JAMES’ WEAK COMPACTNESS THEOREM

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Abstract. The purpose of this note is a careful study of James’ Weak Compactness Theorem. Strikingly, this theorem completely characterizes weak compactness of weakly closed sets in a Banach space by the property that all bounded linear functionals attain their supremum on the set.

To fully appreciate this statement, we first present a simple proof in a familiar finite dimensional setting. Subsequently, we examine the subtleties of compactness in the weak topologies on infinite dimensional normed spaces. Building on this, we then provide detailed proofs of both a separable and non-separable version of James’ Weak Compactness Theorem, following R.E. Megginson (1998).

Using only James’ Weak Compactness Theorem, we proceed to derive alternative proofs for the celebrated James’ Theorem and the equally famous Krein-Šmulian Weak Compactness Theorem as well as another interesting compactness result by Šmulian. Furthermore, we present some insightful applications to measure theory and sequence spaces.

Robert C. James (1918-2004) — Photo by Paul R. Halmos

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1. Introduction

The subject of this note is a thorough study of James' Weak Compactness Theorem by Robert C. James (1964, 1972). Informally, the theorem states that: A weakly closed subset $A$ of a Banach space $X$ is weakly compact if and only if every continuous linear functional on $X$ attains its supremum on $A$.

This theorem is fundamental to a wide range of topics in Functional Analysis and stands as one of the deepest results in the study of weak topologies. Moreover, it has important applications in vector space optimization and has recently found uses in mathematical finance (see e.g. Cascales et al. 2010). As a testimony to its fame, the Encyclopedia of Optimization (Floudas & Pardalos 2009, p. 2091) refers to the theorem as "one of the most beautiful results in functional analysis".

At the same time, however, the proof has a redoutable reputation for being highly inaccessible and it is thus omitted in most textbooks on the subject. The plan for the present exposition is therefore to provide an illuminating and detailed account of the theorem and its proof. By striving for clarity and precision, as opposed to concision, we aim to equip the reader with a complete and comprehensive appreciation of the key aspects of the theorem. As we make no claims of originality, this is indeed the true justification for the existence of this note.

Before proceeding to introduce the basic notation in Section 1.1 below, we first provide the reader with a brief outline of the progression of the paper:

In Sections 2 and 3 we motivate our interest in James' Weak Compactness Theorem and explore the theorem in a finite dimensional setting. In Sections 4 and 5 we then present a comprehensive introduction to topological vector spaces in order to fully appreciate the theory underlying James' Weak Compactness Theorem. The experienced reader may wish to skip these sections and return to them as needed. With the theoretical underpinnings now in place, we devote Section 6 to an investigation of weak compactness in normed spaces. After this, we embark on the heart and soul of this exposition as we present a detailed proof of James' Weak Compactness Theorem in Section 7. We finish Section 7 with three simple, yet insightful, applications of the theorem. In Section 8 we then derive a series of well-established theorems as corollaries of James' Weak Compactness Theorem. Moreover, we provide a series of examples to emphasize that the assumptions of the theorem cannot be relaxed. In addition to this, we discuss the influence of reflexivity on the proof of James' Weak Compactness Theorem and how the ability to apply the theorem will in many cases automatically imply reflexivity. Finally, we finish this notes by proving a weak* analogue of James' Weak Compactness Theorem.

As a final remark, it should be noted that many of our arguments will rely on the Hahn-Banach Theorem and hence the Axiom of Choice is implicitly assumed. We shall not pursue this issue any further.

1.1. Preliminaries

Definition 1.1. Let $X$ be a vector space over $F$, where $F$ denotes either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Suppose $\|\cdot\| : X \to \mathbb{R}_+$ is a norm on $X$. Then the ordered pair $(X, \|\cdot\|)$ is called a normed space. If, in addition, the induced metric $d(x, y) := \|x - y\|$ is complete, then $(X, \|\cdot\|)$ is said to be a Banach space.

When the context is clear, we usually write only $X$, when we are in fact referring to $(X, \|\cdot\|)$. If a norm $\|\cdot\|$ induces a complete metric, then it is sometimes called a Banach norm. The topology on $X$ induced by the metric $d(x, y) := \|x - y\|$ is referred to as the norm topology on $X$. Whenever a topological property of $X$ is mentioned without further qualification, this norm topology is in mind.

All standard properties of normed spaces are assumed to be well-known to the reader. Nevertheless, we shall always mention explicitly any property that we employ. These same principles apply to the properties of linear operators between normed spaces. As a rule of thumb, the material in Chapter 5 of Folland (1999) is considered well-known (this book is used for the first course in Functional Analysis at the University of Copenhagen).
Definition 1.2. Suppose \((X, \| \cdot \|)\) is a normed space. Then we define the closed unit ball of \(X\) as \(\{ x \in X : \| x \| \leq 1 \}\) and denote it by \(B_X\).

The notation \(B_X\) will be used extremely often. It has the advantage that the closed unit balls \(B_X\) and \(B_Y\) of two different normed spaces \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) are easily distinguished without any notational clutter.

Throughout the text, the notions of linear span and convex hull will appear repeatedly. For this reason, we introduce the definitions here and remind the reader of some basic facts.

Definition 1.3. Let \(X\) be a vector space over \(\mathbb{F}\) and let \(A\) be a given subset of \(X\). Then the smallest convex set that includes \(A\) is called the convex hull of \(A\) and is denoted by \(\text{co}(A)\). Analogously, the smallest closed convex set that includes \(A\) is called the closed convex hull of \(A\) and is denoted by \(\overline{\text{co}}(A)\).

It is easily seen that \(\text{co}(A)\) is simply the set of all convex combinations of elements in \(A\). That is, the set of all sums \(\sum_{j=1}^{n} t_j x_j \) where \(x_1, \ldots, x_n \in A\), \(t_1, \ldots, t_n \geq 0\), and \(\sum_{j=1}^{n} t_j = 1\). Moreover, if \(X\) is a normed space, then \(\overline{\text{co}}(A) = \overline{\text{co}}(A)\).

Definition 1.4. Let \(X\) be a vector space over \(\mathbb{F}\) and let \(A\) be a given subset of \(X\). Then the smallest subspace of \(X\) that includes \(A\) is called the linear span of \(A\) and is denoted by \(\text{span}(A)\). Analogously, the smallest closed subspace of \(X\) that includes \(A\) is called the closed linear span of \(A\) and is denoted by \(\overline{\text{span}}(A)\).

Similarly to the case of the convex hull, one easily realizes that \(\text{span}(A)\) is simply the set of all linear combinations \(\sum_{j=1}^{n} \lambda_j x_j \) of elements \(x_1, \ldots, x_n \in A\) where \(\lambda_1, \ldots, \lambda_n \in \mathbb{F}\). In addition, we also have \(\overline{\text{span}}(A) = \overline{\text{span}}(A)\) whenever \(X\) is a normed space.

In the event that \(\mathbb{F} = \mathbb{C}\), we will often be interested in the restriction of multiplication by scalars to \(\mathbb{R}\). This motivates the following definition.

Definition 1.5. Let \(X\) be a vector space over \(\mathbb{F}\). If we restrict the multiplication by scalars map \(m : \mathbb{F} \times X \to \mathbb{R} \times X\), then we obtain a new vector space over \(\mathbb{R}\), which we denote by \(X_\mathbb{R}\). Suppose \(f\) is a real-valued function on \(X\) such that \(f(x + y) = f(x) + f(y)\) for all \(x, y \in X\) and \(f(\lambda x) = \lambda f(x)\) for any \(\lambda \in \mathbb{R}\). Then \(f\) is a linear functional on \(X_\mathbb{R}\) in the usual sense. When this is the case, we say that \(f\) is a real-linear functional on \(X\). Analogously, we sometimes say that \(f\) is a complex-linear functional on \(X\) if it is a linear functional on \(X\) over the field of complex numbers.

Note that if \(\mathbb{F} = \mathbb{R}\), then \(X_\mathbb{R}\) is simply the original vector space. We also emphasize that if \(X\) is a normed space or a Banach space, then so is \(X_\mathbb{R}\) with respect to the same norm. Some useful results about the intimate relationship between real- and complex-linear functionals are given below. It is these properties that make the study of \(X_\mathbb{R}\) fruitful.

Theorem 1.6. If \(f\) is a linear functional on \(X\), then the real part \(u := \text{Re}(f)\) is a real-linear functional on \(X\) with \(f(x) = u(x) - iu(ix)\) for all \(x \in X\). Conversely, if \(u\) is real-linear functional on \(X\), then \(f(x) = u(x) - iu(ix)\) uniquely defines a complex-linear functional \(f\) on \(X\) such that \(u = \text{Re}(f)\). When \(X\) is a normed space, \(f\) is continuous if and only if \(u\) is continuous, in which case \(\| f \| = \| u \|\).

Proof. See Proposition 5.5 in Folland (1999). \(\square\)

Basically, there are two main reasons for our interest in \(X_\mathbb{R}\) and the appertaining real-linear functionals. First of all, the behavior of the real-linear functionals on \(X\) is sufficient for the characterization of weak compactness in James’ Weak Compactness Theorem. Secondly, temporary restriction to \(X_\mathbb{R}\) will simplify our arguments in section 7, where we prove the theorem.

On a completely different note, the next result will play an important role in some of our applications of James’ Weak Compactness Theorem. Although the proof is straightforward, we include it here for completeness (and in lack of a good reference).
Theorem 1.7. Let $X$ be a topological space and suppose that $f : X \to \mathbb{R}$ is a continuous real-valued function. Then $\sup \{ f(x) : x \in A \} = \sup \{ f(x) : x \in \overline{A} \}$ for any subset $A$ of $X$.

Proof. Given a subset $A$ of $X$, let $s := \sup \{ f(x) : x \in \overline{A} \}$. If $A = \emptyset$, then the claim is trivial, so we may assume that $A \neq \emptyset$. Clearly, we have $\sup \{ f(x) : x \in A \} \leq s$ since $A \subseteq \overline{A}$. Hence $s$ is an upper bound of $\{ f(x) : x \in A \}$. It remains to show that $s$ is the smallest upper bound.

Under the assumption that $s < \infty$ (as is sometimes required for the supremum to exist), we need to show that, for every $\varepsilon > 0$, there exists $x \in A$ such that $s - \varepsilon < f(x)$. To this end, let $\varepsilon > 0$ be given. Since $s$ is the supremum of $\{ f(x) : x \in \overline{A} \}$, there exists $a' \in \overline{A}$ such that $s - \varepsilon / 2 < f(a')$. Then, by continuity of $f$, there exists an open neighborhood $U$ of $a'$ such that $|f(x) - f(a')| < \varepsilon / 2$ whenever $x \in U$. In particular, we have $f(a') - \varepsilon / 2 < f(x)$ whenever $x \in U$. Now observe that $U \cap A \neq \emptyset$ by the fact that $a' \in \overline{A}$. Hence there exists $a \in A$ such that $f(a') - \varepsilon / 2 < f(a)$. But then $f(a) > f(a') - \varepsilon / 2 > (s - \varepsilon / 2) - \varepsilon / 2 = s - \varepsilon$ as desired. This proves that $s = \sup \{ f(x) : x \in A \}$ in the cases where $s < \infty$.

The case where $s = \infty$ proceeds similarly. First of all, fix an $N \in \mathbb{N}$. Then there exists an $a' \in \overline{A}$ such that $f(a') > N + 1 / 2$. Moreover, by continuity of $f$, there is an open neighborhood $U$ of $a'$ such that $f(a') - 1 / 2 < f(x)$ whenever $x \in U$. But $U \cap A \neq \emptyset$ since $a' \in \overline{A}$, so this implies that $f(a') - 1 / 2 < f(a)$ for some $a \in A$. Hence $f(a) > f(a') - 1 / 2 > (N + 1 / 2) - 1 / 2 = N$ for some $a \in A$. Since $N \in \mathbb{N}$ was arbitrary, this proves that $\sup \{ f(x) : x \in A \} = \infty$. □

Finally, we remind the reader that the notions of a bounded linear functional (not to be confused with a bounded image) and a continuous linear functional are equivalent. In line with tradition, we shall prefer the term bounded, except when we explicitly wish to stress the continuity (as we e.g. do in Sections 2 and 3 below).
2. Motivation

We begin our journey by considering some motivating theorems from analysis and topology. First and foremost, we present a slight generalization of the Extreme Value Theorem, which is one of the staples of analysis and ordinary calculus.

**Theorem 2.1. (Extreme Value Theorem)** Let \( A \) be a compact subset of a topological space \( X \). Then any continuous real-valued function \( f : A \to \mathbb{R} \) attains its supremum and infimum on \( A \). Analogously, any continuous complex-valued function \( f : A \to \mathbb{C} \) attains the supremum and infimum of its absolute value \( |f| \) on \( A \).

**Proof.** For the first statement, observe that \( f(A) \) is compact in \( \mathbb{R} \) by continuity of \( f \). Hence \( f(A) \) is closed and bounded in \( \mathbb{R} \) by the Heine-Borel Theorem. It follows that \( \sup \{ f(a) \mid a \in A \} = \sup \{ f(A) \} \in f(A) = f(A) \) and likewise for the infimum. The second statement follows from the first by noting that \( |f| = |\cdot| \circ f \) is a continuous real-valued function. □

Perhaps even more interestingly, it turns out that whenever \( X \) is metrizable, the converse of the Extreme Value Theorem is also true. This paves the way for a complete characterization of compactness in metric spaces by means of continuous functions.

**Theorem 2.2.** Let \( X \) be a metrizable topological space and let \( A \) be subset of \( X \). If every continuous real-valued function \( f : A \to \mathbb{R} \) is bounded on \( A \), then \( A \) is compact in \( X \).

**Proof.** We proceed by contraposition and show that if \( A \) is not compact, then there exists an unbounded continuous function \( f : A \to \mathbb{R} \). Since \( A \) inherits the metrizability of \( X \), \( A \) is in particular a Normal space. Hence the Tietze Extension Theorem (Theorem B.1) applies to \( A \), so it is enough to construct an unbounded continuous function \( f : S \to \mathbb{R} \) on a closed subset \( S \) in \( A \). Now, since \( A \) is not compact, it also not sequentially compact by virtue of being metrizable. Therefore, there is an infinite sequence \( (a_n)_{n \in \mathbb{N}} \) in \( A \) with no convergent subsequence. Let \( (a_{n_k})_{k \in \mathbb{N}} \) be a subsequence where no two terms agree. Then \( S := \{a_{n_k} \mid k \in \mathbb{N}\} \) is a countably infinite discrete subset of \( A \). Since it has no limit points in \( A \) (if it had a limit point, then by \( T_1 \)-ness we could choose a subsequence converging to it), it is closed in \( A \). Now let \( f : S \to \mathbb{R} \) be given by \( f(a_{n_k}) = k \), which is well-defined since no two terms in \( S \) are equal. Then \( f \) is clearly unbounded. Also, \( f \) is automatically continuous because \( S \) is discrete. Thus, \( f \) extends to a continuous function on all of \( A \) by the Tietze Extension Theorem (Theorem B.1). Since \( f \) is unbounded on \( S \subseteq A \), it is also unbounded on \( A \). □

By combining Theorems 2.1 and 2.2, we obtain, as promised, a highly instructive characterization of compactness in terms of continuous real-valued functions.

**Theorem 2.3.** Let \( A \) be a subset of a metrizable topological space \( X \). Then the following statements are equivalent.

1. \( A \) is compact
2. Every real-valued continuous function \( f : A \to \mathbb{R} \) attains its supremum and infimum on \( A \)
3. Every real-valued continuous function \( f : A \to \mathbb{R} \) is bounded on \( A \)

**Proof.** The first implication \((1) \Rightarrow (2)\) follows from Theorem 2.1. The second implication \((2) \Rightarrow (3)\) is trivial. Finally, \((3) \Rightarrow (1)\) follows from Theorem 2.2. □

It is an interesting question whether a similar result to Theorem 2.3 can be obtained for continuous linear functionals on normed spaces. A priori, there is no reason to suspect that Theorem 2.3 should generalize to this situation since, on a given normed space, the set of all continuous linear functionals is not even dense in the set of all continuous scalar-valued functions.

In finite dimensions, however, the answer turns out to be a fairly straightforward “yes” if only we demand that \( A \) is closed. This is the topic of Section 3 below. In infinite dimensions, on the other hand, we need to look for a coarser topology with a
natural connection to the continuous linear functionals: This will be the so-called weak topology, which we introduce in Section 5. The weak topology is not only intricately linked to the set of continuous linear functionals, it also shares many useful properties with the metric topologies.

As such, it is worth keeping in mind that it was precisely metrizability that brought us the nice characterization of compactness in Theorem 2.3. In particular, we used the equivalence of compactness and sequential compactness in Theorem 2.2, which is also true in the weak topology by virtue of the remarkable Eberlein-Šmulian Theorem (see Theorem 6.5).
3. The Finite Dimensional Case

In the cases where $X$ is a finite dimensional Banach space, the proof of James’ Weak Compactness Theorem is immensely simplified. The main reason for this simplification is the simple fact that we have the Heine-Borel Theorem at our disposal. Something which we generally do not have in infinite dimensions.

In addition, the norm topology agrees with the weak topology precisely for finite dimensional Banach spaces (see Theorem 5.22), so James’ Weak Compactness Theorem becomes a statement about standard compactness. Thus, the finite dimensional version of James Weak Compactness Theorem is essentially the best we can do if we want to restrict attention to linear functionals in Theorem 2.3. Specifically, we need to restrict ourselves to normed spaces (rather than metric spaces) and we need to require that $A$ is closed.

**Theorem 3.1. (Finite Dimensional James’ Weak Compactness Theorem)** Let $X$ be a finite dimensional Banach space. Suppose that $A$ is a nonempty closed (equivalently, weakly closed) subset of $X$. Then the following statements are equivalent.

1. $A$ is compact (equivalently, weakly compact)
2. Every continuous linear functional $f$ on $X$ attains the supremum of its absolute value $|f|$ on $A$.
3. Every continuous real-linear functional $f$ on $X$ attains the supremum of its absolute value $|f|$ on $A$.
4. Every continuous real-linear functional $f$ on $X$ attains its supremum on $A$.

**Proof.** Observe first that the implications (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), and (1) $\Rightarrow$ (4) all follow from Theorem 2.1. We finish the proof by showing that the implications (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1), and (4) $\Rightarrow$ (3). For this, recall that all linear functionals on $X$ are continuous by the finite dimensionality.

We begin by proving (2) $\Rightarrow$ (1). Since $A$ is assumed closed, it follows from the finite dimensionality of $X$ together with the Heine-Borel Theorem that it suffices to show that $A$ is bounded. Moreover, since any two norms on a finite dimensional Banach space are equivalent, it suffices to prove that $A$ is bounded in the $\ell_2$-norm. To this end, fix a basis $\{e_1, \ldots, e_n\}$ for $X$ and write the elements of $X$ as $x = \lambda_1 e_1 + \cdots + \lambda_n e_n$ where $\lambda_i \in \mathbb{F}$. Then the scalar-valued functions $\pi_i : X \to \mathbb{F}$ defined by $\pi_i(\lambda_1 e_1 + \cdots + \lambda_n e_n) = \lambda_i$ are linear functionals for each $i = 1, \ldots, n$, so it follows from (2) that each $|\pi_i|$ attains its supremum on $A$. But then it holds for every $x \in A$ that

$$\|x\|_2^2 \leq \sum_{i=1}^n |\lambda_i|^2 \leq \sum_{i=1}^n \left( \sup \{|\pi_i(x) : x \in A|\} \right)^2.$$

Hence $A$ is bounded and therefore $A$ is compact.

Now consider the implication (3) $\Rightarrow$ (1). If $\mathbb{F} = \mathbb{R}$, then the real-linear functionals on $X$ are linear functionals in the normal sense. Hence the proof is analogous to the above. If $\mathbb{F} = \mathbb{C}$, we appeal to Theorem 1.6 and write the complex-linear functionals in terms of real-linear functionals. Specifically, we may write $\pi_i(\lambda_1 e_1 + \cdots + \lambda_n e_n) = \text{Re}\lambda_i + i \text{Im}\lambda_i$, where $\lambda_1 e_1 + \cdots + \lambda_n e_n \mapsto \text{Re}\lambda_i$ and $\lambda_1 e_1 + \cdots + \lambda_n e_n \mapsto \text{Im}\lambda_i$ are the real-linear functionals $\text{Re}\pi_i$ and $\text{Im}\pi_i$ on $X$. It follows that

$$\|x\|_2^2 = \left\| \sum_{i=1}^n \lambda_i e_i \right\|_2^2 = \sum_{i=1}^n |\lambda_i|^2 = \sum_{i=1}^n (\text{Re}\lambda_i)^2 + (\text{Im}\lambda_i)^2 \leq \sum_{i=1}^n \left( \sup \{|(\text{Re}\pi_i)(x) : x \in A|\} \right)^2 + \left( \sup \{|(\text{Im}\pi_i)(x) : x \in A|\} \right)^2,$$

so $A$ is bounded in the $\ell_2$-norm. Since $A$ is closed and bounded, we conclude that $A$ is compact.

Finally, we prove the implication (4) $\Rightarrow$ (3). Let $f$ be an arbitrary real-linear functional on $X$ and recall that $\inf \{f(x) : x \in A\} = - \sup \{-f(x) : x \in A\}$. Since $-f$ is also a linear functional on $X$, both the infimum and supremum of $f$ are thus attained.
We conclude that
\[
\sup \{|f(x)| : x \in A\} = \max \left\{ \sup \{f(x) : x \in A\}, \inf \{f(x) : x \in A\} \right\}
\]
is attained for every real linear functional \(f\) on \(X\). This finishes the proof. \(\square\)

Recall the basic fact that every finite dimensional normed space over \(\mathbb{C}\) or \(\mathbb{R}\) is complete (see Theorem 4.7 for a stronger result). Hence there are no restrictions involved in the requirement that \(X\) should be a Banach space in the above theorem. On the other hand, as we shall see in Example 3.2 below, the assumption that the subset \(A\) should be closed in \(X\) is paramount for the implications \((2),(3),(4) \Rightarrow (1)\) to hold. Observe that this assumption was not needed in Theorem 2.3, when we were not restricting our attention to linear functionals.

**Example 3.2.** Let \(X\) be a finite dimensional Banach space. We show that there are subsets in \(X\) on which every continuous linear functional attains its supremum despite the sets not being compact.

To see this, let \(K\) be any closed and bounded subset of \(X\) with nonempty interior. Then \(\sup \{|f(x)| : x \in K\}\) is attained for every continuous linear functional on \(X\) by the Extreme Value Theorem (Theorem 2.1). We claim that the supremum is always attained on the boundary of \(K\). To see this, fix an arbitrary continuous linear functional on \(X\). Notice first that the only constant linear functional is the zero functional, which trivially attains its supremum everywhere on \(K\). Hence we may assume that \(f\) is not constant.

Suppose that the supremum is attained at some \(x_0 \in K^\circ\). Then \(x_0 \in B(x_0, \varepsilon) \subseteq K\) for some \(\varepsilon > 0\). Since \(f\) is not the zero functional, and its supremum is attained at \(x_0\), we must have \(f(x_0) \neq 0\). Take \(\lambda := 1 + \frac{\varepsilon}{2\|x_0\|} > 1\). Then \(\lambda x_0 \in B(x_0, \varepsilon)\) since \(\|\lambda x_0 - x_0\| = \|1 - \lambda\| \|x_0\| = \frac{\varepsilon}{2\|x_0\|} \|x_0\| = \frac{\varepsilon}{2}\). By linearity, we have \(|f(\lambda x_0)| = |\lambda| |f(x_0)| > |f(x_0)|\), which is clearly a contradiction. Hence we conclude that the supremum is attained on the boundary \(\partial K\).

Now let \(C \subseteq K^\circ\) be a proper closed subset of the interior of \(K\). Then \(\partial K \subseteq K \setminus C\). Hence every continuous linear functional on \(X\) still attains its supremum on \(A := K \setminus C\) by the previous result. Yet \(A\) is not compact precisely because it fails to be closed. This clearly stresses the necessity of the assumption that \(A\) should be closed in Theorem 3.1 above. \(\diamond\)
4. Topological Vector Spaces

We do not always have the luxury of a norm, and even when we do, the topology that it induces may not be the most practical for our purposes. Moreover, it is quintessential for a full understanding of normed spaces that we know exactly what properties drive which behavior. Indeed, for many of the key properties of normed spaces, the heart of the matter turns out to be the continuity of the vector space operations rather than any peculiarities of the norm. This leads the way for a natural generalization of normed spaces to vector spaces equipped with a topology that makes addition and scalar multiplication continuous. It is precisely this link between the vector space structure of $X$ and its topology that allows us to join the forces of linear algebra and analysis.

**Definition 4.1.** Let $X$ be a vector space over $\mathbb{F}$. If $\tau$ is a topology on $X$ such that $(x, y) \mapsto xy$ is continuous from $X \times X$ into $X$ and $(\lambda, x) \mapsto \lambda x$ is continuous from $\mathbb{F} \times X$ into $X$, then $\tau$ is a vector topology for $X$ and $(X, \tau)$ is called a Topological Vector Space (abbreviated TVS).

In order to generalize key results from normed spaces, such as the Hahn-Banach Theorems, we also need the existence of convex neighborhoods around every point in $X$. This motivates the following definition.

**Definition 4.2.** Let $(X, \tau)$ be a Topological Vector Space. If $\tau$ has a basis consisting only of convex sets, then $\tau$ is a locally convex topology and $(X, \tau)$ is called a Locally Convex Space (abbreviated LCS).

Observe that a TVS is, in particular, a topological group since the inversion map $x \mapsto -x$ is just multiplication by $-1$, which is assumed continuous. It is a testimony to the sensibility of the above definitions that every normed space is a LCS.

**Theorem 4.3.** Let $X$ be a vector space equipped with a norm topology $\tau_{\|\cdot\|}$. Then $(X, \tau_{\|\cdot\|})$ is a LCS.

**Proof.** The vector space operations are clearly continuous and the collection of open balls constitutes a basis of convex sets. \hfill \Box

We shall need the notion of a balanced set as we go along. Hence we include a formal definition here.

**Definition 4.4.** Let $A$ be any subset of a vector space $X$. If $\alpha A \subseteq A$ whenever $|\alpha| \leq 1$, then we say that $A$ is balanced.

It is immediate from the definition that any open or closed ball in the norm topology centered at $0$ is balanced. As the following theorem shows, this property generalizes to arbitrary topological vector spaces.

**Theorem 4.5.** Let $(X, \tau)$ be a TVS. Then every open neighborhood of $0$ contains a balanced open neighborhood of $0$.

**Proof.** Let $U \in \tau$ be an open neighborhood of $0$. By continuity of scalar multiplication there exist an open ball $B(0, \delta)$ in $\mathbb{F}$ and an open neighborhood $V$ of $0$ in $X$ such that $\lambda V = \{\lambda v : v \in V\} \subseteq U$ whenever $\lambda \in B(0, \delta)$. Now define a new open neighborhood $V' := \bigcup_{\lambda \in \mathbb{F}, |\lambda| \leq \delta} \lambda V$. By construction, $V' \subseteq U$ and $0 \in V'$. Let $\alpha$ be an arbitrary scalar with $|\alpha| \leq 1$ and observe that if $y = \alpha \lambda v$ for some $v \in V$ and some $|\lambda| \leq \delta$, then $|\alpha\lambda| = |\alpha||\lambda| \leq \delta$ implies that $y \in V'$. Hence

$$\alpha V' = \bigcup_{\lambda \in \mathbb{F}, |\lambda| \leq \delta} (\alpha \lambda) V \subseteq \bigcup_{\lambda \in \mathbb{F}, |\lambda| \leq \delta} \lambda V = V'. $$

for any $\alpha \in \mathbb{F}$ with $|\alpha| \leq 1$. Consequently, $V'$ is a balanced open neighborhood of $0$ contained in $U$. \hfill \Box

An extremely useful idea in the theory of metric spaces (and hence normed spaces) is the notion of boundedness. A subset of a metric space is said to be bounded if it is included in an open ball. In a normed space we thus have that a subset $A$ is bounded if
and only if there exists \( M \in \mathbb{R} \) such that \( A \subseteq B(0, r) = rB(0, 1) \) for every \( r > M \). The following definition generalizes this idea to topological vector spaces.

**Definition 4.6.** A subset \( A \) of a TVS \( (X, \mathcal{T}) \) is said to be *bounded* if, for each open neighborhood \( U \in \mathcal{T} \) of 0, there exists \( s_U > 0 \) such that \( A \subseteq s_U U \) for every \( t > s_U \).

By the above remarks, it is clear that a subset of a normed space is bounded according to the metric space definition if and only if it is bounded according to the present definition. In what follows, the term “bounded” shall therefore always refer to Definition 4.4 unless we specifically state otherwise.

We finish this section by showing that the finite dimensional Hausdorff topological vector spaces are very well-known objects. In fact, they are simply copies of the euclidean vector spaces.

**Theorem 4.7.** Suppose \( (X, \mathcal{T}) \) is a finite dimensional Hausdorff TVS. Then \( (X, \mathcal{T}) \) is linearly isomorphic and homeomorphic with \( \mathbb{F}^n \) in the euclidean topology.

**Proof.** Let \( \{v_1, \ldots, v_n\} \) be a basis for \( X \) and let \( \{e_1, \ldots, e_n\} \) be the standard basis for \( \mathbb{F}^n \). Based on this, we define a linear functional \( T : (\mathbb{F}^n, \|\cdot\|_2) \to (X, \mathcal{T}) \) by \( T(e_i) := v_i \) such that

\[
T(\lambda_1 \ldots, \lambda_n) = T(\lambda_1 e_1 + \cdots + \lambda_n e_n) = \lambda_1 v_1 + \cdots + \lambda_n v_n
\]

for each \( (\lambda_1, \ldots, \lambda_n) \in \mathbb{F}^n \). It is immediate that \( T \) is a linear isomorphism from \( \mathbb{F}^n \) onto \( X \). Hence it only remains to show that \( T \) and \( T^{-1} \) are continuous with respect to any Hausdorff vector topology \( \mathcal{T} \) on \( X \) and the euclidean topology \( \mathcal{T}_{\|\cdot\|_2} \) on \( \mathbb{F}^n \).

That \( T \) is continuous follows easily from the \( \mathcal{T}\)-continuity of scalar multiplication and vector addition. To see that \( T^{-1} \) is continuous, consider the unit sphere \( S_{\mathbb{F}^n} \) of \( \mathbb{F}^n \). By the Heine-Borel property of \( \mathbb{F}^n \), \( S_{\mathbb{F}^n} \) is compact. Hence \( T(S_{\mathbb{F}^n}) \) is compact in \( X \) by the continuity of \( T \). Since \( X \) is Hausdorff, \( T(S_{\mathbb{F}^n}) \) is in particular closed in \( X \). Notice that \( 0 = T(0) \notin T(S_{\mathbb{F}^n}) \) since \( 0 \notin S_{\mathbb{F}^n} \). Hence \( X \setminus T(S_{\mathbb{F}^n}) \) is an open neighborhood of 0, so by Theorem 4.5 there is a balanced open neighborhood \( U \) of 0 contained in \( X \setminus T(S_{\mathbb{F}^n}) \). By construction \( U \cap T(S_{\mathbb{F}^n}) = \emptyset \), so \( T^{-1}(U) \cap S_{\mathbb{F}^n} = \emptyset \). Moreover, by linearity of \( T^{-1} \) and balancedness of \( U \), we have \( \alpha T^{-1}(U) = T^{-1}(\alpha U) \subseteq T^{-1}(U) \) whenever \( |\alpha| \leq 1 \). Hence \( T^{-1}(U) \) is balanced.

