

On the finite axiomatizability of some metabelian profinite groups

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A group G is *finitely axiomatizable* (FA) in a class \mathcal{C} if G satisfies a first-order sentence σ such that every \mathcal{C} group satisfying σ is isomorphic to G . Some examples of this phenomenon where \mathcal{C} consists of profinite groups were discussed in [NST]; one of the questions raised in that paper was: *are finitely generated free pro- p groups FA*, in either of the classes profinite groups, pro- p groups?

This is still unknown; a small step in that direction is the following:

Theorem 1 *Each f.g. free metabelian pro- p group on at least two generators is FA in the class of all profinite groups.*

The proof depends on

Theorem 2 *For each $m, d \geq 1$ the profinite wreath product $W_{m,d} = \mathbb{Z}_p^{(m)} \wr \mathbb{Z}_p^{(d)}$ is FA in the class of profinite groups.*

Here,

$$W_{m,d} = \varprojlim_{n \rightarrow \infty} C_{p^n}^{(m)} \wr C_{p^n}^{(d)}.$$

The analogue in the class of abstract groups of Theorem 1 was recently established by Kharlampovich, Miasnikov and Sohrabi; they deduce it from the stronger result that *a free metabelian group is bi-interpretable with \mathbb{Z}* ([KMS] Theorem 30). The proof of this is quite elaborate; it seems plausible that the analogue should hold for free metabelian pro- p groups and \mathbb{Z}_p , but this seems quite difficult.

Facts from Sections 2 and 5 of [NST] will be used without special mention. We also often use the fact \mathcal{H} : *every finitely generated profinite group is Hopfian*, that is, each surjective endomorphism is an isomorphism.

Logical terms (‘formula’, ‘sentence’) all refer to the ordinary first-order language of group theory. As discussed in [NST], ‘isomorphism’ for profinite groups will mean ‘continuous isomorphism’ (among finitely generated profinite groups these are actually equivalent, for non-trivial reasons).

1 A reduction

A subgroup H of a profinite group G is *definably closed* if there is a formula $\phi(x)$ such that

(i) for every profinite group P , the subset

$$\phi(P) := \{s \in P \mid P \models \phi(s)\}$$

is a closed subgroup, and

(ii) $H = \phi(G)$.

Proposition 3 *Let G be a pro- p group. Suppose that G has a definably closed abelian normal subgroup $A \neq 1$ of infinite index such that $A = C_G(a)$ for each $1 \neq a \in A$. Then G satisfies a sentence χ such that for any profinite group H , if $H \models \chi$ then H is a pro- p group.*

For the proof, we combine Lemmas 4.5 and 4.6 of [S] to obtain

Lemma 4 *Let Γ be a profinite group and A a profinite Γ -module such that*

$$\text{for } a \in A, x \in \Gamma, \quad ax = a \implies (a = 0 \vee x = 1), \quad (1)$$

$$pA + A(\Gamma - 1) < A, \quad (2)$$

$$\bigcap_{1 \neq x \in \Gamma} A(x - 1) = 0. \quad (3)$$

Then both Γ and A are pro- p groups.

Now set $\Gamma = G/A$ in Proposition 3. Then the conditions (1), (2) and (3) are satisfied (see the Remark following Lemma 4.6 in [S]). So G satisfies a sentence α such that for any profinite group H satisfying α , H has a closed, definable abelian normal subgroup B , and each of (1), (2) and (3) holds with B for A and H/B for Γ . It follows by the preceding lemma that H is a pro- p group.

2 Ring lemmas

Lemma 5 *Let $R = \mathbb{Z}_p[[\zeta]]$ and $M = R^{(m)}$, a free R -module of rank $m \geq 1$. Let K be an R -submodule of M such that*

- $M/(K + M\zeta) \cong \mathbb{Z}_p^{(m)}$
- $a\zeta \in K \implies a \in K$ for all $a \in M$.

Then $K = 0$.

