Existence results for viscoelastic models with an integral constitutive law

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Summary

Conservation principles

The fluid flows is modeled using

- The equation of conservation of the momentum.
- The equation of the conservation of mass.

Assuming that the flow is incompressible and isothermal, we get

$$\begin{cases} \rho\,(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla \boldsymbol{u}) + \nabla p = \operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f} \\ \operatorname{div}\boldsymbol{u} = 0 \end{cases}$$

- The unknowns: velocity $\boldsymbol{u}$, pressure p, stress $\boldsymbol{\sigma}$.
- The data: density ρ, external body forces $\boldsymbol{f}$.

This system is closed using a constitutive equation connecting the stress $\boldsymbol{\sigma}$ and the deformation $D\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + {}^t\nabla \boldsymbol{u})$.

Many possible closures

✓ For a perfect fluid, we impose $\boldsymbol{\sigma} = 0$
$\rightarrow$ Euler equations

✓ For a Newtonian viscous fluid, $\boldsymbol{\sigma} = 2\eta_s D\boldsymbol{u}$ (viscosity $\eta_s > 0$)
$\rightarrow$ Navier-Stokes equations

✓ For a generalized Newtonian fluid, $\boldsymbol{\sigma} = 2\eta_s(|D\boldsymbol{u}|)D\boldsymbol{u}$

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To take into account some elasticity aspect, appearing for instance in polymer solution, we add an elastic contribution: $\boldsymbol{\sigma} = 2\eta_s D\boldsymbol{u} + \boldsymbol{\tau}$.

✓ Differential models, $\dfrac{\partial \boldsymbol{\tau}}{\partial t} = \cdots$ (example: Oldroyd)

✓ Micro-macro models, $\boldsymbol{\tau} = \displaystyle\int \mathbf{E}(\mathbf{Q}) \otimes \mathbf{Q}\, \psi(\mathbf{Q})\mathrm{d}\mathbf{Q}$ (example: FENE)

✓ Integral model

Integral models

To take into account the past history of the fluid:

$$\boldsymbol{\tau}(t,\boldsymbol{x}) = \int_{-\infty}^{t} m(t-T)\,\mathcal{S}\big(\boldsymbol{F}(T,t,\boldsymbol{x})\big)\,\mathrm{d}T$$

- m is the memory function,
- $\mathcal{S}$ is a model-dependent strain measure,
- $\boldsymbol{F}(T,t,\cdot)$ is the deformation gradient from a time T to a next time t.

The deformation gradient $\boldsymbol{F}$ is coupled with the velocity field $\boldsymbol{u}$:

$$\begin{cases} \partial_t \boldsymbol{F} + \boldsymbol{u}\cdot\nabla\boldsymbol{F} = \boldsymbol{F}\cdot\nabla\boldsymbol{u} \\ \boldsymbol{F}|_{t=T} = \boldsymbol{\delta} \end{cases}$$

Two physical assumptions and examples

Due to physical principles, the functions m and $\mathcal{S}$ must satisfy assumptions.

1°) Fading memory $\Longrightarrow$ m must be positive and decreases to 0.

Examples

- $m(s) = \frac{1}{\lambda}\,\mathrm{e}^{-s/\lambda}$ ($\lambda > 0$ is a relaxation time).
- Use several relaxation times is possible.
- The Doi-Edwards model with Independent Alignment Assumption

$$m(s) = \frac{8}{\pi^2\,\lambda}\sum_{k=0}^{+\infty} \mathrm{e}^{-(2k+1)^2 s/\lambda}$$

In practice, we will assume that

$$(H1) \quad m : \mathbb{R}^+ \longmapsto \mathbb{R} \text{ is positive, decreasing and } \int_0^{+\infty} m(s)\,\mathrm{d}s = 1.$$

Two physical assumptions and examples

2°) Frame indifference $\Longrightarrow \boldsymbol{\tau}$ depends on the Finger tensor $\boldsymbol{B} = {}^t\boldsymbol{F} \cdot \boldsymbol{F}$

Examples

- The "linear" case: $\mathcal{S}(\boldsymbol{F}) = \boldsymbol{B}$
- K-BKZ models

$$\mathcal{S}(\boldsymbol{F}) = \phi_1(I_1, I_2)(\boldsymbol{B} - \boldsymbol{\delta}) + \phi_2(I_1, I_2)(\boldsymbol{\delta} - \boldsymbol{B}^{-1})$$

  where $I_1 = \mathrm{Tr}(\boldsymbol{B})$ and $I_2 = \mathrm{Tr}(\boldsymbol{B}^{-1})$ are the strain invariants.
- PSM models like

$$\mathcal{S}(\boldsymbol{F}) = \frac{\boldsymbol{B}}{1 + \mathrm{Tr}(\boldsymbol{B})}$$