Now observe that \( \|v\|_2 < 1 \) for every \( v \in T^{-1}(U) \). To see this, simply note that if \( \|v\|_2 \geq 1 \) for some \( v \in T^{-1}(U) \), then \( 1/\|v\|_2 \leq 1 \) implies that \( v/\|v\|_2 \in T^{-1}(U) \) by the balancedness of \( T^{-1}(U) \). This clearly contradicts \( T^{-1}(U) \cap S_{\mathbb{F}^n} = \emptyset \).

Finally, let \( \pi_i \circ T^{-1} : X \to \mathbb{F} \) denote the coordinate maps of \( T^{-1} \) and observe that each \( \pi_i \circ T^{-1} \) is linear. Since \( \|v\|_2 < 1 \) for every \( v \in T^{-1}(U) \), we have \( |\pi_i(T^{-1}(x))| \leq \|T^{-1}(x)\|_2 \leq 1 \) for every \( x \in U \). Now let \( \varepsilon > 0 \) be given. By linearity the previous result implies that \( |(\pi_i \circ T^{-1})(x)| \leq \varepsilon \) whenever \( x \in \varepsilon U \). Hence

\[
|\pi_i(T^{-1}(x) - y)| = |(\pi_i \circ T^{-1})(x - y)| < \varepsilon
\]

whenever \( y \in x + \varepsilon U \). Furthermore, note that \( x + \varepsilon U \) is open since \( (X, \mathcal{T}) \) is a TVS. It follows that each coordinate map of \( T^{-1} \) is continuous and therefore \( T^{-1} \) itself is continuous.

It is an immediate consequence of Theorem 4.7 that the unique Hausdorff vector space topology on a finite dimensional vector space is induced by a Banach norm. Furthermore, it follows from Theorems 4.3 and 4.7 that any two norms on a finite dimensional vector space induce the same topology.
5. Weak Topologies and Dual Spaces

The stronger the topology, the easier it is for functions to be continuous. On the other hand, too many open sets makes it difficult for sets to be compact because of finer open covers. If both continuity and compactness is of interest, we thus face a trade-off in our choice of topology. Given a family of functions $\mathfrak{F}$, Theorem 5.1 below suggests a natural choice of topology in this situation.

**Theorem 5.1.** Let $X$ be a set and let $\mathfrak{F}$ be a family of maps $f : X \to Y_f$, where each $Y_f$ is equipped with a topology $\mathcal{S}_f$. Then there exists a unique weakest topology $\mathcal{S}_\mathfrak{F}$ for $X$ such that each $f \in \mathfrak{F}$ is continuous.

**Proof.** Observe that $\mathfrak{S} := \{ f^{-1}(U) : f \in \mathfrak{F}, U \in \mathcal{S}_f \}$ is a subbasis for a topology on $X$ since the union of the elements in $\mathfrak{S}$ equals $X$. Let $\mathcal{S}_\mathfrak{F}$ be the topology generated by $\mathfrak{S}$. Then each $f \in \mathfrak{F}$ is continuous since $\mathfrak{S} \subseteq \mathcal{S}_\mathfrak{F}$. Now suppose $\mathcal{S}_X$ is another topology for $X$ such that each $f \in \mathfrak{F}$ is continuous. Then by definition of continuity, $\mathfrak{S} \subseteq \mathcal{S}_X$. But, by construction, $\mathcal{S}_\mathfrak{F}$ equals the intersection of all topologies for $X$ containing $\mathfrak{S}$. Hence we have $\mathcal{S}_\mathfrak{F} \subseteq \mathcal{S}_X$ as desired. \qed

The topology $\mathcal{S}_\mathfrak{F}$ in Theorem 5.1 is sometimes called the *initial* or weak topology on $X$ with respect to $\mathfrak{F}$. We shall say that $\mathfrak{F}$ is a *topologizing family* for $X$ and denote $\mathcal{S}_\mathfrak{F}$ by $\sigma(X, \mathfrak{F})$.

In functional analysis, we are mainly interested in the cases where $X$ is a vector space and $\mathfrak{F}$ is a vector space of linear functionals. This leads us to introduce the *algebraic dual space* $X^\# := \{ f : X \to \mathbb{F} \mid f \text{ is linear} \}$ consisting of all the linear functionals on $X$. It is easy to see that $X^\#$ is a vector space. For the remainder of this section we shall consider some important properties of the weak topologies induced by subspaces of $X^\#$.

**Theorem 5.2.** Let $X$ be a vector space over $\mathbb{F}$ and consider a subspace $X'$ of the algebraic dual space $X^\#$. Let $\varepsilon > 0$ be given. Then the collection of all “*open balls*"

$$B_\varepsilon(x, A) := \{ y \in X : |f(y) - f(x)| < \varepsilon \forall f \in A \}, \quad x \in X, A \subseteq X'$$

where $A \subseteq X'$ is finite, constitutes a basis for $\sigma(X, X')$. The subcollection of all sets

$$B_\varepsilon(x, f) := \{ y \in X : |f(y) - f(x)| < \varepsilon \}, \quad x \in X, f \in X'$$

forms a subbasis for $\sigma(X, X')$.

**Proof.** Let $\mathfrak{S} := \{ f^{-1}(U) : f \in X', U \in \mathcal{S}_f \}$ be the subbasis for $\sigma(X, X')$ from Theorem 5.1 and let $\mathfrak{B}$ be the basis consisting of all finite intersections of elements in $\mathfrak{S}$. First observe that each $B_\varepsilon(x, A)$ is open with respect to $\sigma(X, X')$ since

$$B_\varepsilon(x, A) \supseteq \bigcap_{f \in A} \{ y \in X : |f(y) - f(x)| < \varepsilon \}$$

$$= \bigcap_{f \in A} f^{-1}(B(f(x), \varepsilon)),$$

which is a finite intersection of elements from $\mathfrak{S}$. Now let $x \in X$ and consider any basis element $\mathcal{O} \in \mathfrak{B}$ with $x \in \mathcal{O}$. Then $\mathcal{O}$ is of the form $f^{-1}_1(U_1) \cap \cdots \cap f^{-1}_n(U_n)$ for some $f_i \in X'$ and $U_i \in \mathcal{S}_f$. For each $i = 1, \ldots, n$ choose an open ball $B(f_i(x), \delta_i) \subseteq U_i$. Let $\delta := \min \{ \delta_1, \delta_2, \ldots, \delta_n \}$ and define $\varphi_i := \frac{\delta}{\delta_i} f_i$, which is again an element of $X'$ in the natural vector space structure on $X^\#$. Then $y \in B_\varepsilon(x, \{ \varphi_1, \ldots, \varphi_n \})$ implies that $\frac{\delta}{\delta_i} |f_i(y) - f_i(x)| = |\varphi_i(y) - \varphi_i(x)| < \varepsilon$ for all $i = 1, \ldots, n$ or equivalently $|f_i(y) - f_i(x)| < \delta$ for all $i = 1, \ldots, n$. Hence $f_i(y) \in B(f_i(x), \delta_i) \subseteq U_i$ for each $i = 1, \ldots, n$. It follows that $B_\varepsilon(x, \{ \varphi_1, \ldots, \varphi_n \})$ is contained in $\mathcal{O}$. Thus, we conclude that the collection of “open balls” $B_\varepsilon(x, A)$ is indeed a basis for $\sigma(X, X')$. The second statement follows from the fact that each $B_\varepsilon(x, A)$ equals the finite intersection $\bigcap_{f \in A} B_\varepsilon(x, f)$. \qed

**Theorem 5.3.** Let $X$ be a set and let $\mathfrak{F}$ be a topologizing family of functions for $X$. Suppose that $(x_\alpha)_{\alpha \in A}$ is a net in $X$ and $x \in X$. Then $x_\alpha \to x$ with respect to $\sigma(X, \mathfrak{F})$ if and only if $f(x_\alpha) \to f(x)$ for every $f \in \mathfrak{F}$.
Proof. Suppose first that $x_\alpha \to x$ with respect to $\sigma(X, \mathfrak{F})$. By definition of $\sigma(X, \mathfrak{F})$, each $f \in \mathfrak{F}$ is continuous. Hence $f(x_\alpha) \to f(x)$ for every $f \in \mathfrak{F}$ as desired. For the converse, assume instead that $f(x_\alpha) \to f(x)$ for every $f \in \mathfrak{F}$. Then, for each $f \in \mathfrak{F}$, it holds for every neighborhood $U$ of $f(x)$ that there is an $\alpha \in A$ such that $f(x_\alpha) \in U$ whenever $\alpha \geq \alpha_f U$. It follows that for every $\varphi^{-1}(V) \in \{ f^{-1}(U) : f \in \mathfrak{F}, U \in \sigma(X, \mathfrak{F}) \}$ with $x \in \varphi^{-1}(V)$, there is an $\alpha \in A$ such that $x_\alpha \in \varphi^{-1}(V)$ whenever $\alpha \geq \alpha_f V$. Now recall from the proof of Theorem 5.1 that $\{ f^{-1}(U) : f \in \mathfrak{F}, U \in \sigma(X, \mathfrak{F}) \}$ is a subbasis for $\sigma(X, \mathfrak{F})$. Hence $x_\alpha \to x$ with respect to $\sigma(X, \mathfrak{F})$ by Theorem B.2.

**Theorem 5.4.** Let $X$ be a vector space and suppose that $X'$ is subspace of the algebraic dual $X^\#$. Then $\sigma(X, X')$ is locally convex, and therefore $(X, \sigma(X, X'))$ is a LCS.

Proof. Let $(x_\alpha), (y_\alpha)$ and $(\lambda_\alpha)$ be nets in $X$ and $F$, respectively. Assume that $x_\alpha \to x$, $y_\alpha \to y$, and $\lambda_\alpha \to \lambda$ for some $x, y \in X$ and $\lambda \in F$. By continuity of multiplication and addition in $F$, we have

$$f(\lambda_\alpha x_\alpha + y_\alpha) = \lambda f(x_\alpha) + f(y_\alpha) \to \lambda f(x) + f(y) = f(\lambda x + y)$$

for every $f \in X'$. Hence $\lambda_\alpha x_\alpha + y_\alpha \to \lambda x + y$ with respect to $\sigma(X, X')$ by Theorem 5.3. Therefore, multiplication by scalars and vector addition are continuous with respect $\sigma(X, X')$, so it is a vector topology. Furthermore, $\{ B(0, r) : r > 0 \}$ is a basis for $F$, so $\mathcal{G} := \{ f^{-1}(B(0, r)) : f \in X', r > 0 \}$ is a subbasis for $\sigma(X, X')$ by the proof of Theorem 5.1. Since each $f^{-1}(B(0, r))$ is convex, the basis generated by $\mathcal{G}$ consists of convex sets. In conclusion, $(X, X')$ is locally convex.

**Theorem 5.5.** Let $X$ be a vector space and suppose that $X'$ is subspace of the algebraic dual $X^\#$. Then a subset $A$ of $X$ is bounded in $(X, \sigma(X, X'))$ if and only if $f(A)$ is bounded in $F$ for every $f \in X'$.

Proof. Suppose first that $A \subseteq X$ is bounded with respect to $\sigma(X, X')$. Then $f^{-1}(B(0, 1))$ is an open neighborhood of $0$ in $X$, so according to Definition 4.6 there exists $s_U > 0$ such that $A \subseteq tf^{-1}(B(0, 1)) = f^{-1}(tB(0, 1)) = f^{-1}(B(0, t))$ for every $t > s_U$. Hence $f(A) \subseteq f(f^{-1}(B(0, t))) \subseteq B(0, t)$ for every $t > s_U$. This implies that $f(A)$ is bounded in $F$.

For the converse, assume that $f(A)$ is bounded in $F$ for every $f \in X'$. Let $U \subseteq X$ be an open neighborhood of $0$. By definition of $\sigma(X, X')$ there is a basis element $f_1^{-1}(B(0, r_1)) \cap \cdots \cap f_n^{-1}(B(0, r_n)) \subseteq U$. Since each $f_i(A)$ is bounded and each $B(0, r_i)$ is an open neighborhood of $0$, there exists $s_i > 0$ such that $f_i(A) \subseteq tB(0, r_i)$ whenever $t > s_i$, for each $i = 1, \ldots, n$. In particular, letting $s_U := \max \{ s_1, \ldots, s_n \}$ we have $f_i(A) \subseteq tB(0, r_i)$ whenever $t > s_U$ for each $i = 1, \ldots, n$. It follows that $A \subseteq \bigcap_{i=1}^n f_i^{-1}(f_i(A)) \subseteq tf_i^{-1}(B(0, r_i))$ whenever $t > s_U$ for each $i = 1, \ldots, n$ and therefore

$$A \subseteq \bigcap_{i=1}^n f_i^{-1}(B(0, r_i)) = t \left( \bigcap_{i=1}^n f_i^{-1}(B(0, r_i)) \right) \subseteq tU$$

for every $t > s_U$. This proves that $A$ is bounded in $(X, \sigma(X, X'))$.

**Theorem 5.6.** Let $X$ be a vector space and suppose that $X'$ is subspace of the algebraic dual $X^\#$. If $X'$ is infinite dimensional, then every nonempty open set $U \in \sigma(X, X')$ is unbounded with respect to $\sigma(X, X')$.

Proof. Fix $U \in \sigma(X, X')$ with $U \neq \emptyset$. Clearly $x + U$ is bounded if and only if $x$ is bounded, so we may assume that $0 \in U$. Then there is a basis element $f_1^{-1}(B(0, r_1)) \cap \cdots \cap f_n^{-1}(B(0, r_n)) \subseteq U$. Now consider the subspace $\bigcap_{j=1}^m \ker f_j$ of $X$ and notice that $\bigcap_{j=1}^m \ker f_j \subseteq f_1^{-1}(B(0, r_1)) \cap \cdots \cap f_n^{-1}(B(0, r_n)) \subseteq U$. Hence it suffices to show that $\bigcap_{j=1}^m \ker f_j$ is unbounded. Since $X'$ is infinite dimensional, we can choose $f \in X'$ which is not a linear combination of $f_1, \ldots, f_n$. Therefore, $\bigcap_{j=1}^m \ker f_j \not\subseteq \ker f$ by Lemma 7.2, so there exists $x_0 \in \bigcap_{j=1}^m \ker f_j \setminus \ker f$. Now, for any $k \in \mathbb{N}$ we have $kx_0 \in \bigcap_{j=1}^m \ker f_j$ and $f(kx_0) = kf(x_0) \neq 0$. It follows that for any $N \in \mathbb{N}$ we can choose $f(kx_0) \in f \left( \bigcap_{j=1}^m \ker f_j \right)$ with $k > N/|f(x_0)|$ so that $|f(kx_0)| = k|f(x_0)| > N$. 

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Hence \( f \left( \bigcap_{j=1}^{n} \ker f_j \right) \) is unbounded in \( F \) and therefore \( \bigcap_{j=1}^{n} \ker f_j \) is unbounded in \((X, \sigma(X, X'))\) by Theorem 5.5.

A final property of \( \sigma(X, X') \) is worth mentioning. By construction, every member of \( X' \) is continuous with respect to \( \sigma(X, X') \). It is not difficult to show, however, that these are in fact all the continuous linear functionals. It follows that the continuous linear functionals with respect to \( \sigma(X, X') \) are precisely the members of \( X' \).

To verify this claim, it suffices to show that every continuous linear functional on \( X \) can be obtained as a linear combination of finitely many members of \( X' \). To this end, we shall need the result that if \( f, f_1, \ldots, f_n \) are linear functionals on a vector space \( X \), then \( f \in \text{span} \{f_1, \ldots, f_n\} \) if and only if \( \bigcap_{j=1}^{n} \ker f_j \subseteq \ker f \). This result is presented as Lemma 7.2 in Section 7, where a full proof is given.

First of all, observe that if \( f \) is any continuous linear functional on \( X \), then the preimage \( f^{-1}(B(0, r)) \) is an open neighborhood of 0 in \( X \) for some fixed \( r > 0 \). Hence there is a basis element \( f^{-1}(B(0, r_1)) \cap \cdots \cap f^{-1}(B(0, r_n)) \) around 0 contained in \( f^{-1}(B(0, r)) \), where \( f_1, \ldots, f_n \in X' \). Now, if \( x \in \bigcap_{j=1}^{n} \ker f_j \), then \( f_j(kx) = f_j(x) = 0 \) for each \( j = 1, \ldots, n \), so \( kx \in f^{-1}(B(0, r_1)) \cap \cdots \cap f^{-1}(B(0, r_n)) \subseteq f^{-1}(B(0, r)) \) for any \( k \in \mathbb{N} \). Thus, if \( f(x) \neq 0 \) we would have \( |f(kx)| = k|f(x)| > r \) whenever \( k > r/|f(x)| \), which clearly contradicts \( kx \in f^{-1}(B(0, r)) \) for all \( k \in \mathbb{N} \). Consequently, \( x \in \ker f \) and therefore \( \bigcap_{j=1}^{n} \ker f_j \subseteq \ker f \). The claim now follows from the aforementioned result.

5.1. The Dual Space

Given a normed space \( X \), the set of bounded linear functionals \( f : X \to F \) constitutes a particularly important subspace of \( X^\# \). This leads us to the following definition.

**Definition 5.7.** Let \( X \) be a normed space. Then the dual space of \( X \) is the normed space \( B(X, F) \), where \( B(X, F) \) is the set of all bounded linear functionals on \( X \) equipped with the operator norm. It is denoted by \( X^* \).

We remind the reader that \( X^* \) is always a Banach space even if \( X \) is not complete. Since the topologizing family \( X^* \) appears so frequently in functional analysis, we reserve a special name for the corresponding topology \( \sigma(X, X^*) \).

**Definition 5.8.** Let \( X \) be a normed space. The topology \( \sigma(X, X^*) \) on \( X \) induced by the topologizing family \( X^* \) is called the weak topology of \( X \).

When the context is clear, we shall denote the weak topology by \( \sigma \) and use the qualifications \( w, \text{ weak}, \text{ or weakly} \) to refer to its topological properties.

**Theorem 5.9.** Let \( X \) be a normed space equipped with the weak topology. Then \((X, \sigma)\) is a completely regular LCS.

**Proof.** First observe that \((X, \sigma(X, X^*))\) is a LCS by Theorem 5.4. Next, notice that \( X^* \) is a separating family of functions by Corollary A.4. Hence Theorem B.3 applies. Consequently, it follows from the complete regularity of \( F \) that \((X, \sigma(X, X^*))\) is completely regular.

**Theorem 5.10.** Let \( X \) be normed space. Suppose \( V \subseteq X \) is a linear subspace of \( X \). Then the weak topology \( \sigma(V, V^*) \) on \( V \) equals the subspace topology that \( V \) inherits as a subspace of \((X, \sigma(X, X^*))\). In particular, if \( A \subseteq V \) is any subset of \( V \), then the subspace topology on \( A \) induced by \( \sigma(X, X^*) \) agrees with the subspace topology on \( A \) induced by \( \sigma(V, V^*) \).

**Proof.** Recall from Theorem 5.2 that, given \( \epsilon > 0 \), the collection \( \mathcal{B}_X \) of “open balls” \( \mathcal{B}_X(x, S) := \{y \in X : \|x^*(y - x)\| < \epsilon \forall x^* \in S \} \) for all \( x \in X \) and all finite subsets \( S \subseteq X^* \) constitutes a basis for \( \sigma(X, X^*) \). As a linear subspace, \( V \) is itself a normed space and hence has a dual space \( V^* \). The collection \( \mathcal{B}_V \) of all “open balls” \( \mathcal{B}_V(x, S) \) for \( x \in V \) and \( S \) a finite subset of \( V^* \) is a basis for \( \sigma(V, V^*) \).

Now consider the basis \( \mathcal{B}_V \) for the subspace topology on \( V \) induced by \( \sigma(X, X^*) \) and fix an arbitrary basis element \( V \cap \mathcal{B}_V(x, S) \). Let \( i : V \hookrightarrow X \) be the inclusion map and
define \( v^*_{x} := x^* \circ i \) for each \( x^* \in S \). Clearly \( v^*_{x} \in V^* \) for each \( x^* \in S \). Notice that \( x, y \in V \) implies \( y - x \in V \) and hence
\[
B^Y_v (x, \{ v^*_{x} : x^* \in S \}) = \{ y \in V : |v^*_{x} (y - x)| < \varepsilon \text{ for all } v^* \text{ with } x^* \in S \} 
\]
\[
= \{ y \in V : |(x^* \circ i) (y - x)| < \varepsilon \text{ for all } x^* \in S \} 
\]
\[
= \{ y \in V : |x^*_v (y - x)| < \varepsilon \text{ for all } x^*_v \text{ with } v^* \in S \} 
\]
\[
= V \cap \{ y \in X : |x^*_v(y - x)| < \varepsilon \text{ for all } x^*_v, \text{ with } v^* \in S \} 
\]
\[
= V \cap B^X_x (x, S). 
\]
It follows that \( B^Y_v \subseteq B^X_x \). 

Conversely, fix an element \( B^Y_v (x, S) \) of \( B^X_x \). Then each \( v^* \in S \) has a Hahn-Banach extension \( x^*_v, \in X^* \) with \( x^*_v \circ i = v^* \). Again it follows from \( y - x \in V \) that
\[
B^Y_v (x, S) = \{ y \in V : |v^*(y - x)| < \varepsilon \text{ for all } v^* \in S \} 
\]
\[
= \{ y \in V : |x^*_v(y - x)| < \varepsilon \text{ for all } x^*_v \text{ with } v^* \in S \} 
\]
\[
= V \cap \{ y \in X : |x^*_v(y - x)| < \varepsilon \text{ for all } x^*_v, \text{ with } v^* \in S \} 
\]
\[
= V \cap B^X_x (x, \{ x^*_v : v^* \in S \}). 
\]
Hence we also have \( B^X \subseteq B^X \). Thus \( B^Y_v = B^X \), which proves the first statement.

For the second statement, note that \( U \subseteq A \) is open in the subspace topology induced by \( \sigma(V, V^*) \) if and only if \( U = A \cap O_V \) for some \( O_V \in \sigma(V, V^*) \). By the first statement, \( \sigma(V, V^*) \) equals the subspace topology on \( V \) induced by \( \sigma(X, X^*) \), from whence it follows that \( O_V \in \sigma(V, V^*) \) if and only if \( O_V = V \cap O_X \) for some \( O_X \in \sigma(X, X^*) \). Therefore, \( U = A \cap O_V \) for some \( O_V \in \sigma(V, V^*) \) if and only if \( U = A \cap V \cap O_X = A \cap O_X \) for some \( O_X \in \sigma(X, X^*) \). But \( U = A \cap O_X \) for some \( O_X \in \sigma(X, X^*) \) if and only if \( U \) is open in the subspace topology induced from \( \sigma(X, X^*) \). This proves the second statement. \( \Box \)

Theorem 5.11. Let \( X \) be a normed space. Suppose \((x_n)\) is a net in \( X \) and \( x \in X \). Then \( x_n \xrightarrow{w} x \) if and only if \( x^*(x_n) \to x^*(x) \) for every \( x^* \in X^* \).

Proof. This follows immediately from Theorem 5.3 by considering \( \mathfrak{F} = X^* \). \( \Box \)

Since \((X, \mathfrak{T}_w)\) is Hausdorff, the weak limit of a sequence is unique. Moreover, norm convergence is easily seen to imply weak convergence.

5.2. The Double Dual and Reflexivity

Given a normed space \( X \), the dual space \((X^*)^*\) of the dual space \( X^* \) is also of significant interest. We denote it by \( X^{**} \) and refer to it as the double dual of \( X \).

Definition 5.12. Let \( X \) be a normed space and define a linear map \( J : X \to X^{**} \) by \( x \mapsto J(x) \) where \((J(x))(x^*) = x^*(x) \) for every \( x^* \in X^* \). Then we say that \( J \) is the canonical embedding from \( X \) into \( X^{**} \).

To see that \( J \) is well-defined, simply note that each \( J(x) : X^* \to F \) is linear by the vector space structure on \( X^* \) and bounded because of the following inequality:
\[
\|J(x)\|_{X^{**}} = \sup \{|x^*(x)| : x^* \in B_{X^*} \} \leq \sup \{ \|x^*\|_{X^*} : \|x\|_X \leq 1 \}. 
\]
The next theorem shows that \( J \) is in fact an isometric embedding of \( X \) in \( X^{**} \).

Theorem 5.13. Let \( X \) be a normed space. Then the canonical embedding \( J : X \to X^{**} \) has the following properties:

(1) \( J : X \to X^{**} \) is linear and injective.

(2) \( X \) is isometrically isomorphic to \( J(X) \subseteq X^{**} \).

(3) \( J(X) \) is closed in \( X^{**} \) if and only if \( X \) is a Banach space.

Proof. First note that \( J \) is linear by the linearity of \( x^* \). Hence (1) and (2) both follow if we can show that \( \|J(x)\|_{X^{**}} = \|x\|_X \) for all \( x \in X \), since this automatically implies injectivity of \( J \). We have already seen that \( \|J(x)\|_{X^{**}} \leq \|x\|_X \) in the above computation. To see the reverse inequality, observe that by Corollary A.3 (with
Y = \{0\}) there exists \(x^*_0 \in X^*\) such that \(\|x^*_0\|_{X^*} = 1\) and \(x^*_0(x) = \|x\|_X\). Hence \(|J(x)|_{X^{**}} = \sup \{|x^*(x) : x^* \in B_{X^*}\} \geq |x^*_0(x)| = \|x\|_X\) as desired. In order to prove (3), recall that a subspace of a complete metric space is closed if and only if it is complete. Hence \(J(X)\) is closed in the Banach space \(X^{**}\) if and only if it is complete. By virtue of (2), \(J(X)\) is complete if and only if \(X\) is complete. This proves (3).

\[\square\]

**Definition 5.14.** A normed space \(X\) is said to be reflexive if \(J(X) = X^{**}\).

It is easy to see that every finite dimensional normed space is reflexive. Indeed, the injectivity of \(J\) together with the fact that \(\dim X = \dim X^* = \dim X^{**}\) forces \(J\) to be surjective.

**Definition 5.15.** Let \(X\) be a normed space. Then the topology \(\sigma(X^*, J(X))\) on \(X^*\) induced by the topologizing family \(J(X)\) is called the weak* topology of \(X^*\).

The weak* topology is often denoted by \(\sigma(X^*, J(X))\) in view of the isometric isomorphism \(J(X) \cong X\). When the context is clear, we shall denote it simply by \(\sigma_{w*}\) and use the qualifications \(w^*\), weak* or weakly* to refer to is topological properties.

**Theorem 5.16.** Let \(X^*\) be the dual space of a normed space \(X\). Then \((X^*, \sigma_{w*})\) is a completely regular LCS.

**Proof.** It follows from Theorem 5.4 that \((X^*, \sigma(X^*, J(X)))\) is a LCS. Moreover, we observe that if \(x^* \neq y^*\), then \(x^*(x_0) \neq y^*(x_0)\) for some \(x_0 \in X\), so \((J(x_0))(x_0) \neq (J(y))(x_0)\). Hence \(J(X) \subseteq X^{**}\) is a separating family of functions on \(X^*\). Therefore, Theorem B.3 applies and we conclude from the complete regularity of \(E\) that \((X^*, \sigma(X^*, J(X)))\) is completely regular. \(\square\)

**Theorem 5.17.** Let \(X^*\) be the dual of a normed \(X\). Suppose \((x^*_\alpha)\) is a net in \(X^*\) and \(x^* \in X^*\). Then \(x^*_\alpha \overset{w^*}{\to} x^*\) if and only if \(x^*_\alpha(x) \to x^*(x)\) for every \(x \in X\).

**Proof.** This follows immediately from Theorem 5.3 by considering \(\mathfrak{H} = J(X)\). \(\square\)

By the Hausdorffness of \((X^*, \sigma_{w^*})\), the weak* limit of a convergent sequence in \(X^*\) is unique. Moreover, one easily sees that norm convergence implies weak* convergence.

### 5.3. Boundedness & Comparison of Topologies

**Theorem 5.18.** Let \(X\) be a normed space. Then \(\sigma(X, X^*) \subseteq \sigma_{\|\cdot\|_X} \) on \(X\). On the dual space \(X^*, \) we have \(\sigma(X^*, J(X)) \subseteq \sigma_{\|\cdot\|_{X^*}} \subseteq \sigma_{\|\cdot\|_{X^*}}\).

**Proof.** For the first statement, simply note that all elements of \(X^*\) are continuous with respect to \(\sigma_{\|\cdot\|_X}\) by definition. Hence \(\sigma(X, X^*) \subseteq \sigma_{\|\cdot\|_X}\) since \(\sigma(X, X^*)\) is the weakest topology with respect to which the elements of \(X^*\) are continuous. For the second statement, first observe that every member of \(X^{**}\) is continuous with respect to \(\sigma_{\|\cdot\|_{X^*}}\). By construction, the weak topology \(\sigma(X^*, X^{**})\) on \(X^*\) is weakest topology such that each \(x^{**} \in X^{**}\) is continuous. Hence \(\sigma(X^*, X^{**}) \subseteq \sigma_{\|\cdot\|_{X^*}}\). Finally, every member of \(J(X) \subseteq X^{**}\) is continuous with respect to \(\sigma(X^*, X^{**})\). However, the weak* topology \(\sigma(X^*, J(X))\) on \(X^*\) is the weakest topology for which every element in \(J(X)\) is continuous. Hence \(\sigma(X^*, J(X)) \subseteq \sigma(X^*, X^{**})\). This completes the proof. \(\square\)

As an immediate consequence of Theorem 5.18, we emphasize that compactness in the norm topology implies weak compactness.

In accordance with Definition 4.6, we shall say that a subset \(A\) of a normed space \(X\) is weakly bounded (respectively, weakly* bounded) if, for each weakly open (respectively, weakly* open) neighborhood \(U\) of zero, there exists \(s_U > 0\) such that \(A \subseteq tU\) whenever \(t > s_U\).

**Theorem 5.19.** Let \(X\) be a normed space. Then a subset \(A \subseteq X\) is norm bounded if and only if it is weakly bounded.
Proof. Suppose that $A$ is bounded in the norm topology. Let $U$ be an arbitrary weakly open neighborhood of $0$. Then $U$ is also open in the norm topology by Theorem 5.18. Hence there exists $s_U > 0$ such that $A \subseteq tU$ for every $t > s_U$ by the assumption that $A$ is norm bounded. Since $U$ was an arbitrary weakly open neighborhood of $0$, $A$ is weakly bounded.