Proof. The quotient map $M \rightarrow M/K$ induces an epimorphism

$$\mathbb{Z}_p^{(m)} \cong M/M\zeta \rightarrow M/(K + M\zeta) \cong \mathbb{Z}_p^{(m)}.$$

This must be an isomorphism (by \mathcal{H}), so $K \subseteq M\zeta$. Thus $K = K \cap M\zeta = K\zeta$, which implies that $K = \bigcap_{n \in \mathbb{N}} K\zeta^n = 0$. ■

Let $R = \mathbb{Z}_p[[\zeta_1, \dots, \zeta_d]]$. Let us call the tuple $(\zeta_1, \dots, \zeta_d)$ a *base* for R (together with p it forms a particular kind of system of parameters for the local ring R). The set of all such bases is denoted $\mathcal{B}(R)$, and for $i < d$ set

$$\mathcal{B}_i(R) = \{(\zeta_1, \dots, \zeta_i) \mid (\zeta_1, \dots, \zeta_d) \in \mathcal{B}(R) \text{ for some } \zeta_{i+1}, \dots, \zeta_d\}.$$

Let $\mathcal{X}_i \subseteq \mathcal{B}_i(R)$ for $1 \leq i < d$. I will call the sequence $(\mathcal{X}_1, \dots, \mathcal{X}_{d-1})$ *rich* if \mathcal{X}_1 contains infinitely many pairwise non-associate elements, and for $i > 1$ and each fixed $(\zeta_1, \dots, \zeta_{i-1}) \in \mathcal{X}_{i-1}$, there are infinitely many distinct ideals of the form

$$\zeta_1 R + \dots + \zeta_i R$$

with $(\zeta_1, \dots, \zeta_i) \in \mathcal{X}_i$.

Lemma 6 *Assume that $d \geq 2$. For $1 \leq i < d$ let \mathcal{X}_i be a subset of $\mathcal{B}_i(R)$ such that $(\mathcal{X}_1, \dots, \mathcal{X}_{d-1})$ forms a rich sequence. Then for $1 \leq i < d$ we have*

$$D_i := \bigcap_{\bar{\zeta} \in \mathcal{X}_i} (\zeta_1 R + \dots + \zeta_i R) = 0.$$

Proof. Suppose first that $i = 1$. As ζ_1 ranges over \mathcal{X}_1 , $\zeta_1 R$ ranges over an infinite set of prime ideals of height 1 in the Noetherian integral domain R , which forces $D_1 = 0$. (This step isn't really necessary: we could allow $i = 1$ in the following argument; but this way may be less confusing.)

Now let $i > 1$ and fix $(\zeta_1, \dots, \zeta_{i-1}) \in \mathcal{X}_{i-1}$. Set

$$Y = \{\zeta_i \in R \mid (\zeta_1, \dots, \zeta_i) \in \mathcal{X}_i\}.$$

Writing $\pi : R \rightarrow \tilde{R} = R/(\zeta_1 R + \dots + \zeta_{i-1} R)$ we have

$$\begin{aligned} D_i \pi &\subseteq \bigcap_{\zeta_i \in Y} (\zeta_1 R + \dots + \zeta_i R) \pi \\ &= \bigcap_{\zeta_i \in Y} \tilde{\zeta}_i \tilde{R}. \end{aligned}$$

This is the intersection of an infinite set of prime ideals of height 1 in the Noetherian integral domain \tilde{R} . It follows that $D_i \pi = 0$, and hence that $D_i \subseteq \zeta_1 R + \dots + \zeta_{i-1} R$. As $(\zeta_1, \dots, \zeta_{i-1})$ ranges over \mathcal{X}_{i-1} these ideals intersect in D_{i-1} .

