Algebraic groups

Fix an algebraically closed field K; by a *variety* I will mean a Zariski-closed subset of $K^{(n)}$ for some n (an 'affine algebraic set').

Topological terms will refer to the Zariski topology.

A *morphism* is a map $f : X \to Y$ between varieties defined by polynomials in the coordinates.

If X is irreducible and Xf is dense in Y one says that f is *dominant*. In this case, Y is also irreducible and Xf contains a dense open subset of Y.

A *linear algebraic group is a* Zariski-closed subgroup of $\mathrm{SL}_n(K)$, for some n

If w is a word and $g_1, \ldots, g_k \in \mathsf{SL}_n(K)$, the entries of the matrix $w(\mathbf{g})$ are given by polynomials in the entries of the matrices g_i; so $f_w : G^{(k)} \to G$ is a morphism.

Theorem 1 (Merzlyakov) *Let G be a linear algebraic group over an algebraically closed field. Then*

- *w has finite width in G,*
- *$w(G)$ is a Zariski-closed subgroup of G;*
- *if G is connected then so is $w(G)$.*

Proof. Assume G *connected* (hence an irreducible variety).

For $j \in \mathbb{N}$ set

$$P_j = (G_{+w} \cdot G_{+w}^{-1})^{*j} = \operatorname{Im} f(j),$$

where

$$f(j) : G^{(2kj)} \to G$$

$$(\mathbf{g}_1, \ldots, \mathbf{g}_{2j}) \mapsto \prod_{i=1}^{j} w(\mathbf{g}_{2i-1}) w(\mathbf{g}_{2i})^{-1}.$$

Now $f(j)$ is dominant as a morphism from $G^{(2kj)}$ into $\overline{P_j}$, so $\overline{P_j}$ is irreducible. Since

$$\overline{P_1} \subseteq \overline{P_2} \subseteq \ldots \subseteq \overline{P_j} \subseteq \overline{P_{j+1}} \subseteq \ldots \qquad (1)$$

and $\dim \overline{P_j} \leq \dim G$ for each j, there exists m such that

$$\dim \overline{P_j} = \dim \overline{P_m} \quad \forall j \geq m.$$

But an irreducible variety can't contain a proper subvariety of the same dimension, so (1) becomes stationary at $\overline{P_m} = T$, say.

Since $f(m) : G^{(2km)} \to T$ is dominant as a map into T, we have $P_m \supseteq U$ for some dense open subset U of T. Now

$$U \subseteq P_m \subseteq w(G) = \bigcup_{j=1}^{\infty} P_j \subseteq T = \overline{U}.$$

It follows that $T = \overline{w(G)}$ is a closed subgroup of G.

Let $y \in T$. Then yU is non-empty and open in T, so $yU \cap U \neq \varnothing$ and hence

$$y \in U \cdot U^{-1} \subseteq P_{2m} \subseteq w(G).$$

Thus

$$T \subseteq P_{2m} \subseteq w(G) \subseteq T.$$

Hence

$$w(G) = \overline{w(G)} = P_{2m} \subseteq G_w^{*4m}.$$

Theorem 2 (Borel) *Let G be a connected semisimple algebraic group over an algebraically closed field, and w a non-trivial word. Then $f_w : G^{(k)} \to G$ is dominant.*
Hence G_{+w} contains a dense open subset of G.

Corollary 1 *Let u, w be non-trivial words. Then $G = G_{+u} \cdot G_{+w}$.*
Every word has positive width 2 in G.

Proof. Let $U \subseteq G_{+u}$ and $W \subseteq G_{+w^{-1}}$ be dense open subsets of G. Let $g \in G$. Then gW is dense so $gW \cap U \neq \varnothing$. Thus

$$g \in U \cdot W^{-1} \subseteq G_{+u} \cdot G_{+w}.$$

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W.l.o.g. K is as large as we like.
Until further notice: fix $n \geq 2$,

$$G = \mathsf{SL}_n(K)$$

χ_g = characteristic polynomial of g.
Define $\chi : G \to K^{n-1}$ by

$$g\chi = (a_1, \ldots, a_{n-1})$$

where

$$\chi_g(T) = T^n - a_1 T^{n-1} + \cdots + (-1)^{n-1} a_{n-1} T + (-1)^n.$$

Define

$$G_{reg} = \{g \in G \mid disc(\chi_g) \neq 0\} = \chi^{-1}(W)$$

where W is the dense open subset of K^{n-1} corresponding to polynomials with non-zero discriminant.

Jordan normal form shows that for each $v \in W$, the fibre $\chi^{-1}(v)$ is exactly one conjugacy class in G.

Lemma 1 *Put* $G_1 = \mathrm{diag}(\mathrm{SL}_{n-1}(K), 1) < G$*. Then* $G_1\chi$ *is a vector space of codimension* 1 *in* K^{n-1}*.*

(Easy)

Lemma 2 G *has a dense subgroup* H *such that* $\chi_h(1) \neq 0$ *for* $1 \neq h \in H$*.*

(Proof to come)

Lemma 3 *If* H *is any dense subgroup of* G *then* $H_{+w} \neq \{1\}$*.*

Proof. We may assume that K contains two algebraically independent elements x and y. Then
$G = \mathrm{SL}_n(K) \geq \mathrm{SL}_2(K) \geq F$ where

$$F = \left\langle \begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix}, \begin{pmatrix} y & 0 \\ x & y^{-1} \end{pmatrix} \right\rangle$$

a non-abelian free group. So $G_{+w} \supseteq F_{+w} \neq \{1\}$. Result follows since H is dense in G.

Proof of Theorem 2 for $G = \mathsf{SL}_n(K)$: by induction on n.