Assume conversely that $A$ is weakly bounded. If $A = \emptyset$ the statement is trivial, so we assume $A \neq \emptyset$. Then $J(A)$ is a nonempty family of bounded linear functionals on the Banach space $X^\ast$. Now notice that $x^\ast(A)$ is bounded in $F$ for each $x^\ast \in X^\ast$ by Theorem 5.5 because $A$ is weakly bounded. Hence

$$\sup \{|(J(x))(x^\ast)| : J(x) \in J(A)\} = \sup \{|x^\ast(x)| : x \in A\} < \infty$$

for every $x^\ast \in X^\ast$. Therefore, the Uniform Boundedness Principle (Theorem A.5) applied to $J(A)$ implies that

$$\sup \{|J(x)| : J(x) \in J(A)\} < \infty.$$  

Since $\|x\| = \|J(x)\|$ by Theorem 5.13, it follows that

$$\sup \{\|x\| : x \in A\} = \sup \{|J(x)| : J(x) \in J(A)\} < \infty.$$  

This proves that $A$ is norm bounded. \hfill $\square$

Corollary 5.20. Let $X$ be a normed space. Then a subset $A \subseteq X$ is norm bounded (or equivalently, weakly bounded) if and only if the image $x^\ast(A)$ is bounded in $F$ for every $x^\ast \in X^\ast$.

Proof. According to Theorem 5.5, $A$ is weakly bounded if and only if $x^\ast(A)$ is bounded in $F$ for every $x^\ast \in X^\ast$. To finish the proof, simply note that $A$ is weakly bounded if and only if it is norm bounded by Theorem 5.19. \hfill $\square$

The analogue of Theorem 5.19 for the weak* topology requires $X$ to be complete. The reason being that we need to apply the Uniform Boundedness Principle to linear functionals on $X$. This was not an issue in Theorem 5.19, where we applied the Uniform Boundedness Principle to the linear functionals on $X^\ast$, since $X^\ast$ is automatically complete even if $X$ is not.

Theorem 5.21. Let $X$ be a Banach space. Then a subset of $A \subseteq X^\ast$ is norm bounded if and only if it is weakly* bounded.

Proof. As in Theorem 5.19, the “only if” part follows from the fact every weakly* open neighborhood of $0$ is also an open neighborhood of $0$ in the norm topology by Theorem 5.18. For the “if” part, assume that $A$ is weakly* bounded. If $A = \emptyset$ the statement is trivial, so we assume $A \neq \emptyset$. Then $A$ is a nonempty family of linear functionals on the Banach space $X$. Since $J(X)$ is the topologizing family for the weak* topology, it follows from Theorem 5.5 that

$$(J(x))(A) = \{(J(x))(x^\ast) : x^\ast \in A\} = \{x^\ast(x) : x^\ast \in A\}$$

is bounded in $F$ for every $J(x) \in J(X)$. In other words, $\{x^\ast(x) : x^\ast \in A\}$ is bounded in $F$ for every $x \in X$. Hence

$$\sup \{|x^\ast(x)| : x^\ast \in A\} < \infty$$

for every $x \in X$, so the Uniform Boundedness Principle (Theorem A.5) applied to $A$ ensures that

$$\sup \{|x^\ast| : x^\ast \in A\} < \infty.$$  

This is precisely the statement that $A \subseteq X^\ast$ is norm bounded. \hfill $\square$

It follows from the preceding results that the study of bounded sets is essentially the same in the weak topologies vis-à-vis the norm topology. As the next two theorems show, when $X$ is finite dimensional, then the topological properties are, in fact, also the same. This, however, is far from the truth when $X$ is infinite dimensional.

Theorem 5.22. Let $X$ be a normed space. Then $\sigma(X, X^\ast) = T_{\|\|}$ if and only if $X$ is finite dimensional.
Proof. Suppose that \( X \) is finite dimensional. Recall from Theorems 4.3 and 5.9 that
\((X, \| \cdot \|)\) and \((X, \sigma(X, X^*))\) are both Hausdorff topological vector spaces. Hence it
follows from Theorem 4.7 that \((X, \| \cdot \|)\) and \((X, \sigma(X, X^*))\) are linearly isomorphic and
homeomorphic to \( \mathbb{F}^n \) in the euclidean topology. In particular, the two topologies agree.
For the converse, suppose instead that \( X \) is infinite dimensional. By Theorem 5.18, it is
always the case that \( \sigma(X, X^*) \subseteq \| \cdot \| \). We claim that \( \| \cdot \| \) is strictly finer than \( \sigma(X, X^*) \).
Since \( X \) is infinite dimensional, the dual space \( X^* \) is also infinite dimensional. Hence
every nonempty \( U \in \sigma(X, X^*) \) is unbounded with respect to \( \sigma(X, X^*) \) by Theorem 5.6.
According to Theorem 5.19, this is equivalent to every nonempty \( U \in \sigma(X, X^*) \) being
unbounded in the norm topology. Since, for example, the open balls of finite radii are
bounded in the norm topology, it follows that they cannot be open in the weak topology.
This proves the claim. \( \Box \)

**Theorem 5.23.** Let \( X^* \) be the dual of a normed space. Then \( \sigma(X^*, X^{**}) = \sigma(X^*, J(X)) \)
if and only if \( X \) is reflexive. And \( \sigma(X^*, J(X)) = \| \cdot \| \) if and only if \( X \) is finite dimen-
sional.

Proof. By the remarks following Theorem 5.6 we have \( \sigma(X^*, X^{**}) = \sigma(X^*, J(X)) \) if
and only if \( X^{**} = J(X) \). Since \( X^{**} = J(X) \) is precisely the definition of reflexivity,
this proves the first statement. Now consider the second statement. By Theorems 4.3,
5.9, and 5.16 all three topologies are Hausdorff vector topologies. Hence they agree
whenever \( X \) is finite dimensional by Theorem 4.7. This proves the “if” part. Conversely,
suppose that \( X \) is infinite dimensional. Then \( \sigma(X^*, X^{**}) \) is strictly coarser than \( \| \cdot \| \)
by Theorem 5.22 and therefore \( \sigma(X^*, J(X)) \) is also strictly coarser than \( \| \cdot \| \) since
\( \sigma(X^*, J(X)) \subseteq \sigma(X^*, X^{**}) \) by Theorem 5.18. \( \Box \)

It follows from Theorem 5.21 that any weakly closed subset of \( X \) is also closed. It also
follows, however, that if \( X \) is infinite dimensional, then there are closed sets which are
not weakly closed. Perhaps surprisingly, the next theorem shows that this discrepancy
never arises for convex sets.

**Theorem 5.24.** (Mazur’s Theorem) Let \( X \) be a normed space. Then the closure
and weak closure of any convex subset of \( X \) are the same. In particular, a convex subset
of \( X \) is closed if and only if it is weakly closed.

Proof. Let \( C \) be an arbitrary convex subset of \( X \). By Theorem 5.18 we have \( \| \cdot \| \subseteq \| \cdot \|_3 \),
so the limit points of \( C \) in the norm topology are also weak limit points of \( C \). Hence \( \overline{C}^{w} \subseteq \overline{C}^{1} \).
We claim that \( \overline{C}^{1} \) is convex. To see this, fix \( x, y \in \overline{C}^{1} \) and \( t \in (0, 1) \). Let
\((x_n), (y_n)\) be sequences in \( C \) converging to \( x \) and \( y \), respectively. Then \( tx_n + (1 - t)y_n \in C \)
for all \( n \in \mathbb{N} \) and \( tx_n + (1 - t)y_n \) converges to \( tx + (1 - t)y \). Hence \( tx + (1 - t)y \in \overline{C}^{1} \)
and so \( \overline{C}^{1} \) is convex as claimed.

Now suppose that \( x_0 \in \overline{C}^{w} \setminus \overline{C}^{1} \). Then \( d(x_0, \overline{C}^{1}) > 0 \) since \( \overline{C}^{1} \) is closed. Seeing
that \( \overline{C}^{1} \) is also convex, it follows from the Hahn Banach Separation Theorem (Theorem
A.8) that there exists \( t \in \mathbb{R} \) such that \( \text{Re} f(x_0) > t > \text{Re} f(x) \) for every \( x \in \overline{C}^{1} \). But
then \( (\text{Re} f)^{-1}((t, \infty)) \cap \overline{C}^{1} = \emptyset \) and \( x_0 \in (\text{Re} f)^{-1}((t, \infty)) \). Since \( \text{Re} f : X \rightarrow \mathbb{R} \) is
weakly continuous (see the proof of Lemma 7.6) and \( (t, \infty) \) is open in \( \mathbb{R} \), we conclude
that \( (\text{Re} f)^{-1}((t, \infty)) \) is a weakly open neighborhood of \( x_0 \), which does not intersect
\( \overline{C}^{1} \). This clearly contradicts \( x_0 \in \overline{C}^{w} \). Hence \( \overline{C}^{w} = \overline{C}^{1} \) as desired. \( \Box \)

It is an immediate consequence of Mazur’s Theorem that the closed unit ball \( B_X \) is
also weakly closed. This result should by no means be taken for granted. For example,
it can be shown that \( B_X \) is in fact the weak closure of the unit sphere \( S_X \) whenever \( X \) is
infinite dimensional. Observe also that any linear subspace of a normed space is convex
and therefore closed if and only if it is weakly closed.
6. Weak Compactness

Compactness is one of the most important topological properties in analysis because of its intimate interplay with continuous functions. The Extreme Value Theorem from Section 2 is arguably the most famous example of this. Another important example is the Heine-Cantor Theorem on uniform continuity.

In finite dimensional normed spaces, the Heine-Borel theorem gives a particularly simple characterization of compactness as equivalent to being closed and bounded. This, however, is by no means true for infinite dimensional normed spaces, where even the closed unit ball fails to be compact. Indeed, it turns out that the finite dimensional normed spaces are characterized precisely by the property that every closed and bounded subset is compact. We shall refer to this property as the Heine-Borel property.

**Theorem 6.1.** Let $X$ be a normed space. Then the following statements are equivalent:

1. $X$ is finite dimensional.
2. $X$ has the Heine-Borel property.
3. The closed unit ball $B_X$ is compact.
4. The unit sphere $S_X$ is compact.

**Proof.** Suppose that $\dim X = n$. Then there is a linear homeomorphism $T : X \to \mathbb{R}^n$ by Theorem 4.7. Let $A$ be any closed and bounded subset of $X$. Clearly $T(A)$ is also closed in $\mathbb{R}^n$. Moreover, since $A$ is bounded we have $A \subseteq tT^{-1}(B(0,1)) = T^{-1}(B(0,t))$ for some $t > 0$. Hence $T(A) \subseteq B(0,t)$, so $T(A)$ is bounded in $\mathbb{R}^n$. It follows that $T(A)$ is compact in $\mathbb{R}^n$ by the Heine-Borel Theorem and therefore $A$ must be compact in $X$. This proves that (1) implies (2). That (2) implies (3) is trivial and the implication (3) $\Rightarrow$ (4) follows from the fact that $S_X$ is a closed subset of $B_X$.

In order to prove the implication (4) $\Rightarrow$ (1), we show that $S_X$ fails to be compact whenever $X$ is infinite dimensional. Since compactness implies sequential compactness in metrizable spaces, it suffices to show that there is a sequence $(x_n)$ in $S_X$ with no convergent subsequence. Specifically, we use induction to construct a linearly independent sequence $(x_n)$ with $\|x_n\| = 1$ for every $n \in \mathbb{N}$ and $\|x_n - x_k\| \geq 1$ whenever $n \neq k$. To begin the induction, choose an arbitrary $x_1 \in S_X$. Now assume that $m$ linearly independent elements $x_1, \ldots, x_m \in S_X$ have been selected such that $\|x_n - x_k\| \geq 1$ whenever $n \neq k$. Then $E_m := \text{span}\{x_1, \ldots, x_m\} \subseteq X$ is a subspace of $X$ with dim$E_m = m$. By Theorem 4.7, the finite dimensionality of $E_m$ implies completeness. Hence $E_m$ is closed since every complete subspace of a metric space is closed. Fix $y_m \in X \setminus E_m$ and let $Y_m := \text{span}\{y_m, E_m\}$. Then $Y_m$ is a normed space of dimension $m + 1$. Since $E_m$ is a closed subspace of $Y_m$ and $y_m \in Y_m \setminus E_m$, it follows from Corollary A.3 that there exists a bounded linear functional $f \in Y_m^*$ such that $f(y_m) = d(y_m, E_m) > 0$, $E_m \subseteq \ker f$, and $\|f\| = 1$. Because $Y_m$ is finite dimensional, the unit sphere $S_{Y_m}$ of $Y_m$ is compact by the implication (1) $\Rightarrow$ (4) proved above. Hence it follows from the Extreme Value Theorem (Theorem 2.1) that $\|f\| = \sup \{\|f(y)\| : y \in S_{Y_m}\}$ is attained. Consequently, we can choose $x_{m+1} \in S_{Y_m} \subseteq S_X$ such that $|f(x_{m+1})| = \|f\| = 1$. Note that $x_{m+1} \notin E_m$ since $|f(x_{m+1})| = 1$ and $E_m \subseteq \ker f$. Hence $x_{m+1}$ is linearly independent of $x_1, \ldots, x_m$. Finally, observe that for every $n < m + 1$ we have $x_n \in E_m \subseteq \ker f$ and therefore

$$\|x_{m+1} - x_n\| \geq \frac{|f(x_{m+1} + x_n)|}{\|f\|} = |f(x_{m+1}) + f(x_n)| = |f(x_{m+1})| = 1.$$ 

This completes the induction and thus finishes the proof.

One of the main motivations for the introduction of the weak and weak* topologies in Section 5 was the greater abundance of compact sets. Hence we may hope to recoup the Heine-Borel property in these topologies. For the dual of a Banach space, this is exactly what happens in the weak* topology.

**Theorem 6.2.** Let $X^*$ be the dual of a Banach space. Then $(X^*, T_{w^*})$ has the Heine-Borel property in the sense that every weakly* closed and weakly* bounded subset is weakly* compact.
Proof. First note that the closed unit ball $\bar{B}_X$ is weak$^*$ compact by the Banach-Alaoglu Theorem (Theorem A.6). Since $(X^*, \mathcal{S}_{w^*})$ is a TVS by Theorem 5.16, the maps $x \mapsto \lambda x$ and $x \mapsto \lambda^{-1} x$ are both continuous with respect to the weak$^*$ topology. Hence $\bar{B}_X$ is homeomorphic to $t \bar{B}_X$ for every $t > 0$ and therefore $t \bar{B}_X$ is weakly$^*$ compact for every $t > 0$. Now suppose that $A \subseteq X^*$ is weakly$^*$ closed and weakly$^*$ bounded. Because $X$ is Banach, Theorem 5.21 ensures that $A$ is also norm bounded. Thus, there exists $t > 0$ such that $A \subseteq t \bar{B}_X \subseteq t \bar{B}_X$, since $\bar{B}_X$ is an open neighborhood of 0 in the norm topology. But then $A$ is a weakly$^*$ closed subset of the weakly$^*$ compact set $t \bar{B}_X$, whereby $A$ itself is weakly$^*$ compact. □

In the weak topology on normed spaces, it turns out that we need the further restriction of reflexivity in order to have the Heine-Borel property. This is proved in Theorem 6.4 as a corollary of the next result.

**Theorem 6.3.** A normed space $X$ is reflexive if and only if the closed unit ball $\bar{B}_X$ is weakly compact.

Proof. Suppose that $X$ is reflexive. Then $J(X) = X^{**}$ by definition of reflexivity. Since $||J(x)||_{X^{**}} = ||x||_X$ by Theorem 5.13 we thus have $J(\bar{B}_X) = B_{X^{**}}$. Now recall that $\bar{B}_X$ is weakly$^*$ compact in $X^{**}$ by the Banach-Alaoglu Theorem (Theorem A.6). It follows that $\bar{B}_X$ is weakly compact in $X$ if we can show that $J$ is a homeomorphism with respect to the weak topology $\sigma(X, X^*)$ on $X$ and the weak$^*$ topology $\sigma(X^{**}, J(X^*))$ on $X^{**}$. To this end, let $(x_\alpha)$ be an arbitrary net in $X$, let $x \in X$ and observe that

$$x_\alpha \overset{w}{\to} x \in X \quad \Leftrightarrow \quad x^*(x_\alpha) \to x^*(x) \quad \text{in } F \quad \text{for every } x^* \in X^* \quad \text{[by Theorem 5.11]}$$

$$\Leftrightarrow \quad (J(x_\alpha))(x^*) \to (J(x))(x^*) \quad \text{in } F \quad \text{for every } x^* \in X^*$$

$$\Leftrightarrow \quad J(x_\alpha) \overset{w^*}{\to} J(x) \quad \text{in } X^{**} \quad \text{[by Theorem 5.17]}.$$ 

Hence $J : X \to X^{**}$ is weak-weak$^*$ continuous and $J^{-1} : X^{**} \to X$ is weak$^*$-weak continuous. That is, $J$ is a homeomorphism and therefore $\bar{B}_X$ is weakly compact.

Conversely, assume that $\bar{B}_X$ is weakly compact. Analogously to the above, $J$ is a homeomorphism from $X$ to $\bar{B}_X$ as a subspace of $X^{**}$ in the weak$^*$ topology. Hence $J(\bar{B}_X)$ is weakly$^*$ compact in $X^{**}$. In particular, $J(\bar{B}_X)$ is weakly$^*$ closed in $X^{**}$ since the weak$^*$ topology is Hausdorff by Theorem 5.16. Furthermore, the fact that $J$ is an isometry by Theorem 5.13, implies that $J(\bar{B}_X) \subseteq B_{X^{**}}$. Now, according to Goldstine’s Theorem (Theorem A.5), $J(\bar{B}_X)$ is weakly$^*$ dense in $B_{X^{**}}$. But $J(\bar{B}_X)$ is closed in the weak$^*$ subspace topology on $B_{X^{**}}$ by the preceding remarks. Hence $J(\bar{B}_X) = B_{X^{**}}$. We claim that this implies surjectivity of $J : X \to X^{**}$. To see this, let $x_0^{**} \in X^{**}$ be arbitrary and notice that $x_0^{**} / \|x_0^{**}\| \in B_{X^{**}}$. Thus, it follows from $J(\bar{B}_X) = B_{X^{**}}$ that there exists $x_0 \in \bar{B}_X$ such that $J(x_0) = x_0^{**} / \|x_0^{**}\|$. But then $J(\|x_0^{**}\| x_0) = \|x_0^{**}\| J(x_0) = x_0^{**}$, which proves that $J$ is surjective. Consequently, $X$ is reflexive. □

**Theorem 6.4.** Let $X$ be a normed space and suppose that $X$ is reflexive. Then $(X, \mathcal{S}_w)$ has the Heine-Borel property in the sense that every weakly closed and weakly bounded subset is weakly compact.

Proof. Assume that $X$ is reflexive. Then $\bar{B}_X$ is weakly compact by Theorem 6.3. Since $X$ is a TVS by Theorem 5.9, it follows that $\bar{B}_X$ and $t \bar{B}_X$ are homeomorphic for any $t > 0$. In particular, $t \bar{B}_X$ is weakly compact for every $t > 0$. Now suppose that $A \subseteq X$ is weakly closed and weakly bounded. Then $A$ is also norm bounded by Theorem 5.19. Since the open unit ball $B_X$ is an open neighborhood of 0 in the norm topology, it follows that there exists $t > 0$ such that $A \subseteq t B_X \subseteq t \bar{B}_X$. But then $A$ is a weakly closed subset of the weakly compact set $t \bar{B}_X$. This immediately implies that $A$ itself is weakly compact. □

In fact, the converse of Theorem 6.4 is also true. To see this, simply note that $\bar{B}_X$ is weakly closed by Mazur’s Theorem. Hence the Heine-Borel property would imply that $\bar{B}_X$ is compact and therefore $X$ would be reflexive by Theorem 6.2. It follows that the weak analogue of the Heine-Borel property actually characterizes reflexivity.
6.1. The Eberlein-Šmulian Theorem

Another striking aspect of the weak topology is its similarity to metric spaces. While the weak topology is not metrizable when $X$ is infinite dimensional, it nevertheless shares some surprisingly nice properties with metric spaces. One of the most beautiful results in this respect is undoubtedly the Eberlein-Šmulian Theorem, which we state here without proof.

**Theorem 6.6.** (The Eberlein-Šmulian Theorem) Suppose $A$ is a subset of a normed space $X$. Then the following statements are equivalent:

1. The set $A$ is weakly compact.
2. The set $A$ is weakly countably compact.
3. The set $A$ is weakly limit point compact.
4. The set $A$ is weakly sequentially compact.

The same is true if “weakly” is replaced by “relatively weakly”.

**Proof.** Recall that compact metric spaces are always complete. By means of the Eberlein-Šmulian Theorem we obtain the same result for the weak topology.

**Theorem 6.7.** Every weakly compact subset of a normed space is complete.

**Proof.** Let $X$ be a normed space and suppose that $A$ is a weakly compact subset of $X$. We need to show that every cauchy sequence in $A$ is convergent. To this end, fix an arbitrary cauchy sequence $(x_n)$ in $A$. Observe that $A$ is weakly limit point compact by the Eberlein-Šmulian Theorem. Hence the infinite subset $\{x_n : n \in \mathbb{N}\}$ in $A$ has a weak limit point $x \in A$. Now fix an arbitrary $x^* \in X^*$ and let $\varepsilon > 0$ be given. Since $(x_n)$ is cauchy, there exists $N \in \mathbb{N}$ such that $\|x_n - x_m\| \leq \varepsilon/(2\|x^*\|)$ whenever $n, m \geq N$. Hence $|x^*(x_n) - x^*(x_m)| \leq \|x^*\| \|x_n - x_m\| < \varepsilon/2$ for all $n, m \geq N$.

Moreover, according to Theorem 5.2, $\{y \in X : \|x^*(y - x)\| < \varepsilon/2\}$ is a basic neighborhood of $x$ in the weak topology. Hence there exist $k \geq N$ such that $x_k \in \{y \in X : \|x^*(y - x)\| < \varepsilon/2\}$. That is, $|x^*(x_k) - x^*(x)| < \varepsilon/2$ for some $k \geq N$. It follows that

$$|x^*(x_n) - x^*(x)| \leq |x^*(x_n) - x^*(x_k)| + |x^*(x_k) - x^*(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$}

Hence $x^*(x_n)$ converges to $x^*(x)$ in $F$. Since $x^* \in X^*$ was arbitrary, we have $x^*(x_n) \rightarrow x^*(x)$ in $F$ for every $x^* \in X^*$. Thus, we conclude that $x_n \stackrel{w}{\rightarrow} x$ in $X$ by Theorem 5.11.

We can, however, do even better than this. As such, we claim that $x_n \rightarrow x$ in norm. To see this, simply observe that if we choose $N$ such that $\|x_n - x_m\| < \varepsilon/2$ for all $n, m > N$ in the above, then the inequality $|x^*(x_n) - x^*(x)| < \varepsilon$ holds for all $x^* \in X^*$ with $\|x^*\| \leq 1$ and $n > N$. Letting $J : X \rightarrow X^{**}$ denote the canonical embedding, we thus have

$$\|J(x_n - x)\|_{X^{**}} = \sup \{\|x^*(x_n - x)\| : x^* \in B_{X^*}\} < \varepsilon$$

for all $n > N$. Hence it follows from Theorem 5.13 that $\|x_n - x\| = \|J(x_n - x)\|_{X^{**}} < \varepsilon$ for all $n > N$. Thus $x_n$ converges to $x$ in norm, which proves that every cauchy sequence in $A$ is convergent. Consequently, $A$ is complete.

We finish the section with an important theorem that relates weak compactness to separability. Specifically, we show that weak compactness is completely determined by the separable closed subspaces of the given normed space. This result will prove itself immensely helpful to our plan of attack in Section 7.

**Theorem 6.8.** Let $A$ be a subset of a normed space $X$. Then $A$ is weakly compact if and only if $A \cap Y$ is weakly compact in $Y$ whenever $Y$ is a separable closed subspace of $X$.

**Proof.** Assume $A$ is weakly compact and let $Y$ be any closed subspace of $X$. Then $Y$ is also weakly closed in $X$ by Theorem 5.24, so $A \cap Y$ is a weakly closed subset of the weakly compact set $A$ in $X$. Hence $A \cap Y$ is weakly compact in $X$ and therefore compact in $Y$ with respect to the subspace topology. By Theorem 5.10, the subspace topology
on $Y$ induced by $\sigma(X, X^*)$ is the same as the weak topology $\sigma(Y, Y^*)$ on $Y$. Thus $A \cap Y$ is weakly compact in $Y$.

Conversely, suppose that $A \cap Y$ is weakly compact in $Y$ for every closed separable subspace $Y$ of $X$. Let $(x_n)$ be an arbitrary sequence in $A$. Then $\overline{\text{span}}\left(\{x_n : n \in \mathbb{N}\}\right)$ is a closed and separable subspace of $X$. Indeed, the closed linear span restricted to the subfield $\mathbb{Q}$ if $F = \mathbb{R}$, or $\mathbb{Q}[i]$ if $F = \mathbb{C}$, is a countable dense subset. Hence $A \cap \overline{\text{span}}\left(\{x_n : n \in \mathbb{N}\}\right)$ is weakly compact in $\overline{\text{span}}\left(\{x_n : n \in \mathbb{N}\}\right)$ and therefore weakly sequentially compact in $\overline{\text{span}}\left(\{x_n : n \in \mathbb{N}\}\right)$ by the Eberlein-Šmulian Theorem. It follows that $(x_n) \subseteq A \cap \overline{\text{span}}\left(\{x_n : n \in \mathbb{N}\}\right)$ has a weakly convergent subsequence (with respect to the weak topology on $\overline{\text{span}}\left(\{x_n : n \in \mathbb{N}\}\right)$, that is) whose limit belongs to $A \cap \overline{\text{span}}\left(\{x_n : n \in \mathbb{N}\}\right)$. In particular, the limit must belong to $A$. By Theorem 5.10, the subsequence is also convergent with respect to the weak topology on $X$. Since $(x_n)$ was arbitrary, it follows that $A$ is weakly sequentially compact in $X$ and therefore weakly compact in $X$ by the Eberlein-Šmulian Theorem.

It should be noted that the first three equivalences in the relative version of the Eberlein-Šmulian Theorem in fact follow from James' Weak Compactness Theorem. To see this, it suffices to show that relative weak countably compactness implies relative weak limit point compactness since the remaining implications then follow easily from standard results in topology. As such, suppose that $A$ is relatively weak countably compact as a subset of a normed space $X$. Then $\overline{A^w}$ is weakly countably compact and hence it is easily seen that $x^* \left(\overline{A^w}\right)$ is also weakly countably compact in $F$ for any $x^* \in X^*$ by weak continuity of $x^*$. By metrizability of $F$ we thus have that $x^* \left(\overline{A^w}\right)$ is weakly compact in $F$. But then $\sup\{x^*(x) : x \in \overline{A^w}\}$ is attained for any $x^* \in X^*$. Since $\overline{A^w}$ is weakly closed, this implies that $\overline{A^w}$ is weakly compact in $X$ by James’ Weak Compactness Theorem. In conclusion, $A$ is relatively weakly compact.

The previous paragraph can in some sense be summarized as saying that James’ Weak Compactness Thereom “almost” implies the Eberlein-Šmulian Theorem. This lends us some reassurance that the Eberlein-Šmulian Theorem is not too big a result to utilize in our proof of James’ Weak Compactness Theorem (as we do in Theorem 7.11 via Theorem 6.7). We leave it to the reader to consider if more can be said about the relationship between these two grand theorems.
7. James’ Weak Compactness Theorem

As emphasized in the introduction, Section 7 is truly the heart and soul of this exposition. For the next many pages we shall dig deeper and deeper into the proof of James’ Weak Compactness Theorem. In doing so, we shall follow the lead of R.C. James’ simplified proof from 1972 as expounded in Megginson (1998).

Our strategy is to first prove a version of James’ Weak Compactness Theorem for separable subsets of the closed unit ball. This is done in Theorem 7.5 but requires a technical lemma in the form of Lemma 7.4. After this, we prove what is essentially a stronger version of Lemma 7.4, namely Lemma 7.10. Together with Theorem 7.5 (and Theorem 6.7 above) this allows us to lose the separability requirement, thereby proving a version of James’ Weak Compactness Theorem for real Banach spaces and balanced subsets of the closed unit ball in Theorem 7.11. Finally, the full version of the theorem is proved in Theorem 7.12 with the help of Theorem 7.11.

We begin by proving the so-called Helly’s Theorem, which plays an important role in Theorem 7.5 below. In order to prove Helly’s Theorem, we need two well-known lemmas. The first lemma is the following standard corollary of the Hahn-Banach Theorem, which we state here without proof.

**Lemma 7.1.** Let $Y$ be a closed subspace of a normed space $X$. If $x \in X \setminus Y$ then there exists a bounded linear functional $f : X \to \mathbb{F}$ such that $f(x) = d(x, Y)$ with $Y \subseteq \ker f$ and $\|f\| = 1$.

**Proof.** See Corollary A.3 in Appendix A. \qed

The second lemma that we need is the following result from linear algebra.

**Lemma 7.2.** Let $f_1, \ldots, f_n$ be a finite family of linear functionals on a vector space $X$. If $f$ is another linear functional on $X$, then $f \in \text{span}\{f_1, \ldots, f_n\}$ if and only if $\bigcap_{i=1}^n \ker f_i \subseteq \ker f$.

**Proof.** Suppose $f \in \text{span}\{f_1, \ldots, f_n\}$. Then $f = \sum_{j=1}^n \lambda_j f_j$ so if $f_j(x) = 0$ for all $j = 1, \ldots, n$ we also have $f(x) = 0$. Hence $\bigcap_{i=1}^n \ker f_i \subseteq \ker f$.

For the converse, assume that $\bigcap_{i=1}^n \ker f_i \subseteq \ker f$. Define a linear operator $T : X \to \mathbb{F}^n$ by $T(x) := (f_1(x), \ldots, f_n(x))$. The linearity of $T$ follows immediately from the linearity of each $f_j$. Moreover, $T(x) = 0$ if and only if $f_j(x) = 0$ for each $j = 1, \ldots, n$, so $\ker T = \bigcap_{i=1}^n \ker f_i$. Now define a linear functional $S : T(X) \to \mathbb{F}$ by $S(T(x)) := f(x)$, where $T(X) \subseteq \mathbb{F}^n$. Linearity follows from

\[
S(\alpha T(x) + \beta T(y)) = S(T(\alpha x + \beta y)) = f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha S(T(x)) + \beta S(T(y)).
\]

To see that $S$ is well-defined, notice that $T(x) = T(y)$ implies $f(x) = f(y)$. Indeed, $T(x) - T(y) = 0$ implies $f_j(x - y) = f_j(x) - f_j(y) = 0$ for each $j = 1, \ldots, n$ so that $x - y \in \bigcap_{i=1}^n \ker f_i \subseteq \ker f$ and hence $f(x) - f(y) = f(x - y) = 0$.