The result follows by induction. ■

Corollary 7 *Suppose $R = \mathbb{Z}_p[[X]]$ where X is the free abelian pro- p group on x_1, \dots, x_d and $d \geq 2$. Let $\mathcal{C} \subseteq X^{(d)}$ denote the set of all bases for X . Then*

$$\bigcap_{\mathbf{y} \in \mathcal{C}} ((y_1 - 1)R + \dots + (y_{d-1} - 1)R) = 0.$$

Proof. The ring R is equal to $\mathbb{Z}_p[[\xi_1, \dots, \xi_d]]$ where $\xi_i = x_i - 1$ ([DDMS] Thm. 7.20). Set

$$\mathcal{X} = \{(y_1 - 1, \dots, y_d - 1) \mid (y_1, \dots, y_d) \in \mathcal{C}\},$$

and let $\pi_i : R^{(d)} \rightarrow R^{(i)}$ denote the projection to the first i factors. Then

$$(\mathcal{X}\pi_1, \dots, \mathcal{X}\pi_{d-1})$$

is a rich sequence. To see this, note that for $i < d$ and $(y_1, \dots, y_{i-1}) \in \mathcal{C}\pi_{i-1}$, the group $\tilde{X} = X/\overline{\langle y_1, \dots, y_{i-1} \rangle}$ is free abelian of rank at least 2, and $\mathbb{Z}_p[[\tilde{X}]]$ is naturally identified with $R/\sum_{j=1}^{i-1} (y_j - 1)R$. Now \tilde{X} has infinitely many 1-generator direct factors $\overline{\langle \tilde{y} \rangle}$, giving rise to infinitely many distinct augmentation ideals $(\tilde{y} - 1)\mathbb{Z}_p[[\tilde{X}]]$, the required condition for a rich sequence. The corollary now follows from the lemma with $i = d - 1$. ■

3 Wreath products

Now we prove Theorem 2.

$W := W_{m,d} = M \rtimes X$ where $X = \overline{\langle x_1, \dots, x_d \rangle}$ say is a free abelian pro- p group and the X -module M is isomorphic to $R^{(m)}$ where $R = \mathbb{Z}_p[[X]] = \mathbb{Z}_p[[\xi_1, \dots, \xi_d]]$, writing $\xi_i = x_i - 1$.

Taking $G = W$ and $A = M$ in Proposition 3, we see that W satisfies a sentence χ such that every profinite group satisfying χ is a pro- p group. So it will suffice to show that W is FA in the class of pro- p groups.

Assume to begin with that $d = 1$, and write $x = x_1$ etc.

Say $M = a_1R \oplus \dots \oplus a_mR$. Then W satisfies a sentence $\Psi(\mathbf{a}, x)$ asserting the following (within the class of pro- p groups):

- The set $\{a_1, \dots, a_m, x\}$ generates W
- $\overline{\langle x \rangle} = C_W(x) \cong \mathbb{Z}_p$,
- $C_W(a_1) = \dots = C_W(a_m) := M$, say,
- M is abelian and normal in W ,
- $M/[M, x] \cong \mathbb{Z}_p^{(m)}$,
- $M \cap C_W(x) = 1$.

(In [NST], §5.1 and §5.4, it is explained how these are expressed in first-order language.)

Now suppose that G is a pro- p group and that $G \models \Psi(\mathbf{b}, y)$ for some $b_1, \dots, b_m, y \in G$. Write B for the (topological) normal closure of $\{b_1, \dots, b_m\}$ and set $Y = \overline{\langle y \rangle}$. Then $Y = C_G(y) \cong \mathbb{Z}_p$, B is an abelian normal subgroup contained in $C_G(b_i)$ for each i , and $G = BY$. It follows that $C_G(b_i) = B.C_Y(b_i) = B$ for each i , because $C_G(b_i) \cap C_G(y) = 1$. Thus $G = B \rtimes Y$. We consider B as an R -module via $x \mapsto y$, and then $B = b_1R + \dots + b_mR$.

Let K be the kernel of the epimorphism $M \rightarrow B$ that sends a_i to b_i for each i . The sentence $\Psi(\mathbf{b}, y)$ implies that $B/B\xi \cong \mathbb{Z}_p^{(m)}$ and that $b\xi = 0 \implies b = 0$. It follows that K satisfies the hypotheses of Lemma 5, and so $K = 0$. Thus B is free of rank m as a module for $\mathbb{Z}_p Y$, and so $G \cong W$.

