semisimple: $N = N'$ and $N/\mathrm{Z}(N)$ is a direct product of non-abelian simple groups.

Assume for simplicity that $\mathrm{Z}(W) = 1$.

Set $K = N$ unless $N > N' > 1$ in which case set $K = N'$.

To prove (6), let $h \in H$. Assume inductively that

$$h = b \cdot \prod_{i=1}^{f_1} \prod_{j=1}^{\delta} [g_{ij}, r_j] \cdot \prod_{j=1}^{f_2} h_j^q \qquad (7)$$

where $h_j, g_{ij} \in H$, $b \in K$ is the 'error term', and the g_{ij} satisfy an *extra* condition

$$\left\langle r_1^{\tau_1(\mathbf{r},\mathbf{g})}, \ldots, r_\delta^{\tau_\delta(\mathbf{r},\mathbf{g})} \right\rangle K = W; \qquad (8)$$

here the τ_j are certain words.

Inductive step: seek $\mathbf{a}_1, \ldots, \mathbf{a}_\delta \in N^{(f_1)}$ and $\mathbf{b} \in N^{(f_2)}$ such that

$$h = \prod_{i=1}^{f_1} \prod_{j=1}^{\delta} [a_{ij} g_{ij}, r_j] \cdot \prod_{j=1}^{f_2} (b_j h_j)^q \qquad (9)$$

and

$$\left\langle r_1^{\tau_1(\mathbf{r},\mathbf{a}.\mathbf{g})}, \ldots, r_\delta^{\tau_\delta(\mathbf{r},\mathbf{a}.\mathbf{g})} \right\rangle = W. \qquad (10)$$

This can be done provided a certain mapping

$$\Phi : N^{(f_1\delta+f_2)} \to K$$

is surjective.

Here Φ is a generalized word over N, got by expanding the commutators and powers in (9) and then cancelling out (7).

The hypothesis (8) is designed precisely to make this happen.

In fact we have to show that (9) has *very many* solutions, so many that some of them are sure to satisfy (10). i.e. *the fibres of* Φ *are sufficiently large*.

The analysis of Φ needs different methods.

Case 1: N *nilpotent*. This comes down to the study of certain quadratic maps over finite fields.

Case 2: N *quasi-semisimple*. Here we study certain equations over a direct product of quasisimple groups.

These are eventually reduced to an equation over *one* quasisimple group:

Theorem 5 *Let $q \in \mathbb{N}$. Suppose that $m \in \mathbb{N}$ and the quasisimple finite group S are sufficiently large. Let $q_1, \ldots, q_m$ be divisors of q and $\beta_1, \ldots, \beta_m$ automorphisms of S. Then there exist inner automorphisms α_i of S such that*

$$S = [S, (\alpha_1\beta_1)^{q_1}] \ldots [S, (\alpha_m\beta_m)^{q_m}].$$

Proof depends on CFSG. When S is of Lie type, it again comes down ultimately to solving equations over finite fields.

One application:

Proposition 2 *Let $q \in \mathbb{N}$. Let N be a quasi-semisimple normal subgroup of a finite group H, and suppose that the simple factors of N are sufficiently large. Then*

$$N \cdot H_q^{*f} = H_q^{*f}$$

where f depends only on q.

As well as being an essential part of the main proof, this can be combined with Proposition 1 to establish

Theorem 6 *The Burnside word x^q is d-restricted for every d.*

Corollary 1 *Every non-commutator word is uniformly elliptic in finite groups.*