Observe that $S$ can be extended to a linear functional $\tilde{S}$ on all of $\mathbb{F}^n$ such that $S = \tilde{S}|_{T(X)}$. To see this, first note that $T(X)$ is a linear subspace of $\mathbb{F}^n$ by the linearity of $T$. Now fix a basis $v_1, \ldots, v_k$ for $T(X)$ and extend it to a basis $v_1, \ldots, v_k, w_{k+1}, \ldots, w_n$ for $\mathbb{F}^n$. Then the linear functional $\tilde{S} : \mathbb{F}^n \to \mathbb{F}$ defined by

\[
\tilde{S}(\alpha_1 v_1 + \cdots + \alpha_k v_k + \alpha_{k+1} w_{k+1} + \cdots + \alpha_n w_n) := S(\alpha_1 v_1 + \cdots + \alpha_k v_k).
\]

clearly does the job. Finally, let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{F}^n$ and notice that

\[
f(x) = \tilde{S}\left( (f_1(x), \ldots, f_n(x)) \right) = \tilde{S}\left( \sum_{j=1}^n f_j(x)e_j \right) = \sum_{j=1}^n \tilde{S}(e_j)f_j(x).
\]

for all $x \in X$. Setting $\lambda_j := \tilde{S}(e_j)$, we have $f = \sum_{j=1}^n \lambda_j f_j$, so $f \in \text{span}\{f_1, \ldots, f_n\}$. This finishes the proof. \qed
Theorem 7.3. (Helly’s Theorem) Let $X$ be a normed space. Let $f_1, \ldots, f_n$ be a finite family of bounded linear functionals on $X$ and let $\lambda_1, \ldots, \lambda_n$ be a corresponding collection of scalars. Then the following two statements are equivalent:

1. There is an $x_0 \in X$ such that $f_i(x_0) = \lambda_i$ for each $i = 1, \ldots, n$
2. There is a nonnegative real number $M \in \mathbb{R}_+$ such that

$$|\alpha_1 \lambda_1 + \cdots + \alpha_n \lambda_n| \leq M \|\alpha_1 f_1 + \cdots + \alpha_n f_n\|$$

for each linear combination $\alpha_1 f_1 + \cdots + \alpha_n f_n$ in span $\{f_1, \ldots, f_n\}$.

If (2) is satisfied, then given any $\varepsilon > 0$ we can choose $x_0$ in (1) such that $\|x_0\| \leq M + \varepsilon$.

Proof. Suppose statement (1) is true. Then there exists $x_0 \in X$ such that $f_i(x_0) = \lambda_i$ for each $i = 1, \ldots, n$. Hence an arbitrary element $\alpha_1 f_1 + \cdots + \alpha_n f_n$ in span $\{f_1, \ldots, f_n\}$ obeys the following inequality:

$$|\alpha_1 \lambda_1 + \cdots + \alpha_n \lambda_n| = |\alpha_1 f_1(x_0) + \cdots + \alpha_n f_n(x_0)|$$

$$\leq \|x_0\| \|\alpha_1 f_1 + \cdots + \alpha_n f_n\|.$$

In turn, statement (2) is satisfied with $M := \|x_0\|$. Note also that $\|x_0\| \leq M + \varepsilon$ for any $\varepsilon > 0$.

For the converse, assume that statement (2) is true. If $\lambda_1 = \cdots = \lambda_n = 0$ then $0 \in X$ is the desired $x_0$. Hence we may assume that at least one $\lambda_j$ is nonzero. In particular, $0 < |\lambda_j| \leq M \|f_j\|$ by (2), so at least one of the linear functionals $f_j$ are nonzero. By rearranging the indices if necessary, we can thus assume that $f_1, \ldots, f_m$ is a maximal linearly independent subcollection of $f_1, \ldots, f_n$. It follows that for each $k = m + 1, \ldots, n$ there are scalars $\alpha_{1,k}, \ldots, \alpha_{m,k} \in \mathbb{R}$ such that $f_k = \sum_{j=1}^m \alpha_{j,k} f_j$.

Suppose we could prove the existence of an $x_0 \in X$ such that $f_k(x_0) = \lambda_k$ for each of the linearly independent $f_1, \ldots, f_m$, and such that $\|x_0\| \leq M + \varepsilon$. Then the fact that $f_k = \sum_{j=1}^m \alpha_{j,k} f_j$ for each $k = m + 1, \ldots, n$ and the fact that the inequality in (2) holds for the full collection $f_1, \ldots, f_n$ would imply that

$$|f_k(x_0) - \lambda_k| = \left|\sum_{j=1}^m \alpha_{j,k} \lambda_j + (-1) \lambda_k\right| \leq M \left|\sum_{j=1}^m \alpha_{j,k} f_j + (-1) f_k\right| = M \|f_k - f_k\| = 0$$

for each $k = m + 1, \ldots, n$. Hence $f_k(x_0) = \lambda_k$ would also hold for $k = m + 1, \ldots, n$. In conclusion, we need only find $x_0 \in X$ such that $f_k(x_0) = \lambda_k$ for each $k = 1, \ldots, m$ and $\|x_0\| \leq M + \varepsilon$ for any $\varepsilon > 0$.

Define a linear operator $T : X \to \mathbb{F}^m$ by $T(x) := (f_1(x), \ldots, f_m(x))$. The linearity of $T$ is an immediate consequence of the linearity of each $f_j$. If $m = 1$, then $T$ is trivially a surjection. We claim that the same is true when $m \geq 2$. By Lemma 7.2, the linear independence of $f_1, \ldots, f_m$ implies that $\bigcap_{j \neq k} \ker f_j \nsubseteq \ker f_k$ for each $k = 1, \ldots, m$. It follows that for each $k = 1, \ldots, m$, we can choose $y_k \in \bigcap_{j \neq k} \ker f_j \setminus \ker f_k$ such that $f_k(y_k) = 1$ and $f_j(y_k) = 0$ whenever $j \neq k$. In turn, $T(y_1) = e_1, \ldots, T(y_m) = e_m$, where $e_i$ denotes the $i$'th standard basis vector of $\mathbb{F}^m$. We conclude that $T$ is a surjection from $X$ onto $\mathbb{F}^m$.

By the surjectivity of $T$, there exists $y_0 \in X$ such that $(f_1(y_0), \ldots, f_m(y_0)) = T(y_0) = (\lambda_1, \ldots, \lambda_m)$. As at least one $\lambda_j$ is nonzero, we have $y_0 \notin \bigcap_{j=1}^m \ker f_j$. Therefore, Lemma 7.1 ensures the existence of a bounded linear functional $f$ on $X$ such that $f(y_0) = d\left(y_0, \bigcap_{j=1}^m \ker f_j\right)$, $\bigcap_{j=1}^m \ker f_j \subsetneq \ker f$, and $\|f\| = 1$. Appealing to Lemma 7.2 once more, it follows from $\bigcap_{j=1}^m \ker f_j \subsetneq \ker f$ that $f \in \text{span} \{f_1, \ldots, f_m\}$. In other words, there are scalars $\beta_1, \ldots, \beta_m$ such that $f = \sum_{j=1}^m \beta_j f_j$. We deduce that

$$\frac{d\left(y_0, \bigcap_{j=1}^m \ker f_j\right) = f(y_0) = \sum_{j=1}^m \beta_j f_j(y_0) = \sum_{j=1}^m \beta_j \lambda_j}{\leq \left|\sum_{j=1}^m \beta_j \lambda_j\right| \leq M \left|\sum_{j=1}^m \beta_j f_j\right| = M \|f\| = M.}$$
Thus, for any \( \varepsilon > 0 \), there exists \( z_0 \in \bigcap_{j=1}^m \ker f_j \) with \( \|y_0 - z_0\| \leq M + \varepsilon \) as the distance would otherwise be larger than \( M \). Since \( z_0 \in \bigcap_{j=1}^m \ker f_j \) we have \( f_k(y_0 - z_0) = f_k(y_0) = \lambda_k \) for each \( k = 1, \ldots , m \). Setting \( x_0 := y_0 - z_0 \) completes the proof. \( \square \)

7.1. The Separable Case

In Theorem 7.5 below we shall prove that if every \( x^* \in X^* \) attains its supremum on a separable and weakly closed subset \( A \) of \( \bar{B}_X \), then that subset is weakly compact. To prove this, we proceed by contraposition. Hence, we shall need a bounded linear functional \( x^* \in X^* \) that does not attain its supremum.

The point of Lemma 7.4 is precisely to ensure the existence of such a linear functional, in the shape of \( \sum_{j=1}^\infty \beta_j y_j^* \), under just the right circumstances. That these circumstances are in fact satisfied when \( A \) is not weakly compact and that \( z^* := \sum_{j=1}^\infty \beta_j y_j^* \) really does not attain its supremum are then the topics of Theorem 7.5.

**Lemma 7.4.** Let \( X \) be a normed space. Let \( 0 < \theta < 1 \) be given and suppose that

(i) \( A \) is a nonempty subset of the closed unit ball \( \bar{B}_X \).

(ii) \( (\beta_n)_{n \in \mathbb{N}} \) is a sequence of strictly positive reals with \( \sum_n \beta_n = 1 \).

(iii) \( (x_n^*)_{n \in \mathbb{N}} \) is a sequence in \( B_{X^*} \) such that \( \sup \{|x_n^*(x)| : x \in A\} \geq \theta \) for every \( x^* \in \text{co} \{(x_n^*: n \in \mathbb{N})\} \).

Then there exists a real number \( \alpha \) with \( 0 < \alpha \leq 1 \) and a sequence \( (y_n^*)_{n \in \mathbb{N}} \) in \( B_{X^*} \) such that

1. \( y_n^* \in \text{co} \{(x_j^*: j \geq n)\} \subseteq B_{X^*} \) for all \( n \in \mathbb{N} \).
2. \( \sup \left\{ \sum_{j=1}^n \beta_j y_j^*(x) : x \in A \right\} = \alpha \).
3. \( \sup \left\{ \sum_{j=1}^n \beta_j y_j^*(x) : x \in A \right\} < \alpha \left(1 - \theta \sum_{j=n+1}^\infty \beta_j \right) \) for every \( n \in \mathbb{N} \).

**Proof.** The proof proceeds in the following manner: First, we employ an inductive argument to construct a sequence \( (y_n^*) \) satisfying (1) along with a sequence \( (\alpha_n) \) whose limit we claim to be the desired \( \alpha \). Then we prove that the inequality \( \theta \leq \alpha \leq 1 \) is indeed satisfied by this limit. Finally, we show that these choices of \( (y_n^*) \) and \( \alpha \) satisfy (2) and (3). For the sake of clarity, we split the proof into four steps.

**Step 1.** First of all, we introduce the seminorm \( |x^*|_A := \sup \{|x^*(x)| : x \in A \} \) defined for every \( x^* \in X^* \). Observe that the supremum is always finite since

\[
|x^*|_A = \sup \{|x^*(x)| : x \in A \} \leq \sup \{|x^*(x)| : x \in \bar{B}_X \} = \|x^*\| < \infty,
\]

where we have used that \( A \subseteq \bar{B}_X \) by (i) and \( \|x^*\| < \infty \) for every \( x^* \in X^* \) by definition of \( X^* \). To see that \( |x^*|_A : X^* \to \mathbb{R}_+ \) indeed defines a seminorm on \( X^* \), we observe that

\[
|\lambda x^*|_A = \sup \{|\lambda x^*(x)| : x \in A \} = |\lambda| \sup \{|x^*(x)| : x \in A \} = |\lambda| |x^*|_A
\]

and

\[
|x^* + y^*|_A = \sup \{|(x^* + y^*)(x)| : x \in A \} = \sup \{|x^*(x)| + |y^*(x)| : x \in A \} \\
\leq \sup \{|x^*(x)| + |y^*(x)| : x \in A \} \\
\leq \sup \{|x^*(x)| : x \in A \} + \sup \{|y^*(x)| : x \in A \} = |x^*|_A + |y^*|_A
\]

**Step 2.** Because the sequence \( (\beta_n) \) in (ii) sums to 1, we can choose a sequence of positive reals \( (\varepsilon_n)_{n \in \mathbb{N}} \) converging to 0 such that

\[
0 < \sum_{k=1}^\infty \frac{\beta_k \varepsilon_k}{\sum_{j=k+1}^\infty \beta_j} \sum_{j=k}^\infty \beta_j < 1 - \theta.
\]

To see this, note that \( \varepsilon_k \) can be chosen smaller than \( (1 - \theta) \sum_{j=k+1}^\infty \beta_j \sum_{j=k}^\infty \beta_j > 0 \).

**Step 3.** We are now ready for the construction of the sequences \( (y_n^*) \) and \( (\alpha_n) \). Our plan is to use induction to prove the existence of a sequence \( (y_n^*)_{n \in \mathbb{N}} \) in \( B_{X^*} \) such that for each \( n \in \mathbb{N} \) we have

\[
y_n^* \in \text{co} \{(x_j^*: j \geq n)\} \subseteq B_{X^*}.
\]

(7.1)
and
\[ \left| \sum_{j=1}^{n-1} \beta_j y_j^* + \left( \sum_{j=n}^{\infty} \beta_j \right) y_n^* \right|_A < \alpha_n(1 + \varepsilon_n) \] (7.2)
where
\[ \alpha_n := \inf \left\{ \left| \sum_{j=1}^{n-1} \beta_j y_j^* + \left( \sum_{j=n}^{\infty} \beta_j \right) y_n^* \right|_A : y^* \in \text{co} \left( \{ x_j^* : j \geq n \} \right) \right\}. \] (7.3)

Note that (7.3) is well-defined since \( X^* \) is a vector space and \( |x^*|_A < \infty \) for every \( x^* \in X^* \).

**Step 3.1.** We begin the induction by noticing that
\[ \alpha_1 = \inf \{ |y^*_1|_A : y^* \in \text{co} \left( \{ x_j^* : j \geq 1 \} \right) \} \geq \theta > 0 \]
by the assumption in (iii). Hence there exists a \( y_1^* \) in \( \text{co} \left( \{ x_j^* : j \geq 1 \} \right) \) so that
\[ |y_1^*_1|_A < \alpha_1(1 + \varepsilon_1). \]
Indeed, \( \alpha_1(1 + \varepsilon_1) \) would otherwise be a larger lower bound than \( \alpha_1 \). This means that (7.1) and (7.2) are satisfied for \( n = 1 \).

**Step 3.2.** Now, let \( n \geq 2 \) be given and suppose that \( y_1^*, \ldots, y_{n-1}^* \) have been selected in accordance with (7.1) and (7.2). Observe that if \( y^* \in \text{co} \left( \{ x_j^* : j \geq n \} \right) \), then
\[
\sum_{j=1}^{n-1} \beta_j y_j^* + \left( \sum_{j=n}^{\infty} \beta_j \right) y^* = \sum_{j=1}^{n-2} \beta_j y_j^* + \beta_n y_n^* + \left( \sum_{j=n}^{\infty} \beta_j \right) y^*
\]
\[= \sum_{j=1}^{n-2} \beta_j y_j^* + \beta_n \left( \frac{\sum_{j=n-1}^{\infty} \beta_j}{\sum_{j=n-1}^{\infty} \beta_j} y_{n-1}^* + \sum_{j=n-1}^{\infty} \beta_j \right). \]
where
\[ \frac{\beta_{n-1}}{\sum_{j=n-1}^{\infty} \beta_j} y_{n-1}^* + \sum_{j=n-1}^{\infty} \beta_j \] since it is a convex combination of the elements \( y_{n-1}^* \) and \( y^* \) in \( \text{co} \left( \{ x_j^* : j \geq n-1 \} \right) \). But then the set, over which \( \alpha_n \) is the infimum, is contained in the set, over which \( \alpha_{n-1} \) is the infimum (see (7.3)). Hence \( \alpha_{n-1} \leq \alpha_n \). Since \( \alpha_1 > 0 \) by Step 3, it follows that \( \alpha_n > 0 \). Therefore, there exists \( y_n^* \) in \( \text{co} \left( \{ x_j^* : j \geq n \} \right) \) satisfying (7.2) by definition of the infimum. This completes the induction.

**Step 4.** For each \( n \in \mathbb{N} \), the set from (7.3), over which \( \alpha_n \) is the infimum, turns out to be bounded above by 1. To see this, recall from Step 1 that \( |x^*|_A \) is a seminorm with \( |x^*|_A \leq \| x^* \| \). Hence it follows from \( y_1^*, y^* \in \overline{B}_X \) and \( \sum_{j=1}^{n-1} \beta_j = 1 \) that
\[
\left| \sum_{j=1}^{n-1} \beta_j y_j^* + \left( \sum_{j=n}^{\infty} \beta_j \right) y^* \right|_A \leq \left| \sum_{j=1}^{n-1} \beta_j y_j^* \right|_A + \left( \sum_{j=n}^{\infty} \beta_j \right) \| y^* \|_A
\]
\[\leq \sum_{j=1}^{n-1} \beta_j \| y_j^* \|_A + \left( \sum_{j=n}^{\infty} \beta_j \right) \| y^* \| \leq \sum_{j=1}^{n-1} \beta_j + \sum_{j=n}^{\infty} \beta_j = 1. \]
By the result in Step 3.2 that \( \alpha_{n-1} \leq \alpha_n \) for each \( n \in \mathbb{N} \), we thus have
\[ 0 < \theta \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq 1. \]
It follows that \( (\alpha_n) \) is increasing and bounded from above, so the sequence converges to its supremum. The limit \( \alpha \) thus exists and satisfies \( \theta \leq \alpha \leq 1 \) as well as \( \alpha_n \leq \alpha \) for all \( n \in \mathbb{N} \).

**Step 4.1.** By construction of \( (\alpha_n) \) we have
\[ \alpha_n \leq \left| \sum_{j=1}^{n-1} \beta_j y_j^* + \left( \sum_{j=n}^{\infty} \beta_j \right) y_n^* \right|_A \leq \alpha_n(1 + \varepsilon_n) \]
for every \( n \in \mathbb{N} \). Recall from Step 2 that \((\varepsilon_n)\) was chosen such that \( \varepsilon_n \to 0 \) for \( n \to \infty \).

Letting \( n \to \infty \), this thus follows from the squeeze theorem that

\[
\alpha = \lim_{n \to \infty} \sum_{j=1}^{n-1} \beta_j y_j^* + (\sum_{j=n}^{\infty} \beta_j) y_{n}^*.
\]

In order to evaluate this limit, we first observe that \(| \cdot |_A\) is Lipschitz continuous. Being a seminorm, it satisfies \(|x^*|_A = |x^* - y^*|_A \leq |x^* - y^*|_A + |y^*|_A\), which in turn implies that \(|x^*|_A - |y^*|_A \leq |x^* - y^*|_A\). Analogously, we have \(|y^*|_A - |x^* - y^*|_A = |x^* - y^*|_A\). The Lipschitz continuity then follows from

\[
|\alpha - \beta|_A = \sum_{j=n}^{\infty} |\beta_j y_j^*|_A \leq \sum_{j=n}^{\infty} \beta_j |y_j^*|_A \leq \sum_{j=n}^{\infty} \beta_j,
\]

which converges to 0 as \( m \) tends to infinity. Likewise, \((\sum_{j=n}^{\infty} \beta_j) y_n^* \to 0\) for \( n \to \infty \) since \(|y_n^*| \leq 1\). It follows from the continuity of \(| \cdot |_A\) and the preceding remarks about convergence that

\[
\alpha = \left| \lim_{n \to \infty} \sum_{j=1}^{n-1} \beta_j y_j^* + \lim_{n \to \infty} \left( \sum_{j=n}^{\infty} \beta_j \right) y_n^* \right|_A = \left( \sum_{j=1}^{\infty} \beta_j y_j^* \right)_A.
\]

This is precisely the statement in (2).

**Step 4.2.** It only remains to show that \((y_n^*)\) and \(\alpha\), as constructed above, satisfy the inequality in (3). To this end, fix an arbitrary \( n \in \mathbb{N} \). Observe that if \( n \geq 2 \), then

\[
\left( \sum_{j=1}^{n} \beta_j y_j^* \right)_A = \sum_{j=1}^{n-1} \beta_j y_j^* + \beta_n y_n^* + \sum_{j=n+1}^{\infty} \beta_j \left( \sum_{j=1}^{n} \beta_j y_j^* \right)_A
\]

\[
= \sum_{j=n}^{\infty} \beta_j \left( \sum_{j=1}^{n-1} \beta_j y_j^* + \left( \sum_{j=n+1}^{\infty} \beta_j \right) y_n^* \right)_A
\]

\[
= \frac{\beta_n}{\sum_{j=n}^{\infty} \beta_j} \left( \sum_{j=1}^{n-1} \beta_j y_j^* + \beta_n y_n^* \right)_A
\]

\[
\leq \frac{\beta_n}{\sum_{j=n}^{\infty} \beta_j} \left( \sum_{j=1}^{n-1} \beta_j y_j^* + \sum_{j=n}^{\infty} \beta_j \left( \sum_{j=1}^{n} \beta_j y_j^* \right)_A \right)
\]

\[
< \frac{\beta_n}{\sum_{j=n}^{\infty} \beta_j} \left( \sum_{j=1}^{n-1} \beta_j y_j^* + \sum_{j=n}^{\infty} \beta_j \left( \sum_{j=1}^{n-1} \beta_j y_j^* \right)_A \right)
\]

\[
= \left( \sum_{j=n+1}^{\infty} \beta_j \right) \left( \frac{\beta_n}{\sum_{j=n+1}^{\infty} \beta_j} \sum_{j=n}^{\infty} \beta_j \left( \sum_{j=1}^{n} \beta_j y_j^* \right)_A \right).
\]

where the strict inequality in (7.4) follows from our construction of \((y_n^*)\) as indicated in formula (7.2) on p. 27.
It follows, by replacing $n$ with $n - k$ in the previous result, that if $n - k \geq 2$, then

\[
\left| \sum_{j=n-k+1}^{n-k} \beta_j y_j^* \right|_A < \left( \frac{\beta_{n-k} \alpha_{n-k} (1 + \varepsilon_{n-k})}{\sum_{j=n-k+1}^{n-k} \beta_j \sum_{j=n-k}^{\infty} \beta_j} + \frac{\left| \sum_{j=1}^{n-1} \beta_j y_j^* \right|_A}{\sum_{j=n-k}^{\infty} \beta_j} \right)
\]

Hence it holds for any $n \geq 2$ that

\[
\left| \sum_{j=1}^{n} \beta_j y_j^* \right|_A < \left( \sum_{j=n+1}^{\infty} \beta_j \right) \left( \frac{\beta_n \alpha_n (1 + \varepsilon_n)}{\sum_{j=n+1}^{\infty} \beta_j} \right) + \left| \sum_{j=1}^{n-1} \beta_j y_j^* \right|_A
\]

\[
< \left( \sum_{j=n+1}^{\infty} \beta_j \right) \left( \frac{\beta_n \alpha_n (1 + \varepsilon_n)}{\sum_{j=n+1}^{\infty} \beta_j} \right) + \left( \sum_{j=n}^{\infty} \beta_j \right) \left( \frac{\beta_{n-1} \alpha_{n-1} (1 + \varepsilon_{n-1})}{\sum_{j=n}^{\infty} \beta_j} \right) + \left| \sum_{j=1}^{n-2} \beta_j y_j^* \right|_A
\]

\[
< \left( \sum_{j=n+1}^{\infty} \beta_j \right) \left( \frac{\beta_n \alpha_n (1 + \varepsilon_n)}{\sum_{j=n+1}^{\infty} \beta_j} \right) + \left( \sum_{j=n}^{\infty} \beta_j \right) \left( \frac{\beta_{n-1} \alpha_{n-1} (1 + \varepsilon_{n-1})}{\sum_{j=n}^{\infty} \beta_j} \right) + \left( \sum_{j=n}^{\infty} \beta_j \right) \left( \frac{\beta_{n-2} \alpha_{n-2} (1 + \varepsilon_{n-2})}{\sum_{j=n}^{\infty} \beta_j} \right) + \left| \sum_{j=1}^{n-3} \beta_j y_j^* \right|_A
\]

\[
< \ldots
\]

\[
< \left( \sum_{j=n+1}^{\infty} \beta_j \right) \left( \sum_{k=2}^{n} \left( \frac{\beta_k \alpha_k (1 + \varepsilon_k)}{\sum_{j=k+1}^{\infty} \beta_j} \right) + \left| \sum_{j=1}^{\infty} \beta_j y_j^* \right|_A \right)
\]

\[
< \left( \sum_{j=n+1}^{\infty} \beta_j \right) \left( \sum_{k=2}^{n} \left( \frac{\beta_k \alpha_k (1 + \varepsilon_k)}{\sum_{j=k+1}^{\infty} \beta_j} \right) + \left| \sum_{j=1}^{\infty} \beta_1 \alpha_1 (1 + \varepsilon_1) \right|_A \right)
\]

\[
= \left( \sum_{j=n+1}^{\infty} \beta_j \right) \sum_{k=1}^{n} \left( \frac{\beta_k \alpha_k (1 + \varepsilon_k)}{\sum_{j=k+1}^{\infty} \beta_j} \right).
\]

For $n = 1$ the above inequality reduces to

\[
\left| \beta_1 y_1^* \right|_A < \left( \sum_{j=2}^{\infty} \beta_j \right) \left( \frac{\beta_1 \alpha_1 (1 + \varepsilon_1)}{\sum_{j=2}^{\infty} \beta_j} \right) = \beta_1 \alpha_1 (1 + \varepsilon),
\]

which is also true since $\left| \beta_1 y_1^* \right|_A = \beta_1 \left| y_1^* \right|_A < \beta_1 \alpha_1 (1 + \varepsilon_1)$ by (7.2). Now, recall from Step 3.1 that $\alpha_n \leq \alpha$ for all $n \in \mathbb{N}$ and recall from Step 2 that

\[
0 < \frac{\beta_k \varepsilon_k}{\sum_{j=k+1}^{\infty} \beta_j \sum_{j=k}^{\infty} \beta_j} < 1 - \theta.
\]
We conclude that for every $n \in \mathbb{N}$ it holds that
\[
\left| \sum_{j=1}^{n} \beta_j y_j^* \right|_A < \left( \sum_{j=n+1}^{\infty} \beta_j \right)^n \left( \sum_{k=1}^{\infty} \beta_k \alpha_k (1 + \varepsilon_k) \beta_{k+1} \sum_{j=k+1}^{\infty} \beta_j \right) \\
\leq \alpha \left( \sum_{j=n+1}^{\infty} \beta_j \right)^n \left( \sum_{k=1}^{\infty} \beta_k \sum_{j=k+1}^{\infty} \beta_j \right) + \beta_k \varepsilon_k \\
< \alpha \left( \sum_{j=n+1}^{\infty} \beta_j \right)^n \left( \sum_{k=1}^{\infty} \beta_k \sum_{j=k+1}^{\infty} \beta_j \right) + (1 - \theta) \\
= \alpha \left( \sum_{j=n+1}^{\infty} \beta_j \right) \left( \sum_{k=1}^{\infty} \frac{1}{\beta_j} \frac{1}{\sum_{j=k+1}^{\infty} \beta_j} - (1 - \theta) \right) \\
= \alpha \left( \sum_{j=n+1}^{\infty} \beta_j \right) \left( \frac{1}{\sum_{j=n+1}^{\infty} \beta_j} - (1 - \theta) \right) \\
= \alpha \left( 1 - \theta \sum_{j=n+1}^{\infty} \beta_j \right) .
\]

This is precisely the statement in (3). Hence the proof is complete.

As already explained, Lemma 7.4 plays an integral role in the proof of the next theorem. Specifically, we shall prove that when a subset is not weakly compact, then the assumptions in Lemma 7.4 are fulfilled, and hence we get a linear functional of the form $z^* = \sum_{j=1}^{\infty} \beta_j y_j^*$, which fails to attain its supremum.

Strictly speaking, we only need the implication (1) $\Rightarrow$ (2) from Theorem 7.5 for the ensuing results in Section 7.2. Nevertheless, we prove all four equivalences. First of all, the extra work involved is negligible and the proof provides a valuable guide for how to proceed in the non-separable case. Secondly, it is well worth the effort to have a complete proof of James’ Weak Compactness Thereom for separable subsets.

**Theorem 7.5.** Let $X$ be a Banach space. Suppose that $A$ is a nonempty, separable, weakly closed subset of the closed unit ball $B_X$. Then the following statements are equivalent:

1. The set $A$ is not weakly compact.
2. There is a $\theta \in \mathbb{R}$ with $0 < \theta < 1$ and a sequence $(x_n^*)$ in $B_{X^*}$ such that $\lim_{n \to \infty} x_n^*(x) = 0$ for each $x \in A$ and $\sup \left\{ |x^*(x)| : x \in A \right\} \geq \theta$ for every $x^* \in \text{co} \left\{ x_n^* : n \in \mathbb{N} \right\}$.
3. There is a $\theta \in \mathbb{R}$ with $0 < \theta < 1$ such that if $(\beta_n)$ is a sequence of strictly positive reals with $\sum_n \beta_n = 1$, then there is an $\alpha \in \mathbb{R}$ with $\theta \leq \alpha \leq 1$ and a sequence $(y_n^*)$ in $B_{X^*}$ such that
   (a) $\lim_{n \to \infty} y_n^*(x) = 0$ for each $x \in A$.
   (b) $\sup \left\{ \sum_{j=1}^{\infty} \beta_j y_j^*(x) : x \in A \right\} = \alpha$.
   (c) $\sup \left\{ \sum_{j=1}^{\infty} \beta_j y_j^*(x) : x \in A \right\} < \alpha (1 - \theta \sum_{j=n+1}^{\infty} \beta_j) \quad \forall n \in \mathbb{N}$.
4. There is a $z^*$ in $X^*$ such that $\sup \left\{ |z^*(x)| : x \in A \right\}$ is not attained.

**Proof.** The plan for the proof is to show that each statement implies the one immediately after it and then close the chain by showing that (4) implies (1). The hard part is proving (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is also deep, but its proof is an easy consequence of Lemma 7.4. The two remaining implications require little work.
(1) \(\Rightarrow\) (2): Assume \(A\) is not weakly compact. Let \(V := \overline{\text{span}}(A)\) and notice that this is again a normed space, so in particular it has a dual space \(V^*\). Now, let \(W\) denote the vector space underlying \(V^*\) and equip \(W\) with the norm \(\|\cdot\|_W: W \to \mathbb{R}_+\) given by \(\|v^*\|_W := \sup \{ |v^*(x)| : x \in A \}\) instead of the usual operator norm. That \(\|v^*\|_W\) is a seminorm follows by the exact same arguments as in Step 1 of the proof of Lemma 7.4. To see that it is a norm, we need to show that if \(\|v^*\|_W = 0\), then \(v^*\) is the zero element in \(W\). So suppose \(\|v^*\|_W = 0\), which means that \(v^*(x) = 0\) for every \(x \in A\), and observe that this implies \(v^*(x) = 0\) for every \(x \in \text{span}(A)\). Since \(A\) is a subset of the normed space \(X\), we have \(\overline{\text{span}}(A) = \text{span}(A)\), and hence \(v^*(x) = 0\) for every \(x \in \overline{\text{span}}(A) = V\) by the continuity of \(v^*\). Thus \(v^*\) is the zero functional on \(V\), which is precisely the zero element of \(W\). We shall refer to the normed space \((W, \|v^*\|_W)\) simply as \(W\).

Now, let \(f: A \to W^*\) be the canonical map from \(A\) into \(W^*\) defined by \((f(x))(v^*) = v^*(x)\) for \(x \in A\) and \(v^* \in W^*\). That is, \(f(x)\) acts on \(v^* \in W\) by evaluation in \(x\). That \(f(x)\) is linear is immediate from the linearity of \(v^*\). Let \(\|\cdot\|_W\), denote the operator norm on \(W^*\). To see that \(f(x)\) is also bounded on \(W^*\), and hence belongs to \(W^*\) as required, notice that \([|f(x)(v^*)| = |v^*(x)|] \leq \sup \{|v^*(x)| : x \in A\} = \|v^*\|_W \leq 1\) for \(v^* \in B_W\).

Therefore, \(\|f(x)\|_W \leq \sup \{|(f(x))(v^*)| : v^* \in B_W\} \leq 1\) for every \(x \in A\), so \(f(x) : W \to F\) is bounded on \(W\) for each \(f(x) \in f(A)\). In particular, we have \(f(A) \subseteq B_{W^*}\).