In practice, we will assume that

$$(H2) \quad \mathcal{S} : \boldsymbol{G} \in \mathcal{L}(\mathbb{R}^d) \longmapsto \mathcal{S}(\boldsymbol{G}) \in \mathcal{L}(\mathbb{R}^d) \text{ is of class } \mathcal{C}^1.$$

Oldroyd as an integral model

In the more simple case, the stress given by

$$\boldsymbol{\tau}(t,\boldsymbol{x}) = \int_{-\infty}^{t} m(t-T)\,\mathcal{S}\big(\boldsymbol{F}(T,t,\boldsymbol{x})\big)\,\mathrm{d}T$$

with $\quad m(s) = \mathrm{e}^{-s} \quad$ and $\quad \mathcal{S}(\boldsymbol{F}) = {}^t\boldsymbol{F}\cdot\boldsymbol{F} - \boldsymbol{\delta}$

where $\quad \begin{cases} \partial_t \boldsymbol{F} + \boldsymbol{u}\cdot\nabla\boldsymbol{F} = \boldsymbol{F}\cdot\nabla\boldsymbol{u} \\ \boldsymbol{F}|_{t=T} = \boldsymbol{\delta} \end{cases}$

It satisfies the well-known Oldroyd constitutive relation:

$$\partial_t\boldsymbol{\tau} + \boldsymbol{u}\cdot\nabla\boldsymbol{\tau} - {}^t\nabla\boldsymbol{u}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\nabla\boldsymbol{u} + \boldsymbol{\tau} = 2D\boldsymbol{u}$$

Remark

For other integral models, there is no equivalent differential law.

Summary

Assumptions and first results

The complete system can be written as

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p - \Delta \boldsymbol{u} = \operatorname{div} \boldsymbol{\tau} + \boldsymbol{f} \\ \operatorname{div} \boldsymbol{u} = 0 \\ \boldsymbol{\tau}(t, \boldsymbol{x}) = \displaystyle\int_{-\infty}^{t} m(t - T)\, \mathcal{S}\big(\boldsymbol{F}(T, t, \boldsymbol{x})\big)\, \mathrm{d}T \\ \partial_t \boldsymbol{F} + \boldsymbol{u} \cdot \nabla \boldsymbol{F} = \boldsymbol{F} \cdot \nabla \boldsymbol{u} \end{cases}$$

Remark

The time T can be view as a parameter. It is interesting to select as independent variable the age $s = t - T$, which is measured relative to the current time t.

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Remark

The time T can be view as a parameter. It is interesting to select as independent variable the age $s = t - T$, which is measured relative to the current time t. We now introduce $\boldsymbol{G}(s, t, \boldsymbol{x}) = \boldsymbol{F}(t - s, t, \boldsymbol{x})$.

The "initial" conditions associated to the system are

$$\boldsymbol{u}\big|_{t=0} = \boldsymbol{u}_0, \qquad \boldsymbol{G}\big|_{t=0} = \boldsymbol{G}_{\mathrm{old}}, \qquad \boldsymbol{G}\big|_{s=0} = \boldsymbol{\delta}.$$

Local result

Theorem 1 (local existence and uniqueness)

Let $T > 0$, $r > 1$ and $p > d = 2$ or 3.

Assume that (H1) and (H2) hold.

If $\boldsymbol{u}_0 \in D_p^r$, $\boldsymbol{G}_{\mathrm{old}} \in L^\infty(\mathbb{R}^+; W^{1,p})$, $\partial_s \boldsymbol{G}_{\mathrm{old}} \in L^r(\mathbb{R}^+; L^p)$, $\boldsymbol{f} \in L^r(0, T; L^p)$, there exists $T_\star \in]0, T]$ and a unique strong solution $(\boldsymbol{u}, p, \boldsymbol{\tau}, \boldsymbol{G})$ in $[0, T_\star]$:

$$
\begin{cases}
\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p - \Delta \boldsymbol{u} = \operatorname{div} \boldsymbol{\tau} + \boldsymbol{f} \\
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\boldsymbol{\tau}(t, \boldsymbol{x}) = \displaystyle\int_0^{+\infty} m(s)\, \mathcal{S}\big(\boldsymbol{G}(s, t, \boldsymbol{x})\big)\, \mathrm{d}s \\
\partial_t \boldsymbol{G} + \partial_s \boldsymbol{G} + \boldsymbol{u} \cdot \nabla \boldsymbol{G} = \boldsymbol{G} \cdot \nabla \boldsymbol{u} \\
\boldsymbol{u}\big|_{t=0} = \boldsymbol{u}_0, \qquad \boldsymbol{G}\big|_{t=0} = \boldsymbol{G}_{\mathrm{old}}, \qquad \boldsymbol{G}\big|_{s=0} = \boldsymbol{\delta}
\end{cases}
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Ideas of proof: fixed point (Schauder's theorem)

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\begin{cases}
\partial_t \boldsymbol{u} + \overline{\boldsymbol{u}} \cdot \nabla \overline{\boldsymbol{u}} + \nabla p - \Delta \boldsymbol{u} = \operatorname{div} \overline{\boldsymbol{\tau}} + \overline{\boldsymbol{f}}, \\
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1. Stokes problem

Ideas of proof: fixed point (Schauder's theorem)

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2. Explicit

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1. Stokes problem
2. Explicit
3. Linear transport equation

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$$\begin{aligned} \mathcal{H}(T, R) = \Big\{ (\boldsymbol{u}, \boldsymbol{G}, \boldsymbol{\tau}) \;;\; & \boldsymbol{u} \in L^r(0, T; W^{2,p}), \quad \partial_t \boldsymbol{u} \in L^r(0, T; L^p), \\ & \boldsymbol{G} \in L^\infty(\mathbb{R}^+ \times (0, T); W^{1,p}), \quad \partial_s \boldsymbol{G},\ \partial_t \boldsymbol{G} \in L^\infty(\mathbb{R}^+; L^r(0, T; L^p)), \\ & \boldsymbol{\tau} \in L^\infty(0, T; W^{1,p}), \quad \partial_t \boldsymbol{\tau} \in L^r(0, T; L^p), \quad \|(\boldsymbol{u}, \boldsymbol{G}, \boldsymbol{\tau})\| \leq R \Big\}. \end{aligned}$$

For T small enough, there exists R such that

$$\Phi : (\overline{\boldsymbol{u}}, \overline{\boldsymbol{G}}, \overline{\boldsymbol{\tau}}) \in \mathcal{H}(T, R) \longmapsto (\boldsymbol{u}, \boldsymbol{G}, \boldsymbol{\tau}) \in \mathcal{H}(T, R).$$

Summary

Some known global existence results

- Small data (very many results)
  ✓ [Guillopé, Saut] *Existence results for the flow of viscoelastic fluids with a differential constitutive law.* Nonlinear Anal. (1990)
  ✓ [Renardy] *An existence theorem for model equations resulting from kinetic theories of polymer solutions.* SIAM J. Math. Anal. (1991)
- Oldroyd with co-rotational assumption
  ✓ [Lions, Masmoudi] *Global solutions for some Oldroyd models of non-Newtonian flows.* Chinese Ann. Math. Ser. B (2000)
- FENE model
  ✓ [Masmoudi] *Global existence of weak solutions to the FENE dumbbell model of polymeric flows.* Inventiones (2013)
- Oldroyd with diffusive stress
  ✓ [Barrett, Süli] *Existence of global weak solutions for some polymeric flow models.* Math. Models Methods Appl. Sci. (2005)
  ✓ [Constantin, Kliegl] *Note on global regularity for two-dimensional Oldroyd-B fluids with diffusive stress.* A.R.M.A. (2012)

Assumptions and results

(H2') $\mathcal{S} : \mathcal{L}(\mathbb{R}^2) \longmapsto \mathcal{L}(\mathbb{R}^2)$ is of class $\mathcal{C}^1$ and satisfies

- $\exists \mathcal{S}_\infty \geq 0 \; ; \; \forall \boldsymbol{G} \in \mathcal{L}(\mathbb{R}^2) \qquad |\mathcal{S}(\boldsymbol{G})| \leq \mathcal{S}_\infty$
- $\exists \mathcal{S}'_\infty \geq 0 \; ; \; \forall \boldsymbol{G} \in \mathcal{L}(\mathbb{R}^2) \qquad |\boldsymbol{G}||\mathcal{S}'(\boldsymbol{G})| \leq \mathcal{S}'_\infty$