Thus $W_{m,1}$ is FA in pro- p groups. Suppose now that $d \geq 2$, and $W = W_{m,d}$. The subgroups X and M are definable by

$$X := C_W(x_1) = \dots = C_W(x_d) \quad (4)$$

$$M := C_W(a_1) = \dots = C_W(a_m). \quad (5)$$

Let $\Phi(\mathbf{a}, \mathbf{x})$ be a first-order formula which asserts (for the pro- p group W) that (4) and (5) hold and

$$\begin{aligned} X &= \overline{\langle x_1, \dots, x_d \rangle} \cong \mathbb{Z}_p^{(d)}, \\ W &= \overline{\langle a_1, \dots, a_m, x_1, \dots, x_d \rangle} \\ [M, M] &= 1, \quad M \triangleleft W \end{aligned}$$

Suppose that G is a pro- p group and $G \models \Phi(\mathbf{b}, \mathbf{y})$ for some $b_i, y_j \in G$. There is an epimorphism $\phi_{\mathbf{b}, \mathbf{y}} : W \rightarrow G$ sending \mathbf{a}, \mathbf{x} to \mathbf{b}, \mathbf{y} respectively. Then $Y = \overline{\langle y_1, \dots, y_d \rangle} \cong \mathbb{Z}_p^d \cong X$, so ϕ induces an isomorphism from X to Y (in view of \mathcal{H}), and so $K_{\mathbf{b}, \mathbf{y}} := \ker \phi_{\mathbf{b}, \mathbf{y}} \leq M$.

Suppose that (t_1, \dots, t_d) is a basis for X . Denote the (topological) normal closure of $\{t_1, \dots, t_{d-1}\}$ in W by $N_{\mathbf{t}}$. So

$$N_{\mathbf{t}} = [M, t_1] \dots [M, t_{d-1}] \overline{\langle t_1, \dots, t_{d-1} \rangle},$$

and it is easy to see that

$$g \in N_{\mathbf{t}} \iff [W, g] \subseteq [M, t_1] \dots [M, t_{d-1}].$$

Thus $N_{\mathbf{t}}$ is definable by a formula $\nu(t_1, \dots, t_d)$, so by the first case, there is a formula $\Upsilon(t_1, \dots, t_d)$ which asserts that $W/N_{\mathbf{t}} \cong W_{m,1}$; this statement is true whenever (t_1, \dots, t_d) is a basis for X .

Finally, let Θ be the sentence asserting, for a pro- p group G , that there exist $b_1, \dots, b_m, y_1, \dots, y_d \in G$ such that (a) $G \models \Phi(\mathbf{b}, \mathbf{y})$ and (b) for each tuple (s_1, \dots, s_d) that generates $Y := C_G(y_1)$, $G \models \Upsilon(\mathbf{s})$.

We have seen that W satisfies Θ . Suppose that the pro- p group G satisfies Θ . Then $\phi := \phi_{\mathbf{b}, \mathbf{y}}$ maps W onto G and X onto Y . Let \mathbf{t} be a basis for X and set $\mathbf{s} = \mathbf{t}\phi$. Then $N_{\mathbf{t}}\phi \leq N_{\mathbf{s}}$, so we have an induced epimorphism $\phi^* : F/N_{\mathbf{t}} \rightarrow G/N_{\mathbf{s}}$. Now $\Upsilon(\mathbf{s})$ asserts that $G/N_{\mathbf{s}} \cong W_{m,1} \cong F/N_{\mathbf{t}}$, and it follows by \mathcal{H} that $K_{\mathbf{b}, \mathbf{y}} \leq N_{\mathbf{t}}$. We know that $K_{\mathbf{b}, \mathbf{y}} \leq M$, and so

$$K_{\mathbf{b}, \mathbf{y}} \leq M \cap N_{\mathbf{t}} = \sum_{i=1}^{d-1} M(t_i - 1).$$

Corollary 7 shows that as \mathbf{t} ranges over all bases for X , these modules intersect in zero. It follows that $K_{\mathbf{b}, \mathbf{y}} = 1$, and so $G \cong W$.