By Corollary A.4, \(V^*\) is a separating family of functions on \(V\). It follows that if \(x \neq y\) then there exists \(v^*_{x,y} \in V^*\) such that \(v^*_{x,y}(x) \neq v^*_{x,y}(y)\) and hence \((f(x))(v^*_{x,y}) \neq (f(y))(v^*_{x,y})\), so \(f(x) \neq f(y)\). In conclusion, \(f\) is injective and therefore bijective onto \(f(A)\). In fact, \(f\) is a homeomorphism onto its image when \(A\) is considered as a subspace of \(X\) in the weak topology and \(f(A)\) is considered as a subspace of \(W^*\) in the weak* topology. Since \(V = \overline{\text{span}}(A)\) is a linear subspace of \(X\) with \(A \subseteq V\), it follows from Theorem 5.10 that the topology which \(A\) inherits as a subspace of \((X, \sigma(X, X^*))\), is equivalent to the topology that it inherits as a subspace of \((V, \sigma(V, V^*))\). This implies that if \((x_\alpha)\) is a net in \(A \subseteq V\) and \(x \in A \subseteq X\), then

\[
x_\alpha \xrightarrow{w^*} x \text{ in } (V, \sigma(V, V^*)) \iff v^*(x_\alpha) \to v^*(x) \text{ in } F \text{ for all } v^* \in W \text{ [by Theorem 5.11]}
\]

\[
\iff (f(x_\alpha))(v^*) \to (f(x))(v^*) \text{ in } F \text{ for all } v^* \in W
\]

\[
\iff (x_\alpha) \xrightarrow{w^*} f(x) \text{ in } (W^*, \sigma(W^*, W)) \text{ [by Theorem 5.17].}
\]

Hence \(f : A \to f(A)\) and \(f^{-1} : f(A) \to A\) are both continuous with respect to these topologies. We conclude that \(A\) as a subspace of \(X\) in the weak topology, is homeomorphic to \(f(A)\) as a subspace of \(W^*\) in the weak* topology.

Recall that, by assumption, \(A\) is not weakly compact in \(X\). Hence \(f(A)\) is not weakly compact in \(W^*\) by the above homeomorphism result. Since \(f(A) \subseteq B_{W^*}\) and \(B_{W^*}\) is weakly compact in \(W^*\) by the Banach-Alaoglu Theorem (Theorem A.6), it follows that \(f(A)\) cannot be weakly* closed in \(W^*\). Thus \(\overline{f(A)}^{w^*} \cap f(A)\) is nonempty. Now fix an element \(F \in \overline{f(A)}^{w^*} \setminus f(A)\) exists. By definition of \(F \in \overline{f(A)}^{w^*}\) there is a net \((f(x_\alpha))\) in \(f(A)\) such that \(f(x_\alpha) \xrightarrow{w^*} F\). Hence \(v^*(x_\alpha) = (f(x_\alpha))(v^*) \to F(v^*)\) in \(F\) for every \(v^* \in W\) by Theorem 5.17. Then \(F(v^*) = v^*(v_0)\) for every \(v^* \in W\) implies that \(v^*(x_\alpha) \to F(v^*) = v^*(v_0)\) in \(F\) for every \(v^* \in W\). By Theorem 5.11 this implies that \(x_\alpha \xrightarrow{w^*} x_0\) in \(W\), from which it follows that \(v_0 \in \overline{A}^{w^*}\). On the other hand, the fact that \(F \notin f(A)\) forces \(v_0 \notin A\). Indeed, \(v_0 \in A\) would imply that \(F(v^*) = v^*(v_0) = (f(v_0))(v^*)\) for all \(v^*\), which in turn would imply that \(F = f(v_0) \in f(A)\). We conclude that \(v_0 \in \overline{A}^{w^*} \setminus A\), which contradicts the fact that \(A\) is weakly closed.

Notice that if \(F \neq 0\), then \(0 \in V\) satisfies \(F(v^*) = 0 = v^*(0)\) for every \(v^* \in W\), which violates the above result. Hence \(F \neq 0\) and in particular \(\|F\|_{W^*} > 0\).
Moreover, it follows from $A \subseteq \bar{B}_X \cap V = \bar{B}_V$ that

$$B_{V^*} = \{ v^* \in W : \|v^*\|_{V^*} \leq 1 \} = \{ v^* \in W : \sup \{|v^*(x)| : x \in \bar{B}_V \} \leq 1 \}$$

$$\subseteq \{ v^* \in W : \sup \{|v^*(x)| : x \in A \} \leq 1 \} = \{ v^* \in W : \|v^*\|_W \leq 1 \} = \bar{B}_W$$

so that

$$\|F\|_{V^*} = \sup \{|F(v^*): v^* \in B_{V^*} \} \leq \sup \{|F(v^*): v^* \in \bar{B}_W \} = \|F\|_W < \infty.$$ 

Thus we have $F \in V^*$ and $\|F\|_W \leq \|F\|_{V^*}$.

Let $J_V : V \to V^{**}$ be the canonical embedding from $V$ into $V^{**}$. Note that $V$ is itself a Banach space as a closed (linear) subspace of the Banach space $X$. Hence $J_V(V)$ is a closed subspace of $V^{**}$ by Theorem 5.13. Since no $v \in V$ satisfies $F(v^*) = v^*(v)$ for every $v^* \in W$ we have $F \neq J_V(v)$ for each $v \in V$ and hence $F \notin J_V(V)$. It follows that $d_{V^*}(F, J_V(V)) > 0$.

Now choose a real number $\Delta$ such that

$$0 < \Delta < d_{V^*}(F, J_V(V)).$$

By the separability of $A$, we can also choose a countable dense subset $\{a_n : n \in \mathbb{N}\}$ in $A$. Let $n \in \mathbb{N}$ be given and consider any set of $n + 1$ scalars $\alpha_1, \ldots, \alpha_{n+1}$. Then we have

$$\|\alpha_1 \Delta + \sum_{j=2}^{n+1} \alpha_j 0\|_{\alpha_1} \Delta \leq \|\alpha_1\| \Delta \frac{\left\|F - J_V\left(-\sum_{j=2}^{n+1} \frac{\alpha_j}{\alpha_1} a_{j,-1}\right)\right\|_{V^*}}{d_{V^*}(F, J_V(V))}.$$ 

$$= \frac{\Delta}{d_{V^*}(F, J_V(V))} \|\alpha_1 F + \sum_{j=2}^{n+1} \alpha_j J_V(a_{j,-1})\|_{V^{**}}.$$ 

Hence the collection of bounded linear functionals $F, J_V(a_1), \ldots, J_V(a_n)$ on $V^*$ and the corresponding collection of scalars $\Delta, 0, \ldots, 0$ satisfy statement (2) in Helly’s Theorem with $M = \frac{\Delta}{d_{V^*}(F, J_V(V))} > 0$. Helly’s Theorem thus assures the existence of a $v_n^* \in V^*$ such that $F(v_n^*) = \Delta, (J_V(a_1))(v_n^*) = 0, \ldots, (J_V(a_n))(v_n^*) = 0$ and such that

$$\|v_n^*\|_{V^*} \leq \frac{\Delta}{d_{V^*}(F, J_V(V))} + \frac{d_{V^*}(F, J_V(V)) - \Delta}{2 \cdot d_{V^*}(F, J_V(V))}.$$ 

Since $n \in \mathbb{N}$ was arbitrary, it follows that for each $n \in \mathbb{N}$ there exists $v_n^* \in V^*$ such that

(i) $\|v_n^*\|_{V^*} < 1$

(ii) $F(v_n^*) = \Delta$

(iii) $v_n^*(a_j) = (J_V(a_j))(v_n^*) = 0$ if $n \geq j$

For each $n \in \mathbb{N}$, let $x_n^* \in X^*$ be the Hahn-Banach extension of $v_n^* : V \to F$ to all of $X$ with $\|x_n^*\|_X = \|v_n^*\|_{V^*} < 1$ by (i). It is immediate from (iii) that $\lim_{n \to \infty} x_n^*(a_j) = 0$ for every $a_j \in \{a_n : n \in \mathbb{N}\}$. Let $\varepsilon > 0$ be given. If $a \in A \setminus \{a_n : n \in \mathbb{N}\}$ then the density of $\{a_n : n \in \mathbb{N}\}$ in $A$ implies that there exists $a_{j_n} \in \{a_n : n \in \mathbb{N}\}$ such that $\|a - a_{j_n}\|_X < \varepsilon$. By $\lim_{n \to \infty} x_n^*(a_{j_n}) = 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that $|x_n^*(a_{j_n})| < \varepsilon$ for all $n > N_\varepsilon$. Together with $\|x_n^*\|_X < 1$ for all $n \in \mathbb{N}$, the previous remarks imply that

$$|x_n^*(a)| \leq |x_n^*(a - a_{j_n})| + |x_n^*(a_{j_n})| \leq \|x_n^*\|_X \cdot |a - a_{j_n}|_X + |x_n^*(a_{j_n})| < \varepsilon + |x_n^*(a_{j_n})| < 2\varepsilon$$

for all $n > N_\varepsilon$. Hence $\lim_{n \to \infty} x_n^*(a) = 0$. In summary, we have thus constructed a sequence $(x_n^*)$ in $\bar{B}_X$ with $\lim_{n \to \infty} x_n^*(x) = 0$ for each $x \in A$.

It remains to show that there exists $0 < \theta < 1$ such that $\sup \{|x^*(x)| : x \in A\} \geq \theta$ whenever $x^* \in \text{co} \{x_n^* : n \in \mathbb{N}\}$. Suppose $x^* \in \text{co} \{x_n^* : n \in \mathbb{N}\}$ and let $v^*$ be the restriction of $x^*$ to $V$. Then it follows from (ii) that

$$\Delta = F(v^*) \leq \|F\|_{W^*}. \|v^*\|_W = \|F\|_{W^*}. \sup \{|v^*(x)| : x \in A\}$$

$$= \|F\|_{W^*}. \sup \{|v^*(x)| : x \in A\}. $$
Letting $\theta := \Delta / \|F\|_{W^*}$, we thus have $\sup \{ |x^*(x)| : x \in A \} \geq \theta$. Finally, observe that

$$0 < \Delta = F(v_1^*) \leq \|F\|_{V^*} \|v_*^*\|_{V} \leq \|F\|_{V^*} \leq \|F\|_{W^*},$$

so that $0 < \theta < 1$ as desired.

(2) $\Rightarrow$ (3): Assume statement (2) is true. That is, assume there exists $\theta \in (0, 1)$ and a sequence $(x_n^*)$ in $\tilde{B}_X$ such that $\lim_{n \to \infty} x_n^*(x) = 0$ for each $x \in A$ and such that $\sup \{ |x^*(x)| : x \in A \} \geq \theta$ for every $x^* \in \co \{ x_n^* : n \in \mathbb{N} \}$. Recall that $A$ is a nonempty subset of $\tilde{B}_X$ and let $(\beta_n)$ be any sequence of positive reals with $\sum_{n=1}^{\infty} \beta_n = 1$. Then Lemma 7.4 ensures the existence of an $\alpha$ with $\theta \leq \alpha \leq 1$ and a sequence $(y_n^*)$ in $\tilde{B}_X$ such that parts (b) and (c) of statement (3) are satisfied.

Now, Lemma 7.4 also ensures that $y_n^* \in \co \{ \{ x_j^* : j \geq n \} \}$ for each $n \in \mathbb{N}$. Together with the assumption that $\lim_{n \to \infty} x_n^*(x) = 0$ for all $x \in A$, this is sufficient for $(y_n^*)$ to also satisfy part (a) of statement 3. To see this, fix an arbitrary $x \in A$ and let $\varepsilon > 0$ be given. Then $\lim_{n \to \infty} x_n^*(x) = 0$ implies the existence of an $N_\varepsilon \in \mathbb{N}$ such that $|x_j^*(x)| < \varepsilon$ for all $j > N_\varepsilon$. Observe that for each $n > N_\varepsilon$ we have $y_n^* = \sum_{j=1}^{k_n} t_{i,n} x_j^*,n$ for some $x_j^*,n, \ldots, x_{j_n}^*,n \in \{ x_j^* : j \geq n > N_\varepsilon \}$, where $t_{i,n}, \ldots, t_{k,n} \geq 0$ and $\sum_{i=1}^{k_n} t_{i,n} = 1$. Hence

$$|y_n^*(x)| = \sum_{i=1}^{k_n} t_{i,n} |x_j^*,n(x)| \leq \sum_{i=1}^{k_n} t_{i,n} |x_j(x)| < \sum_{i=1}^{k_n} t_{i,n} \varepsilon = \varepsilon$$

for all $n > N_\varepsilon$. It follows that $\lim_{n \to \infty} y_n^*(x) = 0$ for each $x \in A$ as desired.

(3) $\Rightarrow$ (4): Fix a sequence $(\beta_n)$ of positive reals with $\sum_{n=1}^{\infty} \beta_n = 1$ and let $\theta, \alpha$, and $(y_n^*)$ have the properties listed in statement (3). Recall from Lemma 7.4 (Step 4.1) that $\sum_{j=1}^{\infty} \beta_j y_j^*$ is convergent, whence $z^* := \sum_{j=1}^{\infty} \beta_j y_j^*$ is a well-defined element of $X^*$. We proceed to show that this particular choice of $z^* \in X^*$ satisfies statement (4) inasmuch as $\sup \{ |z^*(x)| : x \in A \}$ is not attained.

Fix an arbitrary $x_0 \in A$. Since $0 < \theta \leq \alpha$ we have $\theta \alpha > 0$. By part (a) of statement (3), we have $\lim_{n \to \infty} y_n^*(x_0) = 0$. Thus there exists $n \in \mathbb{N}$ such that $|y_n^*(x_0)| < \alpha \varepsilon$ for all $j > n$. Together with parts (b) and (c) of statement (3), this in turn implies that

$$|z^*(x_0)| = \sum_{j=1}^{\infty} \beta_j y_j^*(x_0)$$

$$\leq \left| \sum_{j=1}^{n} \beta_j y_j^*(x_0) \right| + \sum_{j=n+1}^{\infty} \beta_j |y_j^*(x_0)|$$

$$\leq \sup \left\{ \sum_{j=1}^{n} \beta_j y_j^*(x) : x \in A \right\} + \sum_{j=n+1}^{\infty} \beta_j |y_j^*(x_0)|$$

$$\leq \sup \left\{ \sum_{j=1}^{n} \beta_j y_j^*(x) : x \in A \right\} + \alpha \sum_{j=n+1}^{\infty} \beta_j$$

$$= \alpha \left( 1 - \sum_{j=n+1}^{\infty} \beta_j \right) + \alpha \sum_{j=n+1}^{\infty} \beta_j$$

$$= \alpha \left( 1 - \sum_{j=n+1}^{\infty} \beta_j \right) + \alpha \sum_{j=n+1}^{\infty} \beta_j$$

$$= \alpha \sum_{j=n+1}^{\infty} \beta_j$$

$$= \sup \{ |z^*(x)| : x \in A \}. \tag{7.5}$$

The inequality in (7.5) is precisely (c) and the equality in (7.6) is precisely (b). Since $x_0 \in A$ was arbitrary, we conclude that $\sup \{ |z^*(x)| : x \in A \}$ is not attained at any $x \in A$. This completes the proof of (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1): Assume there is a $z^* \in X^*$ such that $\sup \{ |z^*(x)| : x \in A \}$ is not attained. We need to show that this implies $A$ is not weakly compact. To this end, suppose for contradiction that $A$ is weakly compact. Then it follows from the Extreme Value Theorem (Theorem 2.1) that every weakly continuous scalar-valued function attains the supremum of its absolute value on $A$. Now recall that all elements of $X^*$ are continuous with respect to the weak topology by its very definition. Hence $\sup \{ |x^*(x)| : x \in A \}$ is attained for every $x^* \in X^*$. 


This, however, clearly contradicts the existence of a \( z^* \in X^* \) for which the supremum is not attained. Consequently, \( A \) cannot be weakly compact.

Observe that the equivalence of (1) and (4) in Theorem 7.5 is precisely the statement of James’ Weak Compactness Theorem for separable subsets of a Banach space. To see this, one simply needs to realize that whenever the subset \( A \) is weakly compact or the supremum of each linear functional is attained on \( A \), then \( A \) is automatically bounded, and thus there is no loss of generality in assuming that \( A \) is contained in the closed unit ball (see Theorem 7.12 for the details).

### 7.2. Extension to the Non-Separable Case

The goal of the present section is to obtain new versions of Lemma 7.4 and Theorem 7.5, which do not require the subset \( A \) to be separable, but instead only require it to be balanced. Thankfully, we will be able to reuse a lot of the results from Section 7.1.

Before endeavoring to remove the separability requirement, however, we begin by making a simple, yet clever, observation. As the next lemma shows, we lose nothing in terms of the topology by passing to the real Banach space \( X_\mathbb{R} \) induced by a Banach space \( X \) (recall that \( X_\mathbb{R} \) was defined in Definition 1.5).

**Lemma 7.6.** Let \( X \) be a Banach space and let \( X_\mathbb{R} \) be the real Banach space obtained by restricting multiplication by scalars to \( \mathbb{R} \times X \). Then the weak topology \( \sigma (X_\mathbb{R}, (X_\mathbb{R})^*) \) on \( X_\mathbb{R} \) is equivalent to the weak topology \( \sigma (X, X^*) \) on \( X \).

**Proof.** If \( \mathbb{F} = \mathbb{R} \), then \((X_\mathbb{R}, \| \cdot \|) = (X, \| \cdot \|)\), so the statement is trivial. Hence we may assume that \( \mathbb{F} = \mathbb{C} \). Let \((x_\alpha)_{\alpha \in \mathcal{A}} \) be a net in \( X \) and let \( x \in X \). We prove both inclusions.

Suppose \( x_\alpha \to x \) in \((X_\mathbb{R}, \sigma (X_\mathbb{R}, (X_\mathbb{R})^*))\). By construction, every \( u^* \in (X_\mathbb{R})^* \) is \( \sigma (X_\mathbb{R}, (X_\mathbb{R})^*) \) continuous, so \( u^*(x_\alpha) \to u^*(x) \) in \( \mathbb{C} \). For each \( x^* \in X^* \) we have \( x^*(x_\alpha) \to x^*(x) = u^*(x) - iu^*(ix) \) with \( u^* = \text{Re}(x^*) \in (X_\mathbb{R})^* \) by Theorem 1.6. Moreover, \( x \to u^*(ix) \) is clearly also a bounded real-linear functional and hence continuous with respect to \( \sigma (X_\mathbb{R}, (X_\mathbb{R})^*) \). Hence \( u^*(ix_\alpha) \to u^*(ix) \) in \( \mathbb{C} \). It follows that

\[
x^*(x_\alpha) = u^*(x_\alpha) - iu^*(ix_\alpha) \to u^*(x) - iu^*(ix) = x^*(x)
\]

in \( \mathbb{C} \) for all \( x^* \in X^* \). We conclude that every \( x^* \in X^* \) is continuous with respect to \( \sigma (X_\mathbb{R}, (X_\mathbb{R})^*) \). By Theorem 5.1, \( \sigma (X, X^*) \) is the unique weakest topology for which every member of \( X^* \) is continuous, whence \( \sigma (X, X^*) \subseteq \sigma (X_\mathbb{R}, (X_\mathbb{R})^*) \).

Conversely, suppose \( x_\alpha \to x \) in \((X, \sigma (X, X^*))\). Fix an arbitrary \( u^* \in (X_\mathbb{R})^* \) and recall from Theorem 1.6 that there is a unique bounded complex-linear functional \( x^* \in X^* \) such that \( u^* = \text{Re}(x^*) \). Now, by construction of \( \sigma (X, X^*) \) we have \( x^*(x_\alpha) \to x^*(x) \) in \( \mathbb{C} \). Hence \( (x^*(x_\alpha))_{\alpha \in \mathcal{A}} \) is a net in \( \mathbb{C} \) converging to \( x^*(x) \), which means that for all \( r > 0 \) there is an \( \alpha_r \in \mathcal{A} \) such that \( x^*(x_\alpha) \in B(x^*(x), r) \) for every \( \alpha \geq \alpha_r \) in \( \mathcal{A} \). Moreover, we have

\[
|u^*(x_\alpha) - u^*(x)| = |u^*(x_\alpha - x)| = |\text{Re}(x^*(x_\alpha - x))| \\
\leq |x^*(x_\alpha - x)| = |x^*(x_\alpha) - x^*(x)|,
\]

which implies that \( u^*(x_\alpha) \in B(u^*(x), r) \) whenever \( x^*(x_\alpha) \in B(x^*(x), r) \). It follows that given \( r > 0 \), \( u^*(x_\alpha) \in B(u^*(x), r) \) for every \( \alpha \geq \alpha_r \) in \( \mathcal{A} \). Thus the net \( (u^*(x_\alpha)) \) converges to \( u^*(x) \) in \( \mathbb{C} \), so \( u^* \) is continuous with respect to \( \sigma (X, X^*) \). By appealing to Theorem 5.1 again, we conclude that \( \sigma (X_\mathbb{R}, (X_\mathbb{R})^*) \subseteq \sigma (X, X^*) \).

Lemma 7.6 allows us to restrict attention to \( X_\mathbb{R} \) when working with the weak topology on a normed space \( X \). Therefore, we shall only consider real Banach spaces in the remainder of Section 7.2. By means of Lemma 7.6, our results will then be seamlessly translated to the case of complex Banach spaces in the final proof of James’ Weak Compactness Theorem in Section 7.3. Before we can get any further, however, we need to build some extra machinery.
Definition 7.7. Let $X$ be a normed space over $\mathbb{R}$. Suppose that $(x^*_n)_{n\in\mathbb{N}}$ is a bounded sequence in $X^*$. Then we define

$$L(x^*_n) := \left\{ x^* \in X^* \mid x^*(x) \leq \limsup_{n \to \infty} x^*_n(x) \text{ for all } x \in X \right\}$$

and

$$\mathfrak{W}(x^*_n) := \left\{ (y^*_n)_{n \in \mathbb{N}} \subseteq X^* \mid y^*_n \in \text{co} \left\{ \{x^*_j : j \geq n\} \right\} \text{ for each } n \in \mathbb{N} \right\}.$$  

We emphasize that the field in question is $\mathbb{R}$. Hence $\limsup_{n \to \infty} x^*_n(x)$ is well-defined. In Lemma 7.8 below, we list a series of useful properties for the sets in Definition 7.7. In particular, we show in statement (6) of this lemma that $L(x^*_n)$ in fact satisfies

$$L(x^*_n) = \left\{ x^* \in X^* \mid \liminf_{n \to \infty} x^*_n(x) \leq x^*(x) \leq \limsup_{n \to \infty} x^*_n(x) \text{ for all } x \in X \right\}.$$  

This suggests that $L(x^*_n)$ is in some sense a set of potential limit points for $(x^*_n)$. While $\mathfrak{W}(x^*_n)$ is primarily introduced for notational convenience, the set $L(x^*_n)$ is really at the heart of our strategy to lose the separability requirement. We will have more to say about this right after the statement of Lemma 7.10. For now, we shall prove two lemmas about the properties of $L(x^*_n)$ and $\mathfrak{W}(x^*_n)$, which will make our lives considerably easier in the proofs of Lemma 7.10 and Theorem 7.11.

Lemma 7.8. Let $L(x^*_n)$ and $\mathfrak{W}(x^*_n)$ be as in Definition 7.7 above. Then the following statements are true:

1. Every bounded sequence $(x^*_n)$ in $X^*$ satisfies $(x^*_n) \in \mathfrak{W}(x^*_n)$.
2. If $(y^*_n) \in \mathfrak{W}(x^*_n)$, then $(y^*_n) \subseteq B(0, \sup \|x^*_n\| : n \in \mathbb{N})$.
3. If $(y^*_n) \in \mathfrak{W}(x^*_n)$, then $\text{co} \left\{ \{x^*_j : j \geq n\} \right\}$ implies

$$\|y^*_n\| \leq \sup_{j \geq n} \{ \|x^*_j\| : j \geq n \} \leq \sup_{j \in \mathbb{N}} \{ \|x^*_j\| : j \in \mathbb{N} \}$$

for each $n \in \mathbb{N}$. Hence $\{y^*_n : n \in \mathbb{N}\}$ is contained the closed ball with center 0 and radius $\sup \{\|x^*_n\| : n \in \mathbb{N}\}$.

To see that statement (3) is true, fix a sequence $(y^*_n) \in \mathfrak{W}(x^*_n)$ and notice that

$$\text{co} \left( \{x^*_j : j \geq N\} \right) \subseteq \text{co} \left( \{x^*_j : j \geq n\} \right)$$

whenever $N \geq n$. This implies that $y^*_N \in \text{co} \left( \{x^*_j : j \geq n\} \right)$ for every $N \geq n$, so that

$$\text{co} \left( \{y^*_j : j \geq N\} \right) \subseteq \text{co} \left( \{x^*_j : j \geq n\} \right).$$

Thus $(z^*_n) \in \mathfrak{W}(y^*_n)$ implies

$$z^*_n \in \text{co} \left( \{y^*_j : j \geq n\} \right) \subseteq \text{co} \left( \{x^*_j : j \geq n\} \right)$$

for each $n \in \mathbb{N}$, so that $(z^*_n) \in \mathfrak{W}(x^*_n)$. This proves the desired inclusion $\mathfrak{W}(y^*_n) \subseteq \mathfrak{W}(x^*_n)$.

For statement (4) we need to show that $L(y^*_n) \subseteq L(x^*_n)$ whenever $(y^*_n) \in \mathfrak{W}(x^*_n)$. To this end, fix a sequence $(y^*_n) \in \mathfrak{W}(x^*_n)$ and suppose $z^* \in L(y^*_n)$. Then by definition of $z^* \in L(y^*_n)$ we have

$$z^*(x) \leq \limsup_{n \to \infty} y^*_n(x) = \inf_{n \in \mathbb{N}} \left( \sup_{j \geq n} y^*_j(x) \right)$$

for all $x \in X$. Now, for each $n \in \mathbb{N}$ it follows from $(y^*_n) \in \mathfrak{W}(x^*_n)$ that $y^*_j \in \text{co} \left( \{x^*_i : i \geq n\} \right)$ for all $j \geq n$. Hence $y^*_j(x) \leq \sup_{i \geq n} x^*_i(x)$ for all $j \geq n$ and therefore

$$\sup_{i \geq n} y^*_i(x) \leq \limsup_{n \to \infty} x^*_i(x).$$

We conclude that

$$z^*(x) \leq \limsup_{n \to \infty} y^*_n(x) = \inf_{n \in \mathbb{N}} \left( \sup_{j \geq n} y^*_j(x) \right) \leq \inf_{n \in \mathbb{N}} \left( \sup_{j \geq n} x^*_j(x) \right) = \limsup_{n \to \infty} x^*_n(x)$$

for all $x \in X$. That is, $z^* \in L(x^*_n)$. This proves $L(y^*_n) \subseteq L(x^*_n)$.

We now prove statement (5). First observe that, for each $x \in X$, $x^* \in L(x^*_n)$ satisfies

$$x^*(x) \leq \limsup_{n \to \infty} x^*_n(x) \leq \sup_{n \in \mathbb{N}} x^*_n(x) \leq \sup_{n \in \mathbb{N}} \|x^*_n\| \cdot \|x\|_X.$$
In particular, we also have
\[-x^*(x) = x^*(-x) \leq \sup_{n \in \mathbb{N}} \|x_n^*\|_{X^*}, \quad -x = \sup_{n \in \mathbb{N}} \|x_n\|_X, \quad \|x\|_X \]
for all \(x \in X\). Therefore, \(|x^*(x)| \leq \sup_{n \in \mathbb{N}} \|x_n^*\|_{X^*} \cdot \|x\|_X\) for each \(x \in X\). It follows that
\[\|x^*\|_{X^*} = \sup \{|x^*(x)| : x \in B_X\} \leq \sup_{n \in \mathbb{N}} \|x_n^*\|_{X^*} .\]

Finally, statement (6) follows easily from \(\liminf_{n \to \infty} \alpha_n = -\limsup_{n \to \infty} -\alpha_n\). Indeed, if \(x^* \in L(x_n^*)\) then \(x^*(x) \leq \limsup_n x_n^*(x)\) for all \(x \in X\) and therefore \(-x^*(x) = x^*(-x) \leq \limsup_n x_n^*(-x)\) so that \(-\limsup_n x_n^*(-x) \leq x^*(x)\). Hence
\[\liminf_n x_n^*(x) = -\limsup_n -x_n^*(x) = -\limsup_n x_n^*(-x) \leq x^*(x)\]
for all \(x \in X\) as desired. \(\square\)

In what follows, we need \(L(x_n^*)\) to be nonempty. Luckily, the following lemma assures us that this is always the case, as long as the sequence in question is bounded (which is precisely what we required in the definition).

**Lemma 7.9.** Let \(X\) be a normed space over \(\mathbb{R}\). Suppose \((x_n^*)\) is a bounded sequence in \(X^*\). Then \(L(x_n^*)\) is nonempty.

**Proof.** Define a sublinear functional \(p : X \to \mathbb{R}\) on \(X\) by \(p(x) := \limsup_{n \to \infty} x_n^*(x)\). The sublinearity follows immediately from the linearity of \(x_n^*\) together with the well-known properties of the limit superior. Now consider the trivial subspace \(\{0\}\) of \(X\) and let \(z^*\) denote the zero functional on \(\{0\}\). Then \(z^*(0) = 0 = p(0)\), so \(z^*(x) \leq p(x)\) for every \(x \in \{0\}\). Hence the vector space version of the Hahn-Banach Theorem (Theorem A.1) yields the existence of linear (but not necessarily continuous) extension \(x^*\) such that \(x^*(x) \leq p(x)\) for every \(x \in X\). It follows that \(x^*(x) \leq p(x) = \limsup_n x_n^*(x) \leq \sup_{n \in \mathbb{N}} x_n^*(x) \leq \sup_{n \in \mathbb{N}} \|x_n^*\|_{X^*} \cdot \|x\|_X\) for every \(x \in X\) and analogously
\[-x^*(x) = x^*(-x) \leq \sup_{n \in \mathbb{N}} \|x_n^*\|_{X^*} \cdot \|x\|_X .\]
Hence
\[|x^*(x)| \leq \sup_{n \in \mathbb{N}} \|x_n^*\|_{X^*} \cdot \|x\|_X\]
for every \(x \in X\), which together with the boundedness of \((x_n^*)\) implies that
\[\|x^*\|_{X^*} = \sup \{|x^*(x)| : x \in B_X\} \leq \sup_{n \in \mathbb{N}} \|x_n^*\|_{X^*} < \infty .\]
Thus \(x^* : X \to \mathbb{R}\) is bounded, so \(x^* \in X^*\) as desired. As we have already seen that \(x^*(x) \leq \limsup_n x_n^*(x)\) for all \(x \in X\), it follows that \(x^* \in L(x_n^*)\). \(\square\)

We are now ready to embark upon the key technical lemma of this section, in the shape of Lemma 7.10. Before actually proving it, however, we shall take some time to truly appreciate the motivation behind our introduction of \(L(\cdot)\). Hopefully, this will also provide the reader with a firmer grasp of the overall strategy underlying this section.