Examples

- For the PSM model where $\mathcal{S}(\boldsymbol{G}) = \dfrac{{}^t\boldsymbol{G} \cdot \boldsymbol{G}}{1 + \mathrm{Tr}({}^t\boldsymbol{G} \cdot \boldsymbol{G})}$, we have $|\mathcal{S}(\boldsymbol{G})| \leq 1$ and $|\boldsymbol{G}||\mathcal{S}'(\boldsymbol{G})| \leq 2(1 + \sqrt{2})$
- All usual models of type K-BKZ
- Doi-Edwards model with independent alignment approximation

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- All usual models of type K-BKZ ········· OK
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Remark

(H2') is not satisfied for the "linear" case $\mathcal{S}(\boldsymbol{G}) = {}^t\boldsymbol{G} \cdot \boldsymbol{G}$ ········· NO
(consequently, the following result is not proved for the Oldroyd model).

Assumptions and results

Theorem 2 (global existence and uniqueness)

Let $T > 0$, $r > 1$ and $p > d = 2$ such that $\frac{1}{r} + \frac{1}{p} \leq \frac{1}{2}$.
Assume that (H1) and (H2') hold.
If $\boldsymbol{u}_0 \in D^r_p$, $\boldsymbol{G}_{\mathrm{old}} \in L^\infty(\mathbb{R}^+; W^{1,p})$, $\partial_s \boldsymbol{G}_{\mathrm{old}} \in L^r(\mathbb{R}^+; L^p)$, $\boldsymbol{f} \in L^r(0,T; L^p)$, there exists a unique strong solution $(\boldsymbol{u}, p, \boldsymbol{\tau}, \boldsymbol{G})$ on $[0, T]$:

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p - \Delta \boldsymbol{u} = \operatorname{div} \boldsymbol{\tau} + \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \boldsymbol{\tau}(t, \boldsymbol{x}) = \displaystyle\int_0^{+\infty} m(s)\, \mathcal{S}\big(\boldsymbol{G}(s,t,\boldsymbol{x})\big)\, \mathrm{d}s, \\ \partial_t \boldsymbol{G} + \partial_s \boldsymbol{G} + \boldsymbol{u} \cdot \nabla \boldsymbol{G} = \boldsymbol{G} \cdot \nabla \boldsymbol{u}, \\ \boldsymbol{u}\big|_{t=0} = \boldsymbol{u}_0, \qquad \boldsymbol{G}\big|_{t=0} = \boldsymbol{G}_{\mathrm{old}}, \qquad \boldsymbol{G}\big|_{s=0} = \boldsymbol{\delta}. \end{cases}$$

Ideas of proof: main difficulties

✓ **Idea**: obtain additional bounds on the local solution.

✓ **Free**: $\big((\text{H1}) \text{ and } (\text{H2'})\big) \Longrightarrow \|\boldsymbol{\tau}\|_{L^\infty(0,T;L^\infty)} \leq \mathcal{S}_\infty$

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Proof:

$$
\begin{aligned}
|\boldsymbol{\tau}(t,\boldsymbol{x})| &= \int_0^{+\infty} m(s)\, \big|\mathcal{S}\big(\boldsymbol{G}(s,t,\boldsymbol{x})\big)\big| \,\mathrm{d}s \\
&\leq \mathcal{S}_\infty \int_0^{+\infty} m(s)\,\mathrm{d}s && \text{see (H2')} \\
&\leq \mathcal{S}_\infty && \text{see (H1)}
\end{aligned}
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✓ **What else**: $\boldsymbol{\tau} \in L^r(0,T;L^p) \Longrightarrow \nabla \boldsymbol{u} \in L^r(0,T;L^p)$

True for all $1 < r, p < +\infty$ **but false** for $r = p = \infty$

We only have , for all $t \in (0,T)$, and for r, p such that $\frac{1}{r} + \frac{1}{p} \leq \frac{1}{2}$,

$$\|\nabla \boldsymbol{u}(t,\cdot)\|_{L^\infty} \leq C + \|\boldsymbol{g}\|_{L^\infty(0,T;L^\infty)} \ln\big(\mathrm{e} + \|\nabla \boldsymbol{g}\|_{L^r(0,t;L^p)}\big),$$

where $\boldsymbol{g} = \boldsymbol{\tau} - \boldsymbol{u} \otimes \boldsymbol{u}$