4 Free metabelian groups

$F = F_d$ is a free metabelian pro- p group on $d \geq 2$ generators g_1, \dots, g_d . We set $x_i = g_i F'$, $X = F/F' = \langle x_1, \dots, x_d \rangle$ and $A = F'$. Then A is a module for the completed group algebra $R = \mathbb{Z}_p[[X]]$. For $1 \leq j \leq d$ set $X_j = \langle x_1, \dots, x_j \rangle$ and $R_j = \mathbb{Z}_p[[X_j]]$.

Note that R_j is equal to the power series ring $\mathbb{Z}_p[[\xi_1, \dots, \xi_j]]$ where $\xi_i = x_i - 1$ for each i ([DDMS] Thm. 7.20). Thus it is a regular local ring of dimension $1 + j$.

Write $\Delta_{ji} = \sum_{l=1}^i \xi_l R_j$; the unique maximal ideal of R_j is $\mathfrak{m}_j = pR_j + \Delta_{jj}$.

Recall that if $G = \langle h_1, \dots, h_d \rangle$ is a pro- p group then $G' = [h_1, G] \dots [h_d, G]$, a definably closed normal subgroup. In particular, A is definably closed in F .

We will often use the ‘Jacobi identity’ for metabelian groups,

$$[a, b, c][b, c, a][c, a, b] = 1;$$

this follows at once from the Hall-Witt identity when the derived group is abelian.

Putting $u_{ij} = [g_i, g_j]$ we have

Proposition 8 *Each element of A is uniquely expressible as*

$$a = \sum_{1 \leq i < j \leq d} u_{ij} r_{ij} \tag{6}$$

with $r_{ij} \in R_j$ for each i and j .

Proof. The analogue of this result for the abstract free metabelian group, F_0 say, is established in [MR], section 6 (cf. also [B], [BR]). The *existence* of a representation (6), also in the pro- p case, is easily deduced from the Jacobi identity. The proof of *uniqueness* explained in [MR] uses Fox derivatives; these induce mappings $d_j : F'_0 \rightarrow \mathbb{Z}(F_0^{\text{ab}})$ which are F_0 -module homomorphisms and satisfy

$$d_j(u_{ik}) = \begin{cases} 0 & j \neq i, k \\ x_k - 1 & j = i \\ 1 - x_i & j = k \end{cases}.$$

It is easy to verify that each d_j extends by continuity to an R -module homomorphism from A to R , noting that A is the completion of F'_0 w.r.t. the I -adic topology where I is the ideal of $\mathbb{Z}(F_0^{\text{ab}})$ generated by p and ξ_1, \dots, ξ_d , while R is the completion of $\mathbb{Z}(F_0^{\text{ab}})$ w.r.t. the I -adic topology.

Now suppose that a in (6) is equal to 0. We have to show that each r_{ij} is zero. Arguing by induction on d , we may suppose that $r_{ij} = 0$ for all $i < j < d$. Then for $1 \leq j < d$ we have

$$0 = d_j(a) = (x_d - 1)r_{jd},$$

and the result follows. ■

The uniqueness of expression in (6) implies in particular that $ax_1 \neq a$ if $a \neq 0$. It follows by symmetry that $ax_d \neq a$ if $a \neq 0$, so the mapping $a \mapsto a\xi_d$ is injective. Noting that for $i < j < d$ we have $u_{ij}\xi_d = u_{id}\xi_j - u_{jd}\xi_i$, we see that A embeds in a free submodule:

Corollary 9

$$A \cong A\xi_d \leq \bigoplus_{i=1}^{d-1} u_{id}R.$$

It follows in turn that $0 \neq a \in A$ implies $A = C_F(a)$. So we may apply Proposition 3 to find a sentence χ , satisfied by F , such that every profinite group satisfying χ is a pro- p group. Thus to complete the proof of Theorem 1 it will suffice to show that F is FA in the class of pro- p groups.