**Lemma 7.10.** Let \(X\) be a normed space over \(\mathbb{R}\). Let \(0 < \theta < 1\) be given and suppose that

(i) \(A\) is a nonempty balanced subset of the closed unit ball \(B_X\).
(ii) \((\beta_n)_{n \in \mathbb{N}}\) is a sequence of strictly positive reals with \(\sum_n \beta_n = 1\).
(iii) \((x_n^*)_{n \in \mathbb{N}}\) is a sequence in \(B_{X^*}\) such that \(\sup \{|(x^* - w^*) (x) | : x \in A\} \geq \theta\) for every \(x^* \in \text{co} \{x_n^* : n \in \mathbb{N}\}\) and \(w^* \in L(x_n^*)\).

Then there exists a real number \(\alpha \geq \theta \leq \alpha \leq 2\) and a sequence \((y_n^*)_{n \in \mathbb{N}}\) in \(B_{X^*}\) such that whenever \(w^* \in L(y_n^*)\) then it holds that

1. \(\sup \left\{ \sum_{j=1}^n \beta_j (y_j^* - w^*) (x) : x \in A \right\} = \alpha .\)
2. \(\sup \left\{ \sum_{j=1}^n \beta_j (y_j^* - w^*) (x) : x \in A \right\} < \alpha(1 - \theta \sum_{j=n+1}^\infty \beta_j)\) for \(n \in \mathbb{N}\).
Moreover, from assumption (iii). Then the condition that
be chosen such that
The remaining part of the proof proceeds in four steps.

(1) 
, \therefore \text{Step 2 below.}
In Theorem 7.5, \(y_n \to 0\) was used to squeeze the tail of \(\sum_{j=1}^{\infty} \beta_j y_j\) less than some desired number (see the second paragraph of (3) → (4) in the proof of Theorem 7.5).
Obviously, we don’t have quite the same level of control over the tail of \(\sum_{j=1}^{\infty} \beta_j (y_j - w)\) when \((y_n)\) is not convergent to 0. Nonetheless, as we shall see later in Theorem 7.11, \(w \in L(y_n)\) allows us to find certain terms of \(y_n - w\) that are as small as we like, and this turns out to be sufficient for our purposes.

For now, however, it is time to return to the proof of Lemma 7.10, which is the one that follows below.

Proof. As in Lemma 7.4, we introduce the seminorm \(|x^*|_A := \sup \{|x^*(x)| : x \in A\}\) defined for all \(x^* \in X^*\). Recall that this was shown to be continuous and satisfy \(|x^*|_A \leq \|x^*\|_X\), for every \(x^* \in X^*\). Precisely as in Step 2 of Lemma 7.4, we also fix a convergent sequence \((\varepsilon_n)\) of positive reals with \(\lim_{n \to \infty} \varepsilon_n = 0\) such that

\[
0 < \sum_{k=1}^{\infty} \frac{\beta_k \varepsilon_k}{\sum_{j=k+1}^{\infty} \beta_j \sum_{j=k}^{\infty} \beta_j} < 1 - \theta.
\]

The remaining part of the proof proceeds in four steps.

**Step 1.** We wish to construct a sequence of scalars \((\alpha_n)_{n \in \mathbb{N}}\) in \(\mathbb{F}\), a sequence of linear functionals \((y^*_j)_{j \in \mathbb{N}}\) in \(X^*\), and two collections of sequences \((\alpha x^*_j), (x^*_j), (z^*_j), (\alpha z^*_j), \ldots \) in \(X^*\). By means of induction we show that these sequences can

be chosen such that \((\alpha x^*_j)\) is in \(B_X\), and such that, for each \(n \in \mathbb{N}\), we have

(I) \(y_n, (\alpha z^*_j), (\alpha x^*_j)\) lie in \(B_X\).

(II) \((\alpha z^*_j) \in \mathcal{B}_{(n - 1)z^*_j}\)

(III) \((\alpha x^*_j)\) is a subsequence of \((\alpha z^*_j)\)

(IV) \(y_n \in \text{co} \left\{ \{n - 1)x^*_j : j \geq n\} \right\}\)

(V) \(\theta \leq \alpha_n \leq 2\)

(VI) \(\alpha_{n-1} \leq \alpha_n \) if \(n \geq 2\)

Before beginning the induction, first choose \((\alpha x^*_j) := (x^*_j)\), where \((x^*_j)\) is the sequence from assumption (iii). Then the condition that \((\alpha x^*_j)\) should lie in \(B_X\) is satisfied.

In Step 2 below, we proceed to show that the above conditions are all satisfied for the base case \(n = 1\). The inductive step is then completed in Step 3.

**Step 2.** Let \(y^* \in \text{co} \left\{ (\alpha x^*_j) : j \geq 1 \right\}\) and \((\alpha_j) \in \mathcal{B}_{(\alpha x^*_j)}\). Then \((\alpha x^*_j) \subseteq B_X\) implies \(y^* \in B_X\). Also, by definition of \((\alpha_j) \in \mathcal{B}_{(\alpha x^*_j)}\), we have \(v_n \in \text{co} \left( \{x^*_j : j \geq n\} \right) \subseteq B_X\) for each \(n \in \mathbb{N}\), so \((\alpha) \subseteq B_X\). In particular, we thus have \(L(\alpha) \subseteq B_X\) by statement (5) in Lemma 7.8. It follows that if \(w^* \in L(\alpha)\), then

\[
|y^* - w^*|_A \leq |y_j^*|_A + |w^*|_A \leq \|y^*\| + \|w^*\| \leq 2.
\]

Moreover, \((\alpha_j) \in \mathcal{B}_{(\alpha x^*_j)}\) implies that \(L(\alpha_j) \subseteq L(\alpha x^*_j)\) by statement (4) of Lemma 7.8, so if \(w^* \in L(\alpha_j)\), then \(w^* \in L(\alpha x^*_j)\). Thus, assumption (iii) ensures that

\[
|y^* - w^*|_A = \sup \{|(x^* - w^*) (x) : x \in A\} \geq \theta.
\]
We conclude that the set
\[
S_1(y^*, \{v_j^*\}) := \{[y^* - w^*]_A : w^* \in L(v_j^*)\}
\]
is bounded below by \(\theta\) and above by 2. Also, \(L(v_j^*)\) is nonempty according to Lemma 7.9, so \(S_1(y^*, \{v_j^*\})\) is nonempty. In particular,

\[
\alpha_1 := \inf \{\sup S_1(y^*, \{v_j^*\}) : y^* \in \operatorname{co}(\{a x_j^* : j \geq 1\}), (v_j^*) \in \mathcal{W}(a x_j^*)\}
\]
is a well-defined number between \(\theta\) and 2. Thus condition (V) is satisfied for \(n = 1\).

**Step 2.1.** Given our choice of \(\alpha_1\), and the definition of the infimum, we can choose \(y_1^*\) from \(\operatorname{co}(\{a x_j^* : j \geq 1\})\) and \((z_j^*)\) from \(\mathcal{W}(a x_j^*)\) such that
\[
\alpha_1 \leq \sup \{[y_1^* - w^*]_A : w^* \in L(z_j^*)\} < \alpha_1(1 + \varepsilon_1).
\]
This takes care of conditions (II) and (IV). Also, \(y_1^* \in B_X^*\) and \((z_j^*) \subseteq B_X^*\) by the initial remarks in Step 1. All that remains is to choose an appropriate subsequence \((x_j^*)\) of \((z_j^*)\) in \(B_X^*\). Then conditions (I) and (II) would also be satisfied, which would conclude the base case. To this end, notice that because of equation (7.7), and the definition of the supremum, we can choose \(w_1^*\) from \(L(z_j^*)\) such that
\[
\alpha_1(1 - \varepsilon_1) < [y_1^* - w_1^*]_A = \sup \{[y_1^*(x) - w_1^*(x)] : x \in A\}.
\]
Since \(A\) is balanced by assumption (i), we have \(-A = A\), so the absolute value in equation (7.8) can be dropped. Hence it follows from equation (7.8), and the definition of the supremum, that there exists \(x_1 \in A\) such that
\[
\alpha_1(1 - \varepsilon_1) < y_1^*(x_1) - w_1^*(x_1).
\]
Clearly, \(|\lim \inf_j (z_j^*(x_1))| \leq 1\) by the fact that \((z_j^*)\) is contained in \(B_X^*\) and \(x_1 \in A \subseteq \bar{B}_X\). Now choose \(x_j^*(x_1)\) to be a subsequence of \((z_j^*)\) such that \((x_j^*(x_1))\) converges to \(\lim \inf_j (z_j^*(x_1))\). Then \(y_1^*, (z_j^*)\) and \((x_j^*)\) fulfills all the requirements for the base case of \(n = 1\).

**Step 3.** In Step 2 we chose \(\alpha_1, y_1^*, (z_j^*)\) and \((x_j^*)\) in a very particular way. We now show that \(\alpha_n, y_n^*, (z_j^*)\) and \((x_j^*)\) can be chosen according to the same pattern for all \(n \in \mathbb{N}\). The argument is based on strong induction and proceeds through a series of intermediate claims.

**Induction Hypothesis.** Let \(m \geq 2\) be given. Then we assume that \(\alpha_n, y_n^*, (z_j^*)\), and \((x_j^*)\) have already been chosen such that they satisfy conditions (I) through (VI) for each \(n = 1, \ldots, m - 1\). The particular way, in which they are chosen, will be detailed as we move along.

**Step 3.1.** We claim that if \((v_j^*) \in \mathcal{W}(m-1 x_j^*)\), then \((v_j^*)\) is contained in \(B_{X^*}\). It suffices to show that \((m-1 x_j^*)\) is contained in \(B_{X^*}\) since then \(v_n^* \in \operatorname{co}(\{m-1 x_j^* : j \geq n\}) \subseteq B_{X^*}\) for each \(n \in \mathbb{N}\). That \((m-1 x_j^*) \subseteq B_{X^*}\) follows immediately from our induction hypothesis with regard to condition (I).

**Step 3.2.** With a view towards the construction of \((\alpha_j)\), we claim that if \(y^* \in \operatorname{co}(\{m-1 x_j^* : j \geq m\})\) and \((v_j^*) \in \mathcal{W}(m-1 x_j^*)\), then
\[
S_m(y^*, (v_j^*)) := \left\{ \left. [y^* - w^*]_A : w^* \in L(v_j^*) \right\} \right\}
\]
defines a nonempty subset of the closed interval \([0, 2] \subseteq \mathbb{R}\) for each pair \(y^*\) and \((v_j^*)\). In particular, the scalar
\[
\alpha_m := \inf \{\sup S_m(y^*, (v_j^*)) : y^* \in \operatorname{co}(\{m-1 x_j^* : j \geq m\}), (v_j^*) \in \mathcal{W}(m-1 x_j^*)\}
\]
is then a well-defined number between 0 and 2. To see that the claim is indeed true, fix an arbitrary \(y^* \in \operatorname{co}(\{m-1 x_j^* : j \geq m\})\) and an arbitrary \((v_j^*) \in \mathcal{W}(m-1 x_j^*)\). Since \((m-1 x_j^*)\) is in \(B_{X^*}\) by the induction hypothesis, we have \(y^* \in \operatorname{co}(\{m-1 x_j^* : j \geq m\}) \subseteq B_{X^*}\). In addition, the induction hypothesis also ensures that \(y_1^*, \ldots, y_{m-1}^* \in B_{X^*}\).

Furthermore, \((v_j^*) \in \mathcal{W}(m-1 x_j^*)\) implies \((v_j^*) \subseteq B_{X^*}\) by the claim in Step 3.1, so \(L(v_j^*) \subseteq B_{X^*}\) by statement (5) of Lemma 7.8.
It follows from the above that
\[
0 \leq \left| \sum_{j=1}^{m-1} \beta_j y_j^* + \left( \sum_{j=m}^{\infty} \beta_j \right) y^* - w^* \right|_A \leq \left| \sum_{j=1}^{m-1} \beta_j y_j^* + \left( \sum_{j=m}^{\infty} \beta_j \right) y^* \right|_A + |w^*|_A \leq 1 + 1 = 2
\]
for every \( w^* \in L(v_j^*) \) since \( L(v_j^*) \subseteq \tilde{B}_X \) and \( \sum_{j=1}^{m-1} \beta_j y_j^* + \left( \sum_{j=m}^{\infty} \beta_j \right) y^* \) belongs to \( \tilde{B}_X \) as a convex combination of elements in \( \tilde{B}_X \). Recalling that \( L(v_j^*) \) is nonempty by Lemma 7.9, the claim follows.

**Step 3.3.** We claim that \( \mathfrak{V}(m-1,x_j^*) \subseteq \mathfrak{V}(m-2,x_j^*) \) for every \( m \geq 2 \). By Lemma 7.8(3) it suffices to show that \( (m-1,x_n^*) \in \mathfrak{V}(m-2,x_j^*) \). To see that this is the case, simply observe that if \( m \geq 2 \), then \( (m-1,x_j^*)_n \) is a subsequence of \( (m-1,x_j^*)_n \) by the induction hypothesis with regard to (II). Since \( (m-1,x_j^*)_n \in \mathfrak{V}(m-2,x_j^*) \) by the induction hypotheses in relation to (II), we thus have \( m-1,x_n^* \in \text{co} \left( \{m-2,x_j^*: j \geq n\} \right) \) for each \( n \in \mathbb{N} \). This, in turn, implies that \( (m-1,x_n^*) \) itself belongs to \( \mathfrak{V}(m-2,x_j^*) \).

**Step 3.4.** We claim that if \( m \geq 2 \) and \( x^* \in \text{co} \left( \{m-1,x_j^*: j \geq m\} \right) \), then
\[
\frac{\beta_{m-1}}{\sum_{j=1}^{m-1} \beta_j} y_{m-1}^* + \sum_{j=m}^{\infty} \frac{\beta_j}{\sum_{j=1}^{m-1} \beta_j} x^* \in \text{co} \left( \{m-2,x_j^*: j \geq m-1\} \right). \tag{7.10}
\]
To see that this is true, we first observe that \( (m-1,x_j^*)_n \in \mathfrak{V}(m-1,x_j^*) \subseteq \mathfrak{V}(m-2,x_j^*) \) by Lemma 7.8(1) and the claim in Step 3.3. Therefore, \( m-1,x_n^* \in \text{co} \left( \{m-2,x_j^*: j \geq n\} \right) \) for each \( n \in \mathbb{N} \). In particular, \( m-1,x_k^* \in \text{co} \left( \{m-2,x_j^*: j \geq m-1\} \right) \) for all \( k \geq m \). This implies that
\[
x^* \in \text{co} \left( \{m-1,x_j^*: k \geq m\} \right) \subseteq \text{co} \left( \{m-2,x_j^*: j \geq m-1\} \right).
\]
In addition to this, \( y_{m-1}^* \in \text{co} \left( \{m-2,x_j^*: j \geq m-1\} \right) \) by the induction hypothesis with respect to (IV). Since every convex combination of elements in \( \text{co} \left( \{m-2,x_j^*: j \geq m-1\} \right) \) is again in \( \text{co} \left( \{m-1,x_j^*: j \geq m-1\} \right) \), equation (7.10) is true.

**Step 3.5.** We claim that \( \alpha_{m-1} \leq \alpha_m \) as long as \( m \geq 2 \). To see this, first recall the definition of \( S_m \left( y^*, (v_j^*) \right) \) from Step 3.2 and notice that when \( n = 1 \), this definition agrees with the definition of \( S_1 \left( y^*, (v_j^*) \right) \) in Step 2. Now observe that
\[
S_m \left( \sum_{j=1}^{m-1} \beta_j y_j^* + \sum_{j=m}^{\infty} \beta_j x^* \right)
= \left\{ \sum_{j=1}^{m-1} \beta_j y_j^* + \left( \sum_{j=m}^{\infty} \beta_j \right) x^* : w^* \in L(v_j^*) \right\}
= \left\{ \sum_{j=1}^{m-1} \beta_j y_j^* + \beta_{m-1} y_{m-1}^* + \left( \sum_{j=m}^{\infty} \beta_j \right) x^* - w^* \right|_A \in L(v_j^*) \right\}
= \left\{ \sum_{j=1}^{m-1} \beta_j y_j^* + \left( \sum_{j=m}^{\infty} \beta_j \right) x^* - w^* \right|_A : w^* \in L(v_j^*) \right\}
= S_m \left( x^*, (v_j^*) \right).
\]
Recall that the infimum over a subset is at least as large as the infimum over the original set. Hence it follows from the claims in Steps 3.3 and 3.4, together with the
above computation, that
\[
\alpha_{m-1} = \inf \left\{ \sup_{m-1} \{ y^*, (v_j^*) : y^* \in \text{co} \left( \{ m-2x_j^* : j \geq m-1 \} \} , (v_j^*) \in \mathcal{V}(m-2x_j^*) \} \right\}
\]
\[
\leq \inf \left\{ \sup_{m-1} \{ y^*, (v_j^*) : y^* \in \left\{ \sum_{j=m-1}^{\infty} \beta_j y_m^* + \sum_{j=m-1}^{\infty} \beta_j x^* : x^* \in \text{co} \left( \{ m-1x_j^* : j \geq m \} \} , (v_j^*) \in \mathcal{V}(m-1x_j^*) \right\} \right\}
\]
\[
= \inf \left\{ \sup_{m-1} \left( \sum_{j=m-1}^{\infty} \beta_j y_m^* + \sum_{j=m-1}^{\infty} \beta_j x^* , (v_j^*) \right) : x^* \in \text{co} \left( \{ m-1x_j^* : j \geq m \} \} , (v_j^*) \in \mathcal{V}(m-1x_j^*) \right\}
\]
\[
= \inf \left\{ \sup_{m-1} (x^*, (v_j^*)) : x^* \in \text{co} \left( \{ m-1x_j^* : j \geq m \} \} , (v_j^*) \in \mathcal{V}(m-1x_j^*) \right\}
\]
\[
= \alpha_m.
\]
This proves the claim. In particular, condition (VI) is satisfied for \( n = m \).

**Step 3.6.** We claim that \( \theta \leq \alpha_m \leq 2. \) Notice that, because of the claim in Step 3.5, it suffices to show that \( \alpha \geq \theta. \) This, however, was already proved in Step 2, when \( \alpha_1 \) was chosen. Hence condition (V) is satisfied for \( n = m. \)

**Step 3.7** Given our choice of \( \alpha_m, \) and by definition of the infimum, we can choose \( y_m^* \) in \( \text{co} \left( \{ m-1x_j^* : j \geq m \} \} \) and \( (m z_j^*) \) in \( \mathcal{V}(m-1x_j^*) \) such that
\[
\alpha_m \leq \sup \left\{ \sum_{j=1}^{m-1} \beta_j y_j^* + \left( \sum_{j=m}^{\infty} \beta_j \right) y_m^* - w^* \right\} < \alpha_m(1 + \varepsilon_m).
\]

Then conditions (II) and (IV) are satisfied for \( n = m. \) Note also that \( y_m^* \in \mathcal{B}_X \) since \( (m-1 x_j^*) \subset \mathcal{B}_X \) by the induction hypothesis. Likewise, \( (m z_j^*) \subset \mathcal{B}_X \) by the claim in Step 3.1 because \( (m z_j^*) \subset \mathcal{V}(m-1x_j^*) \). Hence the only unresolved issue is our choice of the subsequence \( (m x_j^*) \) of \( (m z_j^*) \), which will automatically satisfy the remaining conditions (I) and (III).

To this end, we begin by writing out the expression behind \( S_m \left( y_m^*, (m z_j^*) \right) \) in the above inequality, which leaves us with
\[
\alpha_m \leq \sup \left\{ \left| \sum_{j=1}^{m-1} \beta_j y_j^* + \left( \sum_{j=m}^{\infty} \beta_j \right) y_m^* - w^* \right| \right\} < \alpha_m(1 + \varepsilon_m). \quad (7.11)
\]

Based on equation (7.11), and by definition of the supremum, we can thus choose \( w_m^* \) in \( L(m z_j^*) \) such that
\[
\alpha_m(1 - \varepsilon_m) < \left| \sum_{j=1}^{m-1} \beta_j y_j^* + \left( \sum_{j=m}^{\infty} \beta_j \right) y_m^* - w_m^* \right|_A.
\]

Now using the fact that \( A \) is balanced by assumption (i), we get that \( A = -A, \) and therefore
\[
\left| \sum_{j=1}^{m-1} \beta_j y_j^* + \left( \sum_{j=m}^{\infty} \beta_j \right) y_m^* - w_m^* \right| = \sup_{x \in A} \left\{ \left| \sum_{j=1}^{m-1} \beta_j y_j^*(x) + \left( \sum_{j=m}^{\infty} \beta_j \right) y_m^*(x) - w_m^*(x) \right| \right\}
\]
\[
= \sup_{x \in A} \left\{ \sum_{j=1}^{m-1} \beta_j y_j^*(x) + \left( \sum_{j=m}^{\infty} \beta_j \right) y_m^*(x) - w_m^*(x) \right\}
\]

Hence there is an \( x_m \in A \) such that
\[
\alpha_m(1 - \varepsilon_m) < \sum_{j=1}^{m-1} \beta_j y_j^*(x_m) + \left( \sum_{j=m}^{\infty} \beta_j \right) y_m^*(x_m) - w_m^*(x_m). \quad (7.12)
\]
By the claim in Step 3.1, \((m_z^*) \in \mathcal{W}(m-1)\) implies \((m_z^*) \subseteq \bar{B}_Y\), so \(x_m \in A \subseteq \bar{B}_X\) ensures that \(\liminf \{m_z^*(x_m)\} \leq 1\). Given this we may choose \((m_x^*)\) as a subsequence of \((m_z^*)\) such that \(\{m_x^*(x_m)\}\) converges to \(\liminf \{m_x^*(x_m)\}\) in \(\bar{B}_Y\). Since our particular choices of \(\alpha_m, y_m^*, (m_z^*)\), and \((m_x^*)\) satisfy all the conditions (I) through (V), the induction is complete.

**Step 4.** With the help of the sequences just constructed we shall now prove that \((y_j^*)\) satisfies (1) and (2) from the statement of the lemma. First we need two intermediate results, which are presented in Steps 4.1 and 4.2. Then we prove (1) in Step 4.3 and (2) in Step 4.4.

**Step 4.1.** We claim that \(L(y_j^*) \subseteq \bigcap_{m=0}^{\infty} L(n_z^*) \subseteq \bigcap_{m=1}^{\infty} L(n_z^*)\). For the second inclusion it is enough to show that \(L(n_x^*) \subseteq L(n_z^*)\) for every \(n \in \mathbb{N}\). To this end, fix \(n \in \mathbb{N}\) and observe that since \((n_x^*)\) is a subsequence of \((n_z^*)\) we have \((n_x^*) \subseteq \mathcal{W}(n_z^*)\). Hence \(L(n_x^*) \subseteq L(n_z^*)\) by Lemma 7.8(4).

For the first inclusion it suffices to show that \(\limsup_{k} \{y_j^*(x)\} \leq \limsup_{k} \{n_x^*(x)\}\) for every \(x \in A\) and every \(n \in \mathbb{N}\). To prove this is the case, fix any \(n \in \mathbb{N}\) and let \(k > n\) be arbitrary. Since \((k_x^*)\) is a subsequence of \((k_z^*)\) by (III), we have \(\{k_x^*: j \geq k\} \subseteq \{k_z^*: j \geq k\}\). Moreover, since \((k_z^*)\) is a subsequence of \((k-1)\) by (II), it holds that \(k_x^*, k_x^*, k_x^*, k_x^*, \ldots \in \{n_x^*: j \geq k\}\) and therefore \(\{k_x^*: j \geq k\} \subseteq \{n_x^*: j \geq k\}\).

It follows that

\[
\co \{n_x^*: j \geq k\} \subseteq \co \{k_x^*: j \geq k\} \subseteq \cdots \subseteq \co \{n_x^*: j \geq k\}.
\]

In particular, we have \(y_k^* \in \{k_x^*: j \geq k\} \subseteq \{n_x^*: j \geq k\}\) for every \(k > n\) by an appeal to (IV) and the above result. Hence \(\sup_{j \geq k} \{y_j^*(x)\} \leq \sup_{j \geq k} \{n_x^*(x)\}\) for every \(k > n\) so that \(\lim_{k \to \infty} \sup_{j \geq k} \{y_j^*(x)\} = \lim_{k \to \infty} \sup_{j \geq k} \{n_x^*(x)\}\). Since \(n \in \mathbb{N}\) was arbitrary we have \(\limsup_{k} \{y_j^*(x)\} \leq \limsup_{k} \{n_x^*(x)\}\) for every \(n \in \mathbb{N}\), which proves the claim.

**Step 4.2.** Suppose that \(w^* \in L(y_j^*)\). Then we claim that

\[
\alpha_n(1 - \varepsilon_n) < \left[ \sum_{j=1}^{n-1} \beta_j y_j^* + \left( \sum_{j=n}^{\infty} \beta_j \right) y_n^* - w^* \right]_A < \alpha_n(1 + \varepsilon_n)
\]  
(7.13)

for every \(n \in \mathbb{N}\). For the second inequality, simply note that if \(w^* \in L(y_j^*)\), then \(w^* \in L(n_x^*)\) for every \(n \in \mathbb{N}\) by the claim in Step 4.1. If \(n \geq 2\), then the conclusion follows from (7.11) since \(w^* \in L(n_z^*)\). Analogously, the case where \(n = 1\) follows from (7.7) since \(w^* \in L(1)\). For the first inequality it suffices to show that (7.12) and (7.9) remain true if \(w_x^*\) is replaced by any \(w^* \in L(y_j^*)\). To prove this, we show that \(w^* \leq w_n^*(x_n)\) for every \(n \in \mathbb{N}\). Now, if \(w^* \leq L(n_y^*)\), then \(w^* \leq L(n_x^*)\) for every \(n \in \mathbb{N}\) by the claim in Step 4.1. Thus, it follows from Lemma 7.8(6) that \(\liminf_{j} \{n_x^*(x_n)\} \leq \liminf_{j} \{n_x^*(x_n)\}\). Recall from our choice of \((n_x^*)\) in Step 3.7 (for \(n \geq 2\) and Step 3 (for \(n = 1\)) that \((n_x^*(x_n)\)) is convergent with \(\liminf_{j} \{n_x^*(x_n)\}\) as its limit. Hence

\[
w^*(x_n) = \lim_{j \to \infty} \{n_x^*(x_n)\} = \liminf_{j \to \infty} \{n_x^*(x_n)\}.
\]

Since \(w_n^*\) was chosen from \(L(n_z^*)\) in Step 3.7 (for \(n \geq 2\)) and Step 3 (for \(n = 1\)), another application of Lemma 7.8(6) yields that \(\liminf_{j} \{n_x^*(x_n)\} \leq w_n^*(x_n)\). From this we conclude that \(w^* \leq w_n^*(x_n)\) as desired.

**Step 4.3.** We now show that statement (1) is satisfied with \(\alpha := \lim_{j \to \infty} \alpha_j\). First of all, it follows from (VI) and (V) that \((\alpha_j)\) is increasing, bounded below by \(\theta\), and bounded above by 2. Hence the sequence is convergent and \(\theta \leq \lim_j \alpha_j \leq 2\). Now fix any \(w^* \in L(y_j^*)\). Then (7.13) applies by the claim in Step 4.2.
Using the same arguments as in Step 4.1 of Lemma 7.4, we conclude from the squeeze theorem and the continuity of $| |_A$ that

$$
\alpha = \lim_{n \to \infty} \left| \frac{1}{n} \sum_{j=1}^{n-1} \beta_j y^*_j + \left( \sum_{j=n}^{\infty} \beta_j \right) y^*_n - w^* \right|_A = \left| \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n-1} \beta_j y^*_j + \left( \sum_{j=n}^{\infty} \beta_j \right) y^*_n - w^* \right|_A = \left| \sum_{j=1}^{\infty} \beta_j (y^*_j - w^*) \right|_A.
$$

This proves statement (1).

**Step 4.4.** It only remains to show that statement (2) is satisfied for the same $\alpha$ as in Step 4.3. The proof of this is completely analogous to the proof in Step 4.2 of Lemma 7.4 with $y^*_j$ replaced by $(y^*_j - w^*)$. Indeed, by virtue of (7.13), the strict inequality in (7.4) remains true when $y^*_j$ replaced by $(y^*_j - w^*)$ inasmuch as

$$
\left| \sum_{j=1}^{n-1} \beta_j (y^*_j - w^*) + \left( \sum_{j=n}^{\infty} \beta_j \right) (y^*_n - w^*) \right|_A = \left| \sum_{j=1}^{n-1} \beta_j y^*_j + \left( \sum_{j=n}^{\infty} \beta_j \right) y^*_n - w^* \right|_A < \alpha n(1 + \varepsilon_n)
$$

for every $n \in \mathbb{N}$. All the other arguments in Step 4.2 of Lemma 7.4 rely only on the properties of $| |_A$, $(\beta_j)$, and $(\varepsilon_k)$, which satisfy all the same conditions here as they did in Lemma 7.4. \qed

For the all-important Theorem 7.11, we shall need the concept of annihilators. Recall that, given a normed space $X$, the *annihilator* of a subset $A \subseteq X$ is denoted by $A^\perp$ and defined as

$$
A^\perp := \{ x^* \in X^* : x^*(x) = 0 \text{ for all } x \in A \}.
$$

In relation to Theorem 7.11, there are two key observations to make. First, if we are only interested in the behavior of a linear functional on a certain subset of $X$, then we may freely add or deduct linear functionals in the annihilator of that subset. This idea motivates the statement in (2) of Theorem 7.11. Secondly, if $(x^*_n)$ is a sequence in $X^*$, then $L(x^*_n)$ is contained in the annihilator of any subset of $\{ x \in X : x^*_n(x) \to 0 \}$. This second observation will allow us to apply Lemma 7.10 in the proof of (2) $\Rightarrow$ (3) below. We also remind the reader that the rôle of $L(\cdot)$ was already discussed after the statement of Lemma 7.10 above.

**Theorem 7.11.** Let $X$ be a Banach space over $\mathbb{R}$. Suppose that $A$ is a nonempty, balanced, weakly closed subset of the closed unit ball $B_X$. Then the following statements are equivalent:

1. The set $A$ is not weakly compact.
2. There is a $\theta \in \mathbb{R}$ with $0 < \theta < 1$, a subset $A_0 \subseteq A$ of $A$, and a sequence $(x^*_n)$ in the closed unit ball $B_X$, with $\lim_{n \to \infty} x^*_n(x) = 0$ for each $x \in A_0$ such that $\sup \{ |x^*(x^*_n(x))| : x^*_n : n \in \mathbb{N} \} \geq \theta$ for every $x^* \in \text{co} \{ x^*_n : n \in \mathbb{N} \}$ and $w^* \in A_0^\perp$.
3. There is a $\theta \in \mathbb{R}$ with $0 < \theta < 1$ such that if $(\beta_n)$ is a sequence of strictly positive reals with $\sum_n \beta_n = 1$, then there is an $\alpha \in \mathbb{R}$ with $\theta \leq \alpha \leq 2$ and a sequence $(y^*_n)$ in $B_{X^*}$ such that whenever $w^* \in L(y^*_n)$ we have
   
   (a) $\sup \left\{ \sum_{j=1}^{\infty} \beta_j (y^*_j - w^*) : x \in A \right\} = \alpha.$
   
   (b) $\sup \left\{ \sum_{j=1}^{\infty} \beta_j (y^*_j - w^*) : x \in A \right\} < \alpha(1 - \theta \sum_{j=n+1}^{\infty} \beta_j), \ n \in \mathbb{N}.$
4. There is a $z^* \in X^*$ such that $\sup \{ |z^*(x)| : x \in A \}$ is not attained.