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Proof: based on the Duhamel formulation

$$\nabla \boldsymbol{u}(t,\boldsymbol{x}) = \mathrm{e}^{\eta t \Delta} \nabla \boldsymbol{u}_0 + \int_0^t \mathrm{e}^{\eta(t-\sigma)\Delta} \mathfrak{P} \Delta(\boldsymbol{\tau} - \boldsymbol{u} \otimes \boldsymbol{u})(\sigma,\boldsymbol{x})\,\mathrm{d}\sigma.$$

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True for all $1 < r, p < +\infty$ **but false** for $r = p = \infty$

We only have , for all $t \in (0,T)$, and for r, p such that $\frac{1}{r} + \frac{1}{p} \leq \frac{1}{2}$,

$$\|\nabla \boldsymbol{u}(t,\cdot)\|_{L^\infty} \leq C + \|\boldsymbol{g}\|_{L^\infty(0,T;L^\infty)} \ln\big(\mathrm{e} + \|\nabla \boldsymbol{g}\|_{L^r(0,t;L^p)}\big),$$

where $\boldsymbol{g} = \boldsymbol{\tau} - \boldsymbol{u} \otimes \boldsymbol{u}$

✓ We must control $\underset{\substack{\uparrow \\ \text{[Constantin \& Seregin]}}}{\|\boldsymbol{u}\|_{L^\infty(0,T;L^\infty)}}$, $\underset{\substack{\uparrow \\ \text{[Lemarié-Rieusset]}}}{\|\nabla \boldsymbol{u}\|_{L^r(0,t;L^p)}}$ and $\underset{\substack{\uparrow \\ ?}}{\|\nabla \boldsymbol{\tau}\|_{L^r(0,t;L^p)}}$

Ideas of proof: main difficulties

✓ Control of $\|\nabla\boldsymbol{\tau}\|_{L^r(0,t;L^p)}$

$$\boldsymbol{\tau}(t,\boldsymbol{x}) = \int_0^{+\infty} m(s)\,\mathcal{S}\big(\boldsymbol{G}(s,t,\boldsymbol{x})\big)\,\mathrm{d}s$$

$$\nabla\boldsymbol{\tau}(t,\boldsymbol{x}) = \int_0^{\infty} m(s)\,\mathcal{S}'(\boldsymbol{G}(s,t,\boldsymbol{x})) : \nabla\boldsymbol{G}(s,t,\boldsymbol{x})\,\mathrm{d}s$$

$$|\nabla\boldsymbol{\tau}(t,\boldsymbol{x})| \le \mathcal{S}'_\infty \int_0^{\infty} m(s)\left|\frac{\nabla\boldsymbol{G}(s,t,\boldsymbol{x})}{|\boldsymbol{G}(s,t,\boldsymbol{x})|}\right|\mathrm{d}s \qquad \text{see (H2')}$$

$$\|\nabla\boldsymbol{\tau}\|^r_{L^r(0,t;L^p)} \le \mathcal{S}'_\infty \underbrace{\int_0^t\int_0^{\infty} m(s)\left\|\frac{\nabla\boldsymbol{G}}{|\boldsymbol{G}|}\right\|^r_{L^p}(s,t)\,\mathrm{d}s\mathrm{d}t}_{y(t)}$$

Ideas of proof: main difficulties

✓ Control of $y(t) = \int_0^t \int_0^\infty m(s) \left\| \frac{\nabla \boldsymbol{G}}{|\boldsymbol{G}|} \right\|_{L^p}^r (s,t) \,\mathrm{d}s\mathrm{d}t$

Lemma 1

$$|\boldsymbol{G}| \geq Cte > 0.$$

Proof:

$$\partial_t \boldsymbol{G} + \partial_s \boldsymbol{G} + \boldsymbol{u} \cdot \nabla \boldsymbol{G} = \boldsymbol{G} \cdot \nabla \boldsymbol{u}$$

$$\partial_t \det \boldsymbol{G} + \partial_s \det \boldsymbol{G} + \boldsymbol{u} \cdot \nabla \det \boldsymbol{G} = \det \boldsymbol{G} \times \operatorname{div} \boldsymbol{u} = 0$$

$$\det \boldsymbol{G} \geq C > 0 \quad \text{(constant along characteristic lines)}$$

Due to the inequality of arithmetic and geometric means, we have

$$|\boldsymbol{G}|^2 = \mathrm{Tr}({}^t\boldsymbol{G} \cdot \boldsymbol{G}) \geq 2\sqrt{\det({}^t\boldsymbol{G} \cdot \boldsymbol{G})} = 2|\det(\boldsymbol{G})| \geq 2C > 0.$$