Now set

$$\begin{aligned} C &= C_{\mathbf{g}} = C_F(g_d) \\ H &= H_{\mathbf{g}} = AC \\ B &= B_{\mathbf{g}} = A\xi_1 + \cdots + A\xi_{d-1} \\ Z &= Z_{\mathbf{g}} = \{a \in A \mid a\xi_d \in B\}. \end{aligned}$$

(If $d = 2$, this means that $Z = B$.)

It follows from Proposition 8 that

$$A/B = Z/B \oplus (D + B)/B$$

where

$$D = \bigoplus_{i=1}^{d-1} u_{id}R,$$

and that $D \cap B = \bigoplus_{i=1}^{d-1} u_{id}(R\xi_1 + \cdots + R\xi_{d-1})$. This implies that

$$A/Z \cong (D + B)/B \cong D/(D \cap B) \cong S^{(d-1)}$$

where $S = \mathbb{Z}_p[[\overline{\langle x_d \rangle}]]$.

Also $C = \langle g_d \rangle$. Thus

$$\frac{H}{Z} = \frac{A}{Z} \overline{\langle g_d \rangle} \cong \mathbb{Z}_p^{(d-1)} \overline{\mathbb{Z}_p} = W_{d-1,1},$$

the wreath product discussed above.

Now all the subgroups mentioned, with the possible exception of D , are definable relative to the parameters (g_1, \dots, g_d) . In view of Theorem 2, there is a formula $\Omega(t_1, \dots, t_d)$ such that $F \models \Omega(\mathbf{g})$ expresses the fact that $C_{\mathbf{g}} = \langle g_d \rangle$ and $H_{\mathbf{g}}/Z_{\mathbf{g}} \cong W_{d-1,1}$.

Let μ be a sentence asserting for a pro- p group G that G is metabelian and that $G/\gamma_3(G) \cong F/\gamma_3(F)$ (recall that $F/\gamma_3(F)$ is FA in pro- p groups by [NST], Theorem 5.15). In particular, if $G \models \mu$ then G is generated by d elements.

Suppose now that G is a pro- p group and that G satisfies

$$\mu \wedge (\forall t_1, \dots, t_d) (\beta_d(t_1, \dots, t_d) \rightarrow \Omega(t_1, \dots, t_d))$$

where $G \models \beta_d(t_1, \dots, t_d)$ iff $G = \overline{\langle t_1, \dots, t_d \rangle}$. Let $\theta : F \rightarrow G$ be an epimorphism and set $K = \ker \theta$.

The induced epimorphism $F/\gamma_3(F) \rightarrow G/\gamma_3(G)$ is an isomorphism by \mathcal{H} , so $K \leq \gamma_3(F) = [A, F]$.

Let $t_i = g_i \theta$ for each i . Then $H_{\mathbf{g}} \theta = H_{\mathbf{t}}$ and $Z_{\mathbf{g}} \theta \subseteq Z_{\mathbf{t}}$, so θ induces an epimorphism $\theta^* : H/Z \rightarrow H_{\mathbf{t}}/Z_{\mathbf{t}}$. Since t_1, \dots, t_d generate G , $G \models \Omega(\mathbf{t})$, so $H_{\mathbf{t}}/Z_{\mathbf{t}} \cong W_{d-1,1} \cong H/Z$, whence θ^* is an isomorphism (by \mathcal{H}); since $K \leq [A, F] \leq H$ it follows that $K \leq Z$. Thus

$$K \leq Z \cap [A, F] = Z \cap (B + A\xi_d) \leq Z \cap D \leq B = B_{\mathbf{g}}.$$

This holds irrespective of the chosen basis \mathbf{g} for F . Now Corollaries 9 and 7 together show that as \mathbf{g} runs over all such bases, the submodules $B_{\mathbf{g}}$ intersect in zero. Thus $K = 1$, and so $G \cong F$.