**Proof.** We prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). The first implication is proved by applying the similar implication (1) $\Rightarrow$ (2) from Theorem 7.5 to a particular separable subset $A_0$ of $A$. The second implication boils down to proving the inclusion $L(x^*_n) \subseteq A_0^\perp$ in order to apply Lemma 7.10. The third implication involves some clever computations and an appropriate choice of $(\beta_n)$, but just as in Theorem 7.5 the idea is simply to control the tail of $z^* = \sum_{j=1}^{n+1} \beta_j (y^*_j - w^*)$ and squeeze the whole thing strictly less than $\alpha$. Finally, the proof that (4) implies (1) is completely analogous to the argument given in Theorem 7.5.

(1) $\Rightarrow$ (2): Suppose that (1) is true. Then it follows from Theorem 6.7 that there exists a separable closed subspace $Y$ of $X$ such that $A \cap Y$ is not weakly compact. Setting $A_0 := A \cap Y$ we have $A_0 \subseteq A \subseteq B_X$. 

Moreover, 0 ∈ A by the assumption that A is balanced and 0 ∈ Y since it is a subspace, so 0 ∈ A ∩ Y. Hence A0 is nonempty. Also, A0 is separable by the separability of Y. Finally, Y is closed in the norm topology, so in particular it is weakly closed, and so A0 is weakly closed. Thus, A0 satisfies all the conditions on A in Theorem 7.5.

By the fact that A0 is not weakly compact, we therefore conclude from the implication (1) ⇒ (2) in Theorem 7.5 that there exists a real number θ ∈ (0, 1) and a sequence of linear functionals (x∗ n) ⊆ BX such that limn→∞ x∗ n(x) = 0 for each x ∈ A0 and such that sup { |x∗ n(x)| : x ∈ A0 } ≥ θ for every x∗ ∈ co{(x∗ n : n ∈ N)}.

Now, if w∗ ∈ A0 1, then w∗(x) = 0 for every x ∈ A0. Together with A0 ⊆ A and the observation, this implies that

\[
\sup \{ |(x^* - w^*)(x)| : x \in A \} = \sup \{ |x^*(x) - w^*(x)| : x \in A \} \\
\geq \sup \{ |x^*(x) - w^*(x)| : x \in A_0 \} \\
= \sup \{ |x^*(x)| : x \in A_0 \} \\
\geq \theta
\]

for every x∗ ∈ co{(x∗ n : n ∈ N)} and w∗ ∈ A0 1. This proves the implication (1) ⇒ (2).

(2) ⇒ (3): Let (βn) be any sequence of strictly positive reals summing to 1. Then A and (βn) satisfy assumptions (i) and (ii) in Lemma 7.10. Now observe that if (2) remains true when A0 1 is replaced by L(x∗ n), then assumption (iii) in Lemma 7.10 would also be satisfied for the given θ ∈ (0, 1). Hence (3) would follow immediately from (2) and the conclusion in Lemma 7.10. Consequently, it suffices to show that L(x∗ n) is a subset of A0 1. To this end, notice that the sequence (x∗ n) in (2) satisfies limn→∞ x∗ n(x) = 0 for all x ∈ A0 ⊆ A. Now recall that if w∗ ∈ L(x∗ n), then lim infn x∗ n(x) ≤ w∗(x) ≤ lim supn x∗ n(x) for every x ∈ X ⊇ A0. Hence w∗(x) = limn→∞ x∗ n(x) = 0 for each x ∈ A0, so L(x∗ n) ⊆ A0 1 as desired.

(3) ⇒ (4): Assume that (3) is true. For the given θ ∈ (0, 1), choose a positive real number Δ such that 0 < Δ < θ 2/2. Now define a sequence of positive reals (βn) by

\[
\beta_n := \frac{2 - Δ}{Δ} \left(\frac{Δ}{2}\right)^n > 0
\]

for each n ∈ N. Note that 0 < \( \frac{Δ}{2} < \frac{θ^2/2}{2} = \frac{θ^2}{4} < \frac{1}{4} \), so \( \sum_{n=1}^{∞} β_n \) is a geometric series with

\[
\sum_{n=1}^{∞} β_n = \frac{2 - Δ}{Δ} \sum_{n=1}^{∞} \left(\frac{Δ}{2}\right)^n = \frac{2 - Δ}{Δ} \frac{\frac{Δ}{2}}{1 - \frac{Δ}{2}} = \frac{2 - Δ}{2 - Δ} = 1.
\]

Let α and (y∗ n) be as in (3) and define z∗ ∈ X∗ by z∗(x) := \( \sum_{j=1}^{∞} β_j (y_j^∗ - w^*)(x) \) for an arbitrary w∗ ∈ L(y∗ n). If we can show that sup { |z∗(x)| : x ∈ A } is not attained, then the proof will be complete. To this end, fix an arbitrary x0 ∈ A. Since w∗ ∈ L(y∗ n), we have \( \lim inf_j y_j^∗(x_0) \leq w^*(x_0) \) so for every ε > 0 there are infinitely many n ∈ N with \( y_n^∗(x_0) < \lim inf_n y_n^∗(x_0) + ε \leq w^*(x_0) + ε \). In particular, the fact that \( θ^2 - 2Δ > 0 \) implies the existence of an n ∈ N with \( y_n^∗(x_0) < w^*(x_0) + (θ^2 - 2Δ) \). Since \( θ ≤ α \) it follows that there is an n ∈ N such that

\[
(y_{n+1}^* - w^*)(x_0) < θ^2 - 2Δ ≤ αθ - 2Δ. \tag{7.14}
\]

Moreover, by Lemma 7.8(5) it also follows from w∗ ∈ L(y∗ n) and (y∗ n) ⊆ Bx∗ that \( ||w^*|| ≤ 1 \). Hence

\[
||y_j^* - w^*|| ≤ ||y_j^*|| + ||w^*|| ≤ 1 + 1 = 2. \tag{7.15}
\]
We conclude from (7.14), (7.15) and statement (b) in (3) that

\[ z^*(x_0) = \sum_{j=1}^{\infty} \beta_j (y_j^* - w^*) (x_0) \]

\[ = \sum_{j=1}^{n} \beta_j (y_j^* - w^*) (x_0) + \sum_{j=n+2}^{\infty} \beta_j (y_j^* - w^*) (x_0) \]

\[ < \sum_{j=1}^{n} \beta_j (y_j^* - w^*) (x_0) + \sum_{j=n+2}^{\infty} \beta_j (y_j^* - w^*) (x_0) \]

\[ \leq \sup \left\{ \left| \sum_{j=1}^{n} \beta_j (y_j^* - w^*) (x) \right| : x \in A \right\} + \sum_{j=n+2}^{\infty} \beta_j (\alpha \theta - 2\Delta) + 2 \sum_{j=n+2}^{\infty} \beta_j \]

\[ \leq \alpha \left( 1 - \theta \sum_{j=n+1}^{\infty} \beta_j \right) + (\alpha \theta - 2\Delta) \beta_{n+1} + 2 \sum_{j=n+2}^{\infty} \beta_j \]

\[ = \alpha - \alpha \theta \sum_{j=n+1}^{\infty} \beta_j + (\alpha \theta - 2\Delta) \beta_{n+1} + 2 \sum_{j=n+2}^{\infty} \beta_j \] \hspace{1cm} (7.16)

Now observe that

\[ \sum_{j=n+2}^{\infty} \beta_j = \sum_{j=n+2}^{\infty} \frac{2-\Delta}{\Delta} \left( \frac{\Delta}{2} \right)^j = \frac{\Delta}{2} \sum_{j=n+1}^{\infty} \frac{2-\Delta}{\Delta} \left( \frac{\Delta}{2} \right)^j \]

\[ = \frac{\Delta}{2} \sum_{j=n+1}^{\infty} \beta_j < \Delta \sum_{j=n+1}^{\infty} \beta_j. \]

Therefore, it follows from (7.16) and statement (a) in (3) that

\[ z^*(x_0) < \alpha - \alpha \theta \sum_{j=n+1}^{\infty} \beta_j + (\alpha \theta - 2\Delta) \beta_{n+1} + 2 \sum_{j=n+1}^{\infty} \beta_j \]

\[ = \alpha - (\alpha \theta - 2\Delta) \sum_{j=n+1}^{\infty} \beta_j \]

\[ = \alpha - (\alpha \theta - 2\Delta) \sum_{j=n+1}^{\infty} \beta_j \]

\[ \leq \alpha \]

\[ = \sup \left\{ \sum_{j=1}^{\infty} \beta_j (y_j^* - w^*) (x) : x \in A \right\} \]

\[ = \sup \left\{ z^*(x) : x \in A \right\}. \]

Finally, recall that \( A \) is balanced by assumption. Hence \( x_0 \in A \) implies that \( -x_0 \in A \). In particular, since the above inequality holds for arbitrary \( x_0 \in A \), we also have \( -z^*(-x_0) = z^*(-x_0) < \sup \{ z^*(x) : x \in A \} \). From this we conclude that \( |z^*(x_0)| < \sup \{ z^*(x) : x \in A \} \) whenever \( x_0 \in A \), so the supremum is not attained.

(4) \Rightarrow (1): Finally, assume that (4) is true. Then there is a weakly continuous linear functional \( z^* \in X^* \) which does not attain its supremum on \( A \). If \( A \) was weakly compact, then this would be a clear contradiction of the Extreme Value Theorem. Hence \( A \) cannot be weakly compact. \( \Box \)

### 7.3. James’ Weak Compactness Theorem

At last, we are now fully equipped to give the proof of James’ Weak Compactness Theorem in all its generality. Most of the hard work has already been done in the previous sections. In short, our primary task is to allow for a final generalization of the equivalence of statements (1) and (4) in Theorem 7.11.
Theorem 7.12. (James’ Weak Compactness Theorem) Let $X$ be a Banach space and suppose that $A$ is a nonempty weakly closed subset of $X$. Then the following statements are equivalent.

1. $A$ is weakly compact in $X$
2. Every bounded linear functional $x^*$ on $X$ attains the supremum of its absolute value $|x^*|$ on $A$.
3. Every bounded real-linear functional $u^*$ on $X$ attains the supremum of its absolute value $|u^*|$ on $A$.
4. Every bounded real-linear functional $u^*$ on $X$ attains its supremum on $A$.

Proof. We first prove that (1) implies both (2), (3), and (4). We then prove that (2) implies (1) and (3) implies (1) such that (1) $\iff$ (2) and (1) $\iff$ (3). Finally, we show that (4) implies (3) such that (1) $\iff$ (4).

1. $\implies$ (2), (3), (4): First recall that all bounded linear functionals on $X$ (i.e. the members of the dual space) are weakly continuous by definition. Likewise, all bounded real-linear functionals on $X$ are weakly continuous by Lemma 7.6. Now assume that (1) is true. Then $A$ is weakly compact. Thus, it follows from the Extreme Value Theorem that every real-valued weakly continuous function attains its supremum and infimum on $A$. This proves (4). In fact, it also proves (3) since the supremum of $|u^*|$ over $A$ is either the supremum of $u^*$ or the absolute value of the infimum of $u^*$, both of which are attained on $A$. Finally, the Extreme Value Theorem also asserts that every complex-valued weakly continuous function attains the supremum and infimum of its absolute value on $A$. This proves (2). We conclude that (1) implies (2), (3), and (4) as desired.

In order to apply Theorem 7.11 in the proof of the final three implications, we need $A$ to be contained in the closed unit ball of $X$. Claims 1 and 2 below justify that this can be assumed without any loss of generality.

Claim 1. We claim that $A$ is bounded whenever (2), (3), or (4) is true. To see this, first suppose that (2) holds and fix any $x^* \in X^*$. Then $|x^*(x_0)| \leq \sup \{|x^*(x)| : x \in A\} < \infty$ for every $x_0 \in A$. Hence $x^*(A)$ is bounded in $\mathbb{F}$ for every $x^* \in X^*$, and therefore $A$ is bounded in $X$ by Corollary 5.20. To prove the claim for (3) and (4), fix an arbitrary $u^* \in (X_\mathbb{R})^*$. If (3) is true, then $|u^*(x_0)| \leq \sup \{|u^*(x)| : x \in A\} < \infty$ for every $x_0 \in A$, so $u^*(A)$ is bounded in $\mathbb{F}$. Suppose instead that (4) is true. Since $u^* \in (X_\mathbb{R})^*$ implies $-u^* \in (X_\mathbb{R})^*$, we have $\inf \{u^*(x) : x \in A\} = -\sup \{-u^*(x) : x \in A\}$. In turn, both the infimum and supremum of $u^*$ is attained, so $u^*(A)$ is bounded in $\mathbb{F}$. In sum, $u^*(A)$ is bounded in $\mathbb{F}$ whenever (3) or (4) holds. Now observe that $x \mapsto u^*(ix)$ is also a bounded real-linear functional on $X$. Hence $u^*(iA)$ is also bounded in $\mathbb{F}$. By Theorem 1.6, each $x^* \in X^*$ can be written as $x^*(x) = u^*(x) - iu^*(ix)$ where $u^* = \Re(x^*) \in (X_\mathbb{R})^*$. Since $|u^*(x) - iu^*(ix)| \leq |u^*(x)| + |u^*(ix)|$, it follows that $x^*(A)$ is bounded for every $x^* \in X^*$. Another application of Corollary 5.20 thus proves that $A$ is weakly bounded in $X$ whenever (3) or (4) is true.

Claim 2. We claim that, when proving the implications (2) $\implies$ (1), (3) $\implies$ (1), and (4) $\implies$ (3), it may be assumed that $A \subseteq B_X$. To see this, notice that by virtue of Claim 1 there is an $r > 0$ such that $A \subseteq B(0,r)$ whenever (2), (3), or (4) is assumed. Now recall that $(X, \Sigma_\mathbb{R})$ is a TVS by Theorem 5.9. Hence the maps $x \mapsto rx$ and $x \mapsto r^{-1}x$ are both continuous in the weak topology. Consequently, $x \mapsto r^{-1}x$ is a homeomorphism from $A$ onto $r^{-1}A$ and therefore $A \subseteq B(0,r)$ and $r^{-1}A \subseteq B(0,1) \subseteq B_X$ are homeomorphic as subsets of $X$ in the weak topology. In particular, $A$ is weakly compact if and only if $r^{-1}A \subseteq B_X$ is weakly compact.

(2) $\implies$ (1): Assume that (2) is true. In accordance with Claim 2 we may in addition assume that $A \subseteq B_X$. Now define

$$E := \bigcap_{x^* \in X^*} \{x \in X : |x^*(x)| \leq \sup \{|x^*(y)| : y \in A\} \} \subseteq B_X.$$ 

Notice that if $x \in A$, then $|x^*(x)| \leq \sup \{|x^*(y)| : y \in A\}$ for every $x^* \in X^*$. Hence $A \subseteq E$ by construction. Also, $E$ is weakly closed as an intersection of weakly closed sets. This follows from the observation that the individual sets are simply preimages under $x^*$ of the closed ball with center 0 and radius $\sup \{|x^*(y)| : y \in A\}$. 

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Finally, $E$ is also balanced. To see this, let $\lambda$ be any scalar with $|\lambda| \leq 1$ and suppose that $y \in \lambda E$. Then $y = \lambda x$ for some $x \in E$, which implies that
\[
|x^*(\lambda x)| = |\lambda| |x^*(x)| \leq |x^*(x)| \leq \sup \{|x^*(y)| : y \in A\}.
\]
for all $x^* \in X^*$. Hence $y \in E$ and therefore $\lambda E \subseteq E$. This proves that $E$ is balanced.

In conclusion, $E$ is a nonempty, balanced, and weakly closed subset of $\bar{B}_X$, which also satisfies $A \subseteq E$.

Note that, by construction of $E$, we have $\sup \{|x^*(x)| : x \in E\} = \sup \{|x^*(x)| : x \in A\}$ for every $x^* \in X^*$. Furthermore, recall that $\sup \{|x^*(x)| : x \in A\}$ is attained for every $x^* \in X^*$ by our initial assumption. Hence $A \subseteq E$ implies that $\sup \{|x^*(x)| : x \in E\}$ is also attained for every $x^* \in X^*$.

We now claim that
\[
\sup \{|x^*(x)| : x \in E\} = \sup \{|\Re(x^*)(x)| : x \in E\}
\]
for every $x^* \in X^*$. To see this, first observe that $|\Re(x^*)(x)| \leq |x^*(x)|$ trivially implies $\sup \{|x^*(x)| : x \in E\} \geq \sup \{|\Re(x^*)(x)| : x \in E\}$. In order to prove the reverse inequality, we fix an arbitrary $x^* \in X^*$ and let $x_0 \in E$ be such that $|x^*(x_0)| = \sup \{|x^*(x)| : x \in E\}$. If $x^*(x_0) = 0$, then the result is trivial, so we may assume that $x^*(x_0) \neq 0$. Thus $\lambda := x^*(x_0)/|x^*(x_0)|$ is a well-defined scalar with $|\lambda| = 1$. Since $|x^*(x_0)|^2 = x^*(x_0)x^*(x_0)$ we have $|x^*(x_0)| = \lambda x^*(x_0)$. In particular, $\lambda x^*(x_0)$ must be real, so $|x^*(x_0)| = \Re(\lambda x^*)(x_0) = \Re(x^*)(\lambda x_0)$. Now notice that $\lambda x_0 \in E$ by the balancedness of $E$. Hence $|x^*(x_0)| \leq \sup \{|\Re(x^*)(x)| : x \in E\}$ and therefore $\sup \{|x^*(x)| : x \in E\} \leq \sup \{|\Re(x^*)(x)| : x \in E\}$. As $x^*$ was arbitrary, this proves the claim in (7.17).

By Theorem 1.6, every $u^* \in (X_R)^*$ is the real part $\Re(x^*)$ of some $x^* \in X^*$. Hence it follows from (7.17) that $\sup \{|u^*(x)| : x \in E\}$ is attained for every $u^* \in (X_R)^*$. Now observe that Theorem 7.11 applies to $X_R$ and $E$ since $X_R$ is a real Banach space (in the original norm) and $E$ is a nonempty, balanced, weakly closed subset of $\bar{B}_X = \bar{B}_{X_R}$. Thus, we conclude from the equivalence of (1) and (4) in Theorem 7.11 that $E$ is weakly compact in $X_R$. By Lemma 7.6, the weak topologies on $X_R$ and $X$ coincide. Hence $E$ is also weakly compact in $X$. Since $A$ is a weakly closed subset of $E$, we conclude that $A$ is weakly compact in $X$ as well.

(3) $\Rightarrow$ (1): Suppose that (3) holds. Then $\sup \{|u^*(x)| : x \in A\}$ is attained for every $u^* \in X^*$. Let $E$ be as above and replace $X, x^*$ by $X_R, u^*$ in the pertaining arguments up until the claim in (7.17). Then we conclude that $\sup \{|u^*(x)| : x \in E\}$ is attained for every $u^* \in (X_R)^*$. Hence $E$ is weakly compact in $X_R$ by Theorem 7.11 and therefore weakly compact in $X$ by Lemma 7.6.

(4) $\Rightarrow$ (3): For the final implication we assume that (4) is true. That is, we assume that $\sup \{|u^*(x)| : x \in A\}$ is attained for every $u^* \in (X_R)^*$. Now observe that $-u^* \in (X_R)^*$ whenever $u^* \in (X_R)^*$. Hence $\sup \{|-u^*(x)| : x \in A\}$ is also attained for every $u^* \in (X_R)^*$. Since
\[
\sup \{|u^*(x)| : x \in A\} = \max \{\sup \{|u^*(x)| : x \in A\}, \sup \{|-u^*(x)| : x \in A\}\}
\]
it follows that $\sup \{|u^*(x)| : x \in A\}$ is attained for every $u^* \in (X_R)^*$ as desired.

7.4. Examples

We finish the section with some simple but insightful applications of James’ Weak Compactness Theorem.

Example 7.13. Consider the Banach space $c_0 := \{(\alpha_n) \in \ell_\infty : \lim_n x_n = 0\}$ equipped with the $\ell_\infty$-norm. We claim that the closed unit $\bar{B}_{c_0}$ is not weakly compact. To see this, recall that $c_0^* \cong \ell_1$ and observe that $(2^{-n}) \in \ell_1$ with $\|(2^{-n})\|_1 = 1$ since $\sum_{n=1}^\infty |2^{-n}| = \frac{1/2}{1-1/2} = 1$. It follows that if $x^* \in c_0^*$ is the bounded linear functional represented by $(2^{-n}) \in \ell_1$, then $\sup \{|x^*(\alpha_n)| : (\alpha_n) \in \bar{B}_{c_0}\} = \|x^*\| = 1$. 

Now notice that \( \| (\alpha_n)\|_\infty = 1 \) implies that \( |\alpha_n| \leq 1 \) for all \( n \in \mathbb{N} \) while \( (\alpha_n) \in c_0 \) implies that \( |\alpha_n| < 1 \) eventually. Hence

\[
|x^*(\alpha_n)| = \left| \sum_{n=1}^{\infty} \frac{1}{2^n} \alpha_n \right| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} |\alpha_n| < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1
\]

for all \( (\alpha_n) \in B_{c_0} \). Consequently, \( \sup \{ |x^*(\alpha_n)| : (\alpha_n) \in B_{c_0} \} \) is not attained and therefore it is an immediate consequence of James’ Weak Compactness Theorem that \( B_{c_0} \) cannot be weakly compact.

We shall now apply the full force of James’ Weak Compactness Theorem to provide an alternative proof of what Diestel & Uhl (1977, p. 14) refers to as a key structure theorem for countably additive vector measures on \( \sigma \)-algebras.

**Example 7.14.** Let \( (\Omega, \mathcal{F}) \) be a measurable space and let \( (X, \| \cdot \|) \) be a real Banach space. A map \( \mu : \mathcal{F} \to X \) is said to be a vector measure if \( \mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) \) whenever \( E_1 \) and \( E_2 \) are disjoint members of \( \mathcal{F} \). Let \( (E_j) \) be any sequence of pairwise disjoint members of \( \mathcal{F} \) such that \( \bigcup_{n=1}^\infty E_j \in \mathcal{F} \). If \( \mu(\bigcup_{n=1}^\infty E_j) = \sum_{j=1}^\infty \mu(E_j) \), where the infinite sum is convergent in the norm on \( X \), then \( \mu \) is said to be a countably additive (or \( \sigma \)-additive) vector measure.

It is proved in Corollary 7 of Diestel & Uhl (1977, p.14) that a countably additive vector measure \( \mu \) has a relatively weakly compact range. This, of course, is a strong result in relation to the characterization of the values \( \mu(E) \in X \) for \( E \in \mathcal{F} \). In the following, we present a short and non-technical proof using James’ Weak Compactness Theorem.

Let \( \mu(\mathcal{F}) \) denote the range of \( \mu \) in \( X \). Fix an arbitrary \( x^* \in X^* \) and observe that \( x^* \circ \mu : \mathcal{F} \to \mathbb{R} \) defines a signed measure on \( (\Omega, \mathcal{F}) \). Indeed, if \( (E_j) \) is a sequence of pairwise disjoint members of \( \mathcal{F} \) such that \( \bigcup_{n=1}^\infty E_n \in \mathcal{F} \), then

\[
\sum_{j=1}^n x^*(\mu(E_j)) = x^*\left( \sum_{j=1}^n \mu(E_j) \right) \to x^*\left( \sum_{j=1}^\infty \mu(E_j) \right) = x^*\left( \mu\left( \bigcup_{n=1}^\infty E_j \right) \right)
\]

by the continuity of \( x^* \) since \( \sum_{j=1}^\infty \mu(E_j) \) is convergent. Consequently,

\[
x^* \circ \mu\left( \bigcup_{n=1}^\infty E_j \right) = \sum_{j=1}^\infty x^* \circ \mu(E_j).
\]

Because \( x^* \circ \mu \) is a signed measure on \( (\Omega, \mathcal{F}) \), there is a Hahn-Decomposition of \( \Omega \) (Theorem A.9). That is, there exists \( P, N \in \mathcal{F} \) such that \( \Omega = P \cup N \) and \( P \cap N = \emptyset \) with \( x^* \circ \mu(E) \geq 0 \) for every \( E \subseteq P \) and \( x^* \circ \mu(E) \leq 0 \) for every \( E \subseteq N \). It follows that for any \( E \in \mathcal{F} \), we have

\[
x^* \circ \mu(E) = x^* \circ \mu((E \cap P) \cup (E \cap N)) \leq x^* \circ \mu(E \cap P) + x^* \circ \mu(E \cap N) 
\]

\[
\leq x^* \circ \mu(E \cap P) \leq x^* \circ \mu(P)
\]

Hence \( \sup \{ x^*(\mu(E)) : E \in \mathcal{F} \} \) is attained at \( P \). By the weak continuity of \( x^* \), it follows that

\[
\sup \left\{ x^*(x) : x \in \overline{\mu(\mathcal{F})}^\ast \right\} = \sup \{ x^*(x) : x \in \mu(\mathcal{F}) \} = \sup \{ x^*(\mu(E)) : E \in \mathcal{F} \} = \sup \{ x^* \circ \mu(E) : E \in \mathcal{F} \} = x^* \circ \mu(P).
\]

Since \( x^* \in X^* \) was arbitrary, we conclude that \( \sup \left\{ x^*(x) : x \in \overline{\mu(\mathcal{F})}^\ast \right\} \) is attained for every \( x^* \in X^* \). Seeing as \( \overline{\mu(\mathcal{F})}^\ast \) is weakly closed, it follows from James’ Weak Compactness Theorem that \( \overline{\mu(\mathcal{F})}^\ast \) is weakly compact. This finishes the proof.
One of the very first encounters with compactness in analysis is the case of the closed intervals on the real line. If a real $L_p$-space is partially ordered by the relation $f \geq g$ if $f \geq g$ a.e., then a similar situation arises, where the order intervals $[f, \psi] := \{f \in L_p : \varphi \leq f \leq \psi\}$ turn out to be weakly compact. This is in fact intimately linked to the result in Example 7.14, since a weakly compact order interval in a Banach lattice (of which the $L_p$-spaces are special cases) is always the range of a countably additive vector measure (Diestel & Uhl, 1977, p. 275).

**Example 7.15.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and consider the real $L_p$-space $L_p(\Omega, \mathcal{F}, \mu)$ with $1 \leq p < \infty$. If $p = 1$ it is assumed that $\mu$ is $\sigma$-finite (such that the dual of $L_1$ is $L_\infty$). Recall that the $L_p$-spaces are Banach spaces. Define

$$[\varphi, \psi] := \{f \in L_p(\Omega, \mathcal{F}, \mu) : \varphi \leq f \leq \psi \text{ } \mu\text{-a.e.}\}$$

for $\varphi, \psi \in L_p$ such that $\varphi \leq \psi \text{ } \mu\text{-a.e.}$ We claim that $[\varphi, \psi]$ is weakly compact. First recall that $L_p^* \cong L_q$ where $p$ and $q$ are conjugate. Fix $x^* \in L_p^*$ and let $g$ be the corresponding element in $L_q$. Then $x^*(f) = \int f g d\mu$ whenever $f \in L_p$. Now define $h \in [\varphi, \psi]$ by

$$h(x) := \begin{cases} \varphi(x), & x \in \{g < 0\} \\ \psi(x), & x \in \{g \geq 0\} \end{cases}.$$

Then it holds for any $f \in [\varphi, \psi]$ that

$$x^*(f) = \int_\Omega f g d\mu = \int_{\{g < 0\}} f g d\mu + \int_{\{g \geq 0\}} f g d\mu \leq \int_{\{g < 0\}} \varphi g d\mu + \int_{\{g \geq 0\}} \psi g d\mu = \int_{\{g < 0\}} \varphi h d\mu + \int_{\{g \geq 0\}} \psi h d\mu = \int_\Omega h g d\mu = x^*(h).$$

Hence $\sup \{x^*(f) : f \in [\varphi, \psi]\}$ is attained at $h \in [\varphi, \psi]$. Since $x^* \in L_p^*$ was arbitrary, it follows that $\sup \{x^*(f) : f \in [\varphi, \psi]\}$ is attained for every $x^* \in X^*$.

Moreover, we claim that $[\varphi, \psi]$ is convex and closed in the $L_p$-norm. The convexity is clear, but the closedness requires a little work. Let $(f_n)$ be a convergent sequence in $[\varphi, \psi]$ with $f_n \to f \in L_p(\Omega, \mathcal{F}, \mu)$ in the $L_p$-norm. Then there exists a sequence of measurable sets $(F_n)$ in $\mathcal{F}$ such that, for each $n \in \mathbb{N}$, $F_n^c$ is a null set and $\varphi(x) \leq f_n(x) \leq \psi(x)$ for all $x \in F_n$. Moreover, there is a subsequence $(f_{n_k})$ such that $f_{n_k} \to f$ a.e. by a standard result in Measure Theory. That is, there exists $F \in \mathcal{F}$ such that $F^c$ is a null set and $f_{n_k}(x) \to f(x)$ for each $x \in F$. Now define $A := F \cap (\bigcap_n F_n)$ and observe that $A^c = F^c \cup (\bigcup_n F_n^c)$ is again a null set. Observe that, for all $x \in A$, we have $\varphi(x) \leq f_{n_k}(x) \leq \psi(x)$ for each $k \in \mathbb{N}$ and $f_{n_k}(x) \to f(x)$. Thus, by the Squeeze Theorem, we have $\varphi(x) \leq f(x) \leq \psi(x)$ for every $x \in A$, where $A^c$ is a null set. In other words, the limit $f$ of $(f_n)$ belongs to $[\varphi, \psi]$, so $[\varphi, \psi]$ is closed as claimed.

Since $[\varphi, \psi]$ is closed and convex it is also weakly closed by Mazur’s Theorem (Theorem 5.24). Consequently, it follows from James’ Weak Compactness Theorem that $[\varphi, \psi]$ is weakly compact.

In the event that $1 < p < \infty$ the $L_p$ spaces are reflexive and so the result in Example 7.15 could have been obtained much less profoundly by simply appealing to the Heine-Borel Property of reflexive normed spaces. The space $L_1$, on the other hand, is in general not reflexive, so in this case the Heine-Borel Theorem could not have been applied. Hence Example 7.15 is by no means a trivial result and it serves well to illustrate the force of James’ Weak Compactness Theorem.

Moreover, it is a famous result of Dunford that the relatively weakly compact subsets of $L_1$ are uniformly integrable (See Diestel & Uhl, 1977, p. 105). Hence the previous example also serves to show that the ordered intervals in $L_1$ are always uniformly integrable.
8. Corollaries, Counterexamples and Comments

We begin the section by considering the celebrated James' Theorem on the reflexivity of Banach spaces (James, 1964), which can now be obtained as an easy corollary of James' Weak Compactness Theorem.

**Theorem 8.1. (James' Theorem)** Let $X$ be a Banach space. Then $X$ is reflexive if and only if every bounded linear functional on $X$ is norm-attaining.

**Proof.** Assume that every linear functional on $X$ is norm-attaining. This is simply the statement that $\|x^*\| = \sup \{|x^*(x)| : x \in B_X\}$ is attained for every $x^* \in X^*$. Now recall that $B_X$ is weakly closed by Mazur’s Theorem (Theorem 5.24). Hence it follows from James’ Weak Compactness Theorem that $B_X$ is weakly compact. By Theorem 6.2 this is equivalent to reflexivity of $X$.