Recall: $G_1 = \mathsf{SL}_{n-1}(K) < G$.

Let $X = \overline{G_{+w}}$ be the closure of G_{+w} in G. We have to show that $X = G$.

If $n = 2$ then $G_1 = \{1\} \subseteq X$.

If $n > 2$, inductive hypothesis gives

$$G_1 = \overline{G_{1,+w}} \subseteq \overline{G_{+w}} = X.$$

Set $Y = \overline{X\chi} \subseteq K^{n-1}$, let H be the subgroup given in Lemma 2, and take $h \in H_{+w} \setminus \{1\}$.

Then $h \in X$ but $h\chi \notin G_1\chi$. Hence

$$K^{n-1} \supseteq Y \supsetneq G_1\chi.$$

As Y is an irreducible variety and $\dim(G_1\chi) = n-2$ it follows that $Y = K^{n-1}$.

Thus $\chi_{|X} : X \to K^{n-1}$ is dominant, and so $X\chi$ contains a dense open subset U of K^{n-1}.

Claim: $\chi^{-1}(U \cap W) \subseteq X$.

Suppose $g \in G$ and $g\chi \in U \cap W$. Then $g \in G_{reg}$ and $g\chi = x\chi$ for some $x \in X$.

Therefore g is conjugate to x, hence $g \in X$.

Finally:

both U and W open and dense in K^{n-1}

$\Rightarrow \chi^{-1}(U \cap W)$ is non-empty and open in G

$\Rightarrow \chi^{-1}(U \cap W)$ dense in G

$\Rightarrow X = G$.

rk(G): dimension of maximal torus in G.
Isogeny: epimorphism with finite kernel.

Theorem 3 *Let G be a connected simple algebraic group over K. If G is not isogenous to $\mathrm{SL}_n(K)$ for any n then G has a closed connected semisimple subgroup H such that $\dim(H) < \dim(G)$ and $\mathrm{rk}(H) = \mathrm{rk}(G)$.*

Proof *(sketch)* The root system Φ of G is not of type A_n. Say $\{\alpha_1, \ldots, \alpha_l\}$ is a basis and

$$d = \sum_{i=1}^{l} d_i \alpha_i$$

is the dominant root. Then, for a suitable ordering,

$$-d,\ \alpha_2, \ldots, \alpha_l$$

is the basis of a closed subsystem Ψ of Φ.
H is the subgroup of G corresponding to Ψ (Borel's argument).

Larsen's argument:

$$\mathsf{SL}_2^{(n)} < \mathsf{Sp}_{2n},$$

$$\mathsf{SL}_2^{(2n)} = \mathsf{Spin}_4^{(n)} < \mathsf{Spin}_{4n} < \mathsf{Spin}_{4n+1},$$

$$\mathsf{SL}_2^{(2n-2)} \times \mathsf{SL}_4 = \mathsf{Spin}_4^{(n-1)} \times \mathsf{Spin}_6 < \mathsf{Spin}_{4n+2} < \mathsf{Spin}_{4n+3},$$

$$E_6 > A_2^{(3)}$$
$$E_7 > A_1 \times A_3^{(2)}$$
$$E_8 > A_4^{(2)}$$
$$F_4 > A_2^{(2)}$$
$$G_2 > A_2$$

Proof of Theorem 2

W.l.o.g. G *connected and simple, not isogenous to* $\mathsf{SL}_n(K)$.

Let H be a semisimple subgroup as given in Theorem 3.

Arguing by induction on $\dim(G)$ we may suppose that H_{+w} is dense in H.

Now let T be a maximal torus in H. Then T is also a maximal torus of G, and

$$T \subseteq H = \overline{H_{+w}} \subseteq \overline{G_{+w}}.$$

Fact: the union of all conjugates of T is a dense subset of G (it contains the regular semisimple elements).

So

$$\overline{G_{+w}} \supseteq \overline{\bigcup_{g \in G} T^g} = G.$$

Proof of Lemma 2. We may assume that K contains either $L = \mathbb{Q}_p$ for some prime p (when char(K) = 0) or $L = \mathbb{F}_p((t))$ (when char(K) = 0).

There exists a *central division algebra* Δ over L of index n.

K is algebraically closed:

$$\Delta \otimes_L K \cong \mathsf{M}_n(K).$$

Identify Δ with an L-subalgebra of $\mathsf{M}_n(K)$, and put

$$H = \Delta \cap \mathsf{SL}_n(K).$$

Then

$$\begin{aligned} 1 \neq h \in H \Rightarrow h - 1 &\text{ invertible} \\ &\Rightarrow \chi_h(1) \neq 0. \end{aligned}$$

H is dense in $\mathsf{SL}_n(K)$: indeed $H = \mathcal{H}(L)$ where $\mathcal{H}$ is an L-form of SL_n, i.e. $\mathcal{H}(K) = \mathsf{SL}_n(K)$.

This concludes our discussion of algebraic groups over an algebraically closed field K. We have seen that if G is simple then every word has width 2 in $G(K)$. What about other fields? The question of verbal width in groups such as $\mathsf{SL}_n(F)$ when F is an algebraic number field seems not to have been explored, and should be interesting. The following is deduced by Borel and Larsen from Theorem 2:

Corollary 2 *If G is a simple algebraic group defined over $\mathbb{R}$ and w is a non-trivial word then $G(\mathbb{R})_{+w}$ contains a non-empty set that is open in the real topology of $G(\mathbb{R})$.*

I expect that this also holds with $\mathbb{Q}_p$ (and the p-adic topology) in place of $\mathbb{R}$.

Finite fields: next lecture!