This completes the proof of Theorem 1.

5 The case $d = 2$

There is a much simpler proof when $d = 2$. Assume now that F is the free metabelian pro- p group on generators g, h . Adapting the notation of the preceding section, write $A = F'$, $x = Ag$, $y = Ah$, $u = [g, h]$, $R = \mathbb{Z}_p[[F/F']] = \mathbb{Z}_p[[\xi, \eta]]$ where $\xi = x - 1$, $\eta = y - 1$. Thus

$$A = uR \cong R$$

by Proposition 8, and

$$\begin{aligned} \gamma_3(F) &= [A, g][A, h] \\ &= A\xi + A\eta \end{aligned}$$

using additive notation for the F/F' -module A .

Set

$$H = F/\gamma_3(F),$$

this is the free class-2 nilpotent pro- p group (the Heisenberg group over \mathbb{Z}_p).

Let $\Psi(s, t)$ be a first-order formula such that for a pro- p group G and $s, t \in G$, $G \models \Psi(s, t)$ if and only if

1. s and t generate G
2. $B := [G, s][G, t]$ is abelian (recall that given $\mathbf{1}$., B is in fact the derived group of G)
3. $G/[B, s][B, t] \cong H$ (note that given $\mathbf{2}$., $[B, s][B, t] = \gamma_3(G)$)

4. For $a, b \in B$,

$$[a, s][b, t] = 1 \iff a = [c, t] \wedge b = [c^{-1}, s] \text{ for some } c \in B.$$

(Here we use the fact that H is FA in the class of pro- p groups, a special case of [NST], Theorem 5.15.)

Now I claim (a) F satisfies $\Psi(g, h)$ and (b) if G is a pro- p group G , $s, t \in G$, and $G \models \Psi(s, t)$ then $F \cong G$ by a map sending g to s and h to t .

This shows that F is FA in the class of pro- p groups; as above we quote Proposition 3 to infer that F is FA in the class of all profinite groups.

Proof of (a). Only condition **4.** needs comment. Writing A additively, this asserts for $a, b \in A$ that

$$a\xi + b\eta = 0 \iff a = c\eta \wedge b = -c\xi \text{ for some } c \in A.$$

As $A \cong R$, this follows from the fact that $R\xi \cap R\eta = R\xi\eta$ (while neither of ξ, η is a zero-divisor).

Proof of (b). Now G is a metabelian pro- p group generated by s and t , so there exists an epimorphism $\theta : F \rightarrow G$ with $g\theta = s$ and $h\theta = t$. In view of \mathcal{H} , Condition **3.** implies that the induced epimorphism $F/\gamma_3(F) \rightarrow G/\gamma_3(G)$ is an isomorphism. It follows that $\ker \theta := K$ is contained in $\gamma_3(F) = [A, g][A, h]$.

Suppose now that $w = [a, g][b, h] \in K$. Then

$$1 = [a, g][b, h]\theta = [a', s][b', t]$$

where $a' = a\theta, b' = b\theta \in A\theta = B$. According to **4.**, there exists $c' \in B$ such that $a' = [c', t]$ and $b' = [c'^{-1}, s]$. Say $c' = c\theta$ for some $c \in A$ (this exists because θ maps A onto B). Then

$$a = [c, h]w_1, \quad b = [c^{-1}, g]w_2$$

with $w_1, w_2 \in K$. Thus translating into additive notation we have

$$\begin{aligned} w &= [[c, h]w_1, g][[c^{-1}, g]w_2, h] \\ &= (c\eta + w_1)\xi + (-c\xi + w_2)\eta \\ &= w_1\xi + w_2\eta. \end{aligned}$$

It follows that $K \subseteq [K, F]$. As F is a pro- p group this forces $K = 1$, so θ is an isomorphism as required.

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