Conversely, suppose that $X$ is reflexive. Then $B_X$ is weakly compact by Theorem 6.2. Hence, James’ Weak Compactness Theorem ensures that $\|x^*\| = \sup \{|x^*(x)| : x \in B_X\}$ is attained for every $x^* \in X^*$. Consequently, every bounded linear functional on $X$ is norm-attaining. □

**Example 8.2.** We claim that the Banach space $(c_0, \|\cdot\|_\infty)$ is not reflexive. To see this, simply recall from Example 7.13 that the bounded linear functional $(2^{-n}) \in \ell_1 \cong c_0^*$ does not attain its norm. Hence $c_0$ is not reflexive by James’ Theorem.

As another striking example of the power of James’ Weak Compactness Theorem, we shall now present an alternative and highly instructive proof of the famous Krein-Šmulian Weak Compactness Theorem.

**Theorem 8.3. (The Krein-Šmulian Weak Compactness Theorem)** Let $X$ be a Banach space. Then the closed convex hull of any weakly compact subset of $X$ is itself weakly compact.

**Proof.** Fix any weakly compact subset $A$ of $X$. Then $A$ is in particular weakly closed, since the weak topology is Hausdorff by Theorem 5.9. Hence James' Weak Compactness Theorem ensures that $\sup \{|x^*(x)| : x \in A\}$ is attained for every $x^* \in X^*$. Now fix an arbitrary $x^* \in X^*$ and let $x_0 \in A$ be such that $|x^*(x_0)| = \sup \{|x^*(x)| : x \in A\}$. Observe that if $x \in \text{co}(A)$, then $x = \sum_{i=1}^n \lambda_i x_i$ for some $x_1, \ldots, x_n \in A$ and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$. Hence

$$|x^*(x)| = |x^*\left(\sum_{i=1}^n \lambda_i x_i\right)| = \left|\sum_{i=1}^n \lambda_i x^*(x_i)\right| \leq \sum_{i=1}^n \lambda_i |x^*(x_i)|$$

$$\leq \sum_{i=1}^n \lambda_i \max \{|x^*(x_j)| : x_1, \ldots, x_n \in A\}$$

$$= \max \{|x^*(x_j)| : x_1, \ldots, x_n \in A\}$$

$$\leq \sup \{|x^*(y)| : y \in A\}.$$ 

It follows that $\sup \{|x^*(x)| : x \in \text{co}(A)\} \leq \sup \{|x^*(y)| : y \in A\}$. Since $X$ is normed, we have $\overline{\text{co}}(A) = \overline{\text{co}}(A)$ and therefore

$$\sup \{|x^*(x)| : x \in \overline{\text{co}}(A)\} \leq \sup \{|x^*(y)| : y \in A\}$$

by the continuity of $x^*$. Now recall that $x_0 \in A \subseteq \overline{\text{co}}(A)$ satisfies

$$|x^*(x_0)| = \sup \{|x^*(y)| : y \in A\}.$$ 

Hence $\sup \{|x^*(x)| : x \in \overline{\text{co}}(A)\}$ is attained at $x_0 \in \overline{\text{co}}(A)$. Since $x^* \in X^*$ was arbitrary, we conclude that $\sup \{|x^*(x)| : x \in \overline{\text{co}}(A)\}$ is attained for every $x^* \in X^*$. Finally, since $\overline{\text{co}}(A)$ is a closed and convex, it follows from Mazur’s Theorem (Theorem 5.24) that $\overline{\text{co}}(A)$ is also weakly closed. Thus, $\overline{\text{co}}(A)$ is weakly compact by James’ Weak Compactness Theorem. □
As a final application of James’ Weak Compactness Theorem, we provide an alternative proof of a very nice theorem by Smulian (1939). What is so nice about this particular theorem, is that it presents a characterization of weak compactness for convex subsets of a Banach space without any reference to the dual space.

**Theorem 8.4.** A convex subset \( C \) of a Banach space \( X \) is weakly compact if and only if \( \bigcap_{n=0}^{\infty} C_n \) is nonempty whenever \( C_1, C_2, \ldots \) is a nested sequence (that is, \( C_1 \supseteq C_2 \supseteq \cdots \)) of nonempty convex subsets of \( C \) that are closed in \( C \).

**Proof.** First suppose that \( C \) is a convex and weakly compact subset of \( X \). Then \( C \) is weakly closed in \( X \) and therefore also closed in \( X \). Now, for each \( n \in \mathbb{N} \), \( C_n \) is closed in \( X \) by assumption and therefore also closed in \( X \). But then the convexity of \( C_n \) implies that \( C_n \) is weakly closed in \( X \) for every \( n \in \mathbb{N} \) by Mazur’s Theorem (Theorem 5.24).

In other words, \( C \) is weakly compact and each \( C_n \) is a weakly closed subset of \( C \). Since the sequence \( C_1, C_2, \ldots \) is nested, it follows from Cantor’s Intersection Theorem (Theorem B.5) that \( \bigcap_{n=0}^{\infty} C_n \) is nonempty.

For the converse implication, fix an arbitrary convex subset \( C \) of \( X \) and suppose that \( \bigcap_{n=0}^{\infty} C_n \) is nonempty whenever \( C_1, C_2, \ldots \) is a nested sequence of nonempty convex subsets of \( C \) that are closed in \( C \).

First we claim that \( C \) is weakly closed in \( X \). Since \( C \) is convex it suffices to show that \( C \) is closed in \( X \) by Mazur’s Theorem (Theorem 5.24). To see this, suppose that \( x_0 \in \overline{C} \). Then by definition we have \( \overline{B}(x_0, \frac{1}{n}) \cap C \neq \emptyset \) for every \( n \in \mathbb{N} \). Hence the sets \( \overline{B}(x_0, \frac{1}{n}) \cap C \) form a nested sequence of nonempty convex sets that are closed in \( C \).

In turn, \( \bigcap_{n=0}^{\infty} \overline{B}(x_0, \frac{1}{n}) \cap C \) must be nonempty by our initial assumption. But clearly \( \bigcap_{n=0}^{\infty} \overline{B}(x_0, \frac{1}{n}) = \{x_0\} \), which implies \( x_0 \in C \). Consequently, \( \overline{C} = C \) and hence \( C \) is closed in \( X \) as claimed.

Next we claim that \( s := \sup \{ u^*(x) : x \in C \} \) is finite for any bounded real-linear functional \( u^* \) on \( X \) (this is not clear a priori since we do not know if \( C \) is bounded). To see this, fix an arbitrary \( u^* \in (X^*)^* \) and notice that the preimages \( u^{-1}([n, \infty)) \) are closed and convex. In turn, the sets \( u^{-1}([n, \infty)) \cap C \) form a nested sequence of convex sets that are closed in \( C \). Hence, if each \( u^{-1}([n, \infty)) \) is nonempty, then \( \bigcap_{n=0}^{\infty} u^{-1}([n, \infty)) \cap C \) is nonempty by our initial assumption and so there exists \( x_0 \in C \) with \( u^*(x_0) = \infty \), which is clearly a contradiction. Thus, \( s \leq N \) for some \( N \in \mathbb{N} \).

Now, for all \( n \in \mathbb{N} \) there exists \( x \in C \) such that \( u^*(x) \geq s - 1/n \) by definition of the supremum \( s \). Hence \( u^{-1}([s - 1/n, \infty)) \cap C \) is nonempty for every \( n \in \mathbb{N} \) and these sets clearly form a nested sequence. Since each set \( u^{-1}([s - 1/n, \infty)) \cap C \) is also convex and closed in \( C \), it follows from our initial assumption that \( \bigcap_{n=0}^{\infty} u^{-1}([s - 1/n, \infty)) \cap C \) is nonempty. Thus, there exists \( x_0 \in C \) such that \( s \geq u^*(x_0) \geq s - 1/n \) for all \( n \in \mathbb{N} \), which implies \( u^*(x_0) = s \). That is, \( u^* \) attains its supremum at \( x_0 \in C \).

Since \( C \) is weakly closed, and each bounded real-linear functional \( u^* \) attains its supremum on \( C \), James’ Weak Compactness Theorem ensures that \( C \) is weakly compact. \( \square \)

8.1. The Assumption of Weak Closedness

It is assumed in the statement of James’ Weak Compactness Theorem that the subset in question is weakly closed. The below example shows that this assumption is indeed necessary for the theorem to hold.

**Example 8.5.** Consider again the Banach space \( (c_0, \| \cdot \|_\infty) \). Let \( e_n = (0, \ldots, 0, 1, 0, \ldots) \) denote the \( n \)th standard unit vector in \( c_0 \) and let \( A := \{ n^{-1}e_n : n \in \mathbb{N} \} \). We show that every bounded linear functional on \( c_0 \) attains its supremum on \( A \) despite the fact that \( A \) is not weakly compact. Since \( A \) also fails to be weakly closed, this serves to illustrate the importance of the assumption that \( A \) should be weakly closed in order to apply James’ Weak Compactness Theorem.

First recall that \( x \in c_0 \) is a weak limit point of \( A \) if and only if there is a net in \( A \setminus \{x\} \) that converges weakly to \( x \). Now observe that \( \| n^{-1}e_n \|_\infty = \| n^{-1} \|_\infty = n^{-1} \to 0 \), so \( n^{-1}e_n \) converges to 0 in the norm topology and hence also in the weak topology.
Thus, $A$ is not weakly closed (and not closed either). By the Hausdorffness of the weak topology, this in particular implies that $A$ is not weakly compact.

As in Example 7.13, recall that $c_0^* \cong \ell_1$. Now fix an arbitrary $x^* \in c_0^*$ represented by $(\alpha_j) \in \ell_1$ and observe that

$$x^*(n^{-1}e_n) = \sum_{j=1}^{\infty} \alpha_j n^{-1} e_n = \frac{\alpha_n}{n}.$$  

We need to show that $s := \sup \{ |\alpha_n| / n : n \in \mathbb{N} \}$ is attained. If there are only finitely many $\alpha_n \neq 0$, then the supremum is trivially attained. So assume there are infinitely many $\alpha_n \neq 0$ and suppose, for contradiction, that the supremum is not attained. By definition of the supremum, there exists $k \in \mathbb{N}$ such that $|\alpha_k| / k > \frac{s}{2}$. If only finite terms $|\alpha_k| / k$ satisfy this inequality, then the supremum would be attained at the maximum of these. Hence there are infinitely many $k \in \mathbb{N}$ such that $|\alpha_k| > k\frac{s}{2}$. But then $||\alpha_j||_1 = \sum_{j=1}^{\infty} |\alpha_j| = \infty$, which contradicts $(\alpha_j) \in \ell_1$. It follows that sup $\{ |\alpha_n| / n : n \in \mathbb{N} \}$ is indeed attained for every $x^* \in c_0^*$.

As the next example shows, even closedness in the norm topology is a too weak restriction on the subset.

**Example 8.6.** Given an infinite dimensional reflexive Banach space $X$, we claim that norm closedness of a subset $A \subseteq X$ is not enough for the implication $(2) \Rightarrow (1)$ in James’ Weak Compactness Theorem to hold. To see this, we consider the unit sphere $S_X$.

First of all, $S_X$ is clearly closed in the norm topology. On the other hand, $S_X$ is not weakly closed since $X$ is infinite dimensional. To prove this, we show that $0$ belongs to the weak closure of $S_X$ (in fact, the closure equals $B_X$). Notice that by the proof of Theorem 5.6, every weakly open neighborhood $U$ of $0$ contains a nontrivial subspace $Y$ (equal to the intersection of the kernels of a finite collection of bounded linear functionals). Since $Y$ is nontrivial, there exists $x_0 \in Y \subseteq U$ and thus $\frac{1}{\|x_0\|} x_0 \in Y \subseteq U$. Hence $U \cap S_X$ is nonempty for every weakly open neighborhood $U$ of $0$. This proves that the weak closure of $S_X$ contains $0$. Notice that, in particular, $S_X$ cannot be weakly compact by the Hausdorffness of the weak topology.

Now observe that since $X$ is reflexive, each $x^* \in X^*$ attains supremum on $B_X$ by James’ Theorem. Fix $x^* \in X^*$ and suppose that the supremum is attained at some $x_0$ with $\|x_0\| < 1$. Then $\frac{1}{\|x_0\|} > 1$, so

$$\left| \frac{1}{\| x_0 \|} x^* (\frac{1}{\| x_0 \|} x_0) \right| = \frac{1}{\| x_0 \|} |x^* (x_0)| > |x^* (x_0)|,$$

where $\frac{1}{\| x_0 \|} x_0 \in S_X \subseteq B_X$, which is a contradiction. Hence the supremum must be attained at a point in $S_X$. It follows that every bounded linear functional on $X$ attains its supremum on $S_X$ even though $S_X$ is not weakly compact.

**8.2. The Assumption of Completeness**

In the next example we pay a final visit to the Banach space $(c_0, \| \cdot \|_\infty)$ from Examples 7.13, 8.2, and 9.1. Specifically, we provide an example where James Weak Compactness Theorem fails in an incomplete, albeit dense, subspace of $c_0$.

**Example 8.7.** Let $c_{00} := \{ (\alpha_n) \in \ell_\infty : \alpha_n \neq 0 \text{ for only finitely many } n \in \mathbb{N} \}$ denote the subset of $c_0$ consisting of sequences with finite support. We first show that $c_{00}$ is not complete in the $\ell_\infty$ norm. To see this, consider the sequence $\left( \sum_{j=1}^{n} \frac{1}{j} e_j \right)$ where $e_j$ denotes the $j$'th standard unit vector in $c_{00}$. Assume that $n > m$ and observe that

$$\left\| \sum_{j=1}^{n} \frac{1}{j} e_j - \sum_{j=1}^{m} \frac{1}{j} e_j \right\|_\infty = \left\| (0, \ldots, 0, \frac{1}{m+1}, \ldots, \frac{1}{n}, 0 \ldots) \right\|_\infty = \frac{1}{m+1} \rightarrow 0.$$

Hence $\left( \sum_{j=1}^{n} \frac{1}{j} e_j \right)$ is a cauchy sequence in $c_{00}$. Suppose that the sequence has its limit $x = (\xi_j) \in c_{00}$. Then there exists $n \in \mathbb{N}$ such that $\xi_j = 0$ for all $j > n$. But then $\left\| (\sum_{j=1}^{n} \frac{1}{j} e_j) - x \right\|_\infty \geq \frac{1}{n+1} > 0$ for all $j > n$, which is a contradiction.
Consequently, the limit must have infinitely many nonzero terms and therefore belongs to $c_0 \setminus c_{00}$. We conclude that $c_{00}$ is not complete.

Next we claim that $c_{00}^\ast \cong l_1$. In order to prove this, we use that $c_{00}$ is dense in $c_0$ to construct an isometric isomorphism from $c_0^\ast$ onto $c_{00}^\ast$. The conclusion then follows from the well-known result that $c_0^\ast \cong l_1$. Define a linear map $T : c_0^\ast \to c_{00}^\ast$ by $x^* \mapsto x^* \circ \iota$, where $\iota : c_{00} \hookrightarrow c_0$ denotes the natural injection of $c_{00}$ into $c_0$. Trivially, the map is well-defined since $x^* \circ \iota$ is a composition of continuous maps. Surjectivity follows immediately from the Hahn-Banach Theorem (since $c_{00}$ is dense in $c_0$ this can be proved directly without Hahn-Banach). To see that $T$ is injective, notice that if $x^*(x) = y^*(x)$ for all $x \in c_{00}$, then $x^*(x) = y^*(x)$ for all $x \in \overline{c_{00}} = c_0$ by continuity. Finally, $B_{c_{00}}$ is dense in $B_{c_0}$, so $\|x^* \circ \iota\|_{c_{00}^\ast} = \|x^\ast\|_{c_0^\ast}$. This proves the claim.

Now consider the subset $K := \{0\} \cup \{n^{-1}e_n : n \in \mathbb{N}\}$ of $c_{00}$. We claim that $K$ is compact. As argued in Example 8.5, the sequence $(n^{-1}e_n)$ converges to 0 in norm. Let $\mathcal{U}$ be an open cover of $K$. Take an open neighborhood $U_0$ of 0 in $\mathcal{U}$. Then there exists $k \in \mathbb{N}$ such that $n^{-1}e_n \in U_0$ for all $n > k$. Choosing neighborhoods $U_1, \ldots, U_k$ of $1^{-1}e_1, \ldots, k^{-1}e_k$ in turn yields a finite subcover $\{U_0, U_1, \ldots, U_k\}$.

What we are really interested in, however, is the closed convex hull of $K$ as a subset of $c_{00}$. It is worth emphasising that if we take the closed convex hull of $K$ in the complete space $c_0$ (instead of $c_{00}$), then it is immediately seen to be compact by Mazur’s Compactness Theorem (Theorem A.10).

Nevertheless, we claim that $\overline{c}(K)$ is not even weakly compact when taken in the incomplete subspace $c_{00}$. To see this, first observe that $\sum_{j=1}^n 2^{-j} < 1$. Hence $x_n := \sum_{j=1}^n 2^{-j}e_j$ belongs to $\overline{c}(K)$ for each $n \in \mathbb{N}$ since 0 $\in K$. Because compactness always implies limit point compactness, it suffices to show that the infinite subset $\{x_n : n \in \mathbb{N}\} \subseteq \overline{c}(K)$ has no weak limit point in $\overline{c}(K)$.

Suppose, for contradiction, that there exists a weak limit point $x = (\xi_j) \in \overline{c}(K)$. By weak continuity, $x^*(x)$ would then be a limit point of $\{x^*(x_n) : n \in \mathbb{N}\}$ for each $x^* \in c_{00}^\ast$. Now take the $j$th standard unit vector $e_j$ in $l_1 \cong c_{00}^\ast$ and observe that $e_j(x_n) = 0$ when $n < j$ while $e_j(x_n) = 2^{-j}j^{-1}e_j$ for all $n \geq j$. Thus, we must have $e_j(x) = 2^{-j}j^{-1}e_j$ in order for $e_j(x)$ to be a limit point of $\{e_j(x_n) : n \in \mathbb{N}\}$. But then $\xi_j = 2^{-j}j^{-1}e_j$ for each $j \in \mathbb{N}$. Hence the weak limit point $x = (\xi_j)$ has infinitely many nonzero terms. This is clearly a contradiction, since $x$ is then not even an element in $c_{00}$. We conclude that $\overline{c}(K)$ cannot be weakly compact.

Finally, we claim that every $x^* \in c_{00}^\ast$ attains its supremum on $\overline{c}(K)$ despite the fact that $\overline{c}(K)$ is not weakly compact. By an argument analogous to the one given in the proof of the Krein-Smulian Weak Compactness Theorem (Theorem 8.3), it suffices to show that $\sup \{|x^*(x)| : x \in K\}$ is attained for every $x^* \in c_{00}^\ast$. That is the case follows immediately from the compactness of $K$. Alternatively, one can proceed analogously to Example 8.5 and reach the desired conclusion directly.

Since $\overline{c}(K)$ is closed and convex by definition, Mazur’s Theorem (Theorem 5.24) ensures that $\overline{c}(K)$ is weakly closed. Thus, the only assumption in James’ Weak Compactness Theorem, which is not satisfied, is the completeness of $c_{00}$. Consequently, this assumption cannot be dropped.

Notice that Example 8.7 also shows that the assumption of completeness is necessary in the Krein-Smulian Weak Compactness Theorem. This provides some comfort that we did not lose out on anything by using James Weak Compactness Theorem to prove it in Theorem 8.3.

It is important to emphasize that the true problem in Example 8.7 is really one of incompleteness. Recall from Theorem 6.6 that the weakly compact subsets of a normed space are always complete. In that sense, $c_{00}$ is simply too incomplete to contain its limit points.

Another way of seeing this issue directly is by noting that the sequence $\left(\sum_{j=1}^n 2^{-j}j^{-1}e_j\right)$ in $\overline{c}(K)$ is cauchy but its limit $\sum_{j=1}^\infty 2^{-j}j^{-1}e_j$ is not in $\overline{c}(K)$ for the very reason that $c_{00}$ is not complete. Indeed, $\sum_{j=1}^\infty 2^{-j}j^{-1}e_j$ would have belonged to the closed convex hull in the Banach space $c_0$. 

\*
8.3. The Rôle of Reflexivity

Recall from Section 3 that James’ Weak Compactness was proved quite easily for finite dimensional Banach spaces by a simple application of the Heine-Borel Theorem. As we have already hinted at in Section 3, it is really the ability to apply the Heine-Borel Theorem that is the key to a non-technical proof of the theorem rather than, for example, finite dimensionality.

Now recall from the remarks after Theorem 6.4 that the reflexive normed spaces are precisely the ones that have the Heine-Borel Property when equipped with the weak topology. As such, James’ Weak Compactness Theorem is not at all surprising for reflexive spaces. To see that this is indeed the case, let \( A \) be a weakly closed subset of a reflexive Banach space \( X \) and suppose that \( x^* \) attains its supremum on \( A \) for every \( x^* \in X^* \). In order to show that \( A \) is weakly compact we only need to show that \( A \) is bounded because \( X \) has the Heine-Borel Property. To do this, observe that \( x^*(A) \) is bounded for each \( x^* \in X^* \) by the assumption that each \( x^* \in X^* \) attains its supremum on \( A \) (and hence also attains its infimum by linearity). By Corollary 5.30 this implies that \( A \) is bounded. Consequently, \( A \) is weakly compact by the Heine-Borel Theorem.

Thus, what complicates our proof in Section 7 is simply that if \( X \) is not reflexive, then it is not enough for a set to be weakly closed and bounded in order to be weakly compact. The next theorem exhibits just how intimate the connection between reflexivity and James’ Weak Compactness Theorem really is.

**Theorem 8.8.** Let \( X \) be a Banach space. Suppose that there is a subset \( A \) of \( X \) with nonempty interior such that every \( x^* \in X^* \) attains its supremum on \( A \). Then \( X \) is reflexive.

*Proof.* Let \( A \) be as in the theorem. Then \( \overline{A}^w \) is weakly closed and \( \sup \{ x^*(x) : x \in \overline{A}^w \} = \sup \{ x^*(x) : x \in A \} \) is still attained for every \( x^* \in X^* \). Hence \( \overline{A}^w \) is weakly compact by James’ Weak Compactness Theorem. Since \( A \) has nonempty interior we can choose \( r > 0 \) such that \( \overline{B}(0,r) \) is contained in \( \overline{A}^w \). Now, \( \overline{B}(0,r) \) is weakly closed by convexity and therefore \( \overline{B}(0,r) \) is weakly compact by the weak compactness of \( \overline{A}^w \). Since \( \overline{B}(0,r) \) is homeomorphic to \( B_X \) in the weak topology, it follows that \( B_X \) is weakly compact. But then \( X \) is reflexive by Theorem 6.3. This completes the proof. \( \square \)

It follows from Theorem 8.8 that if James’ Weak Compactness Theorem can be applied to a set with nonempty interior, then the Banach space in question is necessarily reflexive. Similarly, if James’ Weak Compactness Theorem can be applied to a subset of a non-reflexive Banach space, then that subset must have empty interior.

As an example of this, recall Example 7.15. There it was shown that the order intervals \( [\varphi,\psi] \) of \( L_1(\Omega,F,\mu) \) are weakly compact. Now, if \( F \) has infinitely many members, then \( L_1(\Omega,F,\mu) \) is not reflexive and so the order intervals \( [\varphi,\psi] \) must have empty interior according to Theorem 8.8. The reader should verify that this is indeed the case.

8.4. JAMES’ WEAK* COMPACTNESS THEOREM

Given the dual space \( X^* \) of a Banach space \( X \), it is interesting to consider the analogue of James’ Weak Compactness Theorem for the weak* topology on \( X^* \). First recall from Section 5.3 that the weak* topology is coarser than the weak topology, so compactness should be more easily achieved. More importantly, however, recall from Theorem 6.2 that \( (X^*,\mathfrak{F}_w) \) always has the Heine-Borel Property when \( X \) is Banach. Thus, in light of the discussion in Section 8.3, the case with which the following result is obtained should come as no surprise.

**Theorem 8.9.** (James’ Weak* Compactness Theorem) Let \( X \) be a Banach space. Suppose that \( A \) is a nonempty weakly* closed subset of \( X^* \). Then the following statements are equivalent.

1. \( A \) is weakly* compact in \( X^* \).
2. Every \( J(x) \in J(X) \subseteq X^{**} \) attains the supremum of its absolute value on \( A \).
Since $J(x)(x^*) = x^*(x)$, the statement in (2) simply says that $\sup \{|x^*(x)| : x^* \in A\}$ is attained for every $x \in X$.

Proof. Suppose that $A$ is weakly* compact. By definition of the weak* topology, every $J(x) \in J(X) \subseteq X^{**}$ is weakly* continuous. Hence (2) follows from the Extreme Value Theorem.

For the converse, suppose that $\sup \{|x^*(x)| : x^* \in A\}$ is attained for every $x \in X$. Then $\sup \{|x^*(x)| : x^* \in A\}$ is finite for each $x \in X$, so the Uniform Boundedness Principle (Theorem A.5) ensures that $\sup \{|x^*(x)| : x^* \in A\}$ is finite. This means that $A$ is contained in the closed ball with center 0 and radius $\sup \{|x^*(x)| : x^* \in A\}$. Hence $A$ is bounded in the norm topology and therefore also weakly* bounded by Theorem 5.19. Now recall that $A$ is weakly* closed by assumption. Since $A$ is both weakly* bounded and weakly* closed, we conclude from Theorem 6.4 that $A$ is weakly* compact. □

It should be noted that the name “James’ Weak* Compactness Theorem” has been invented for our purposes only and does not appear in the literature. In fact, the theorem is not really mentioned anywhere, except as an exercise in Megginson (1998).
Theorem A.1. (The Hahn-Banach Theorem for Vector Spaces). Let $X$ be a vector space over $\mathbb{R}$. Suppose that $p$ is a sublinear functional on $X$ and $f_0$ is a linear functional on a subspace $Y$ of $X$ such that $f_0(y) \leq p(y)$ for every $y \in Y$. Then there is a linear functional $f$ on all of $X$ such that the restriction of $f$ to $Y$ is $f_0$ and $f(x) \leq p(x)$ for every $x \in X$.

Proof. See Theorem 1.9.5 in Megginson (1998).

Theorem A.2. (The Hahn-Banach Theorem for Normed Spaces). Let $X$ be a normed space over $\mathbb{F}$. Suppose that $f_0$ is a bounded linear functional on a subspace $Y$ of $X$. Then there is a bounded linear functional $f$ on all of $X$ such that the restriction of $f$ to $Y$ is $f_0$ and $\|f\| = \|f_0\|$.

Proof. See Theorem 1.9.6 in Megginson (1998).

Corollary A.3. Let $Y$ be a closed subspace of a normed space $X$. If $x \in X \setminus Y$ then there exists a bounded linear functional $f : X \to F$ such that $f(x) = d(x, Y)$, $Y \subseteq \ker f$, and $\|f\| = 1$.


Corollary A.4. Suppose $x, y \in X$ are distinct elements of a normed space $X$. Then there is a bounded linear functional $x^* \in X^*$ such that $x^*(x) \neq x^*(y)$.


Theorem A.5. (The Uniform Boundedness Principle). Let $\mathfrak{F}$ be a family of bounded linear operators from a Banach space $X$ into a normed space $Y$. If the supremum $\sup \{|T(x)| : T \in \mathfrak{F}\}$ is finite for each $x \in X$, then $\sup \{|T| : T \in \mathfrak{F}\}$ is also finite.

Proof. See Theorem 1.6.9 in Megginson (1998).

Theorem A.6. (The Banach-Alaoglu Theorem). Let $X$ be a normed space. Then the closed unit ball $B_{X^*}$ of $X^*$ is weakly* compact.


Theorem A.7. (Goldstone’s Theorem). Let $X$ be a normed space and let $J : X \to X^{**}$ be the canonical embedding of $X$ in $X^{**}$. Then $J(B_X)$ is weakly* dense in $B_{X^{**}}$.


Theorem A.8. (Hahn Banach Separation Theorem) Let $X$ be a normed space. If $C$ is a nonempty convex subset of $X$ and $x_0 \in X$ satisfies $d(x_0, C) > 0$, then there exists a bounded linear functional $f$ on $X$ such that $\text{Re} f(x_0) > \sup \{\text{Re} f(x) : x \in C\}$. In particular, there exists $t \in \mathbb{R}$ such that $\text{Re} f(x_0) > t > \text{Re} f(x)$ for every $x \in C$.


Theorem A.9. (The Hahn Decomposition Theorem) Let $(\Omega, \mathcal{F})$ be a measurable space. If $\mu$ is a signed measure on $(\Omega, \mathcal{F})$, then there exists a positive set $P \in \mathcal{F}$ with $\mu(E) \geq 0$ for every $E \subseteq P$ and a negative set $N \in \mathcal{F}$ with $\mu(E) \leq 0$ for every $E \subseteq N$ such that $P \cup N = \Omega$ and $P \cap N = \emptyset$.

Proof. See Theorem 3.3 in Folland (1999).

Theorem A.10. (Mazur’s Compactness Theorem) Let $X$ be a Banach space. Then the closed convex hull of any compact subset of $X$ is itself compact.

Appendix B. List of Topology Theorems

Theorem B.1. (The Tietze Extension Theorem) Let \( X \) be a normal \((T_4)\) space and suppose that \( A \) is a closed subset of \( X \). Then any continuous map of \( A \) into \( \mathbb{R} \) may be extended to a continuous map of all of \( X \) into \( \mathbb{R} \).

Proof. See Theorem 35.1 in Munkres (2000).

Theorem B.2. Let \((X, \mathcal{T})\) be a topological space and let \( S \) be a subbasis for \( \mathcal{T} \). Suppose that \((x_\alpha)_{\alpha \in A}\) is a net in \( X \) and \( x \in X \). Then \( x_\alpha \to x \) if and only if, for every \( U \in S \) containing \( x \), there is an \( \alpha_U \in A \) such that \( x_\alpha \in U \) whenever \( \alpha \geq \alpha_U \).


Theorem B.3. Let \( X \) be a set and let \( \mathcal{F} = \{f : X \to (Y_f, \mathcal{T}_f)\} \) be a separating topologizing family of functions for \( X \). If each \((Y_f, \mathcal{T}_f)\) is \( T_0 \), \( T_1 \), Hausdorff \((T_2)\), regular \((T_3)\), or completely regular \((T_{3\frac{1}{2}})\), then so is \((X, \sigma(X, \mathcal{F}))\).

Proof. See Proposition 2.4.8 in Megginson (1998).

Theorem B.4. (The Heine-Borel Theorem) Let \( A \) be a subset of \( \mathbb{R}^n \). Then \( A \) is compact if and only if it is closed and bounded in the euclidean metric.

Proof. See Theorem 27.3 in Munkres (2000).

Theorem B.5. Let \( X \) be a topological space. Then \( X \) is compact if and only if, for every collection \( \mathcal{C} \) of closed sets in \( X \) having the finite intersection property, the intersection \( \bigcap_{C \in \mathcal{C}} C \) is nonempty. In particular, if we have a nested sequence \( C_1 \supseteq C_2 \supseteq \cdots \) of nonempty closed sets in a compact space \( X \), then the collection \((C_n)\) automatically has the finite intersection property, and hence the intersection \( \bigcap_n C_n \) is nonempty.

References