Ideas of proof: main difficulties

✓ Control of $y(t) = \int_0^t \int_0^\infty m(s) \left\| \frac{\nabla \boldsymbol{G}}{|\boldsymbol{G}|} \right\|_{L^p}^r (s,t)\, \mathrm{d}s\mathrm{d}t$

Lemma 1

$$|\boldsymbol{G}| \geq Cte > 0.$$

Lemma 2

$$y'(t) \leq C + y(t) + y(t) \|\nabla \boldsymbol{u}\|_{L^\infty(0,T;L^\infty)} + \|\nabla^2 \boldsymbol{u}\|^r_{L^r(0,t;L^p)}.$$

Proof: "Classical" estimates (Cauchy-Schwarz, Hölder, Young inequalities) from the definition of $\boldsymbol{G}$:

$$\partial_t \boldsymbol{G} + \partial_s \boldsymbol{G} + \boldsymbol{u} \cdot \nabla \boldsymbol{G} = \boldsymbol{G} \cdot \nabla \boldsymbol{u}$$

Ideas of proof: main difficulties

✓ Control of $y(t) = \int_0^t \int_0^\infty m(s) \left\| \frac{\nabla \boldsymbol{G}}{|\boldsymbol{G}|} \right\|_{L^p}^r (s,t) \, \mathrm{d}s \mathrm{d}t$

Lemma 1

$$|\boldsymbol{G}| \geq Cte > 0.$$

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$$y'(t) \leq C + y(t) + y(t) \|\nabla \boldsymbol{u}\|_{L^\infty(0,T;L^\infty)} + \|\nabla^2 \boldsymbol{u}\|_{L^r(0,t;L^p)}^r.$$

Lemma 3

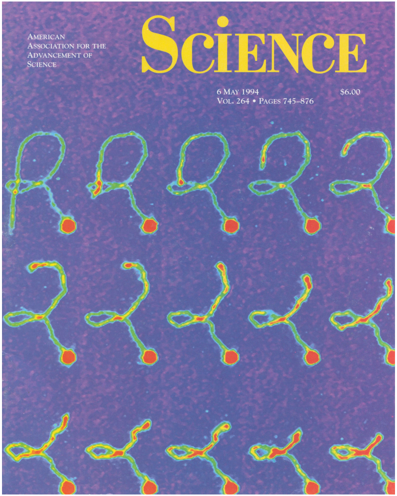
$$y'(t) \leq C(\mathrm{e} + y(t)) \ln(\mathrm{e} + y(t)).$$

We conclude that $y(t) \leq \mathrm{e}^{\mathrm{e}^{Ct}}$ for all $t \in (0, T)$. □

Summary

The Doi-Edwards model

✓ M. Doi and S.F. Edwards wrote a series of papers (1978, 1980) expanding the concept of reptation introduced by P.G. de Gennes in 1971.



The Doi-Edwards model

✓ They obtain a model describing the dynamics of flexible polymers in melts and concentrated solutions:

$$
\begin{cases}
\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p - \Delta\boldsymbol{u} = \operatorname{div}\boldsymbol{\tau} \\
\operatorname{div}\boldsymbol{u} = 0 \\
\boldsymbol{\tau}(t,\boldsymbol{x}) = \displaystyle\int_{-\frac12}^{\frac12} \boldsymbol{S}(t,\boldsymbol{x},\ell)\,\mathrm{d}\ell \\
\boldsymbol{S}(t,\boldsymbol{x},\ell) = -\displaystyle\int_0^{+\infty} \partial_s K(s,t,\boldsymbol{x},\ell)\,\mathcal{S}(\boldsymbol{G}(s,t,\boldsymbol{x}))\,\mathrm{d}s \\
\partial_t \boldsymbol{G} + \boldsymbol{u}\cdot\nabla\boldsymbol{G} + \partial_s\boldsymbol{G} = \boldsymbol{G}\cdot\nabla\boldsymbol{u} \\
\partial_t K + \boldsymbol{u}\cdot\nabla K + \partial_s K + \Big(\nabla\boldsymbol{u} : \displaystyle\int_0^{\ell}\boldsymbol{S}\Big)\partial_\ell K - \partial_\ell^2 K = 0
\end{cases}
$$

$$
\text{where}\quad \mathcal{S}(\boldsymbol{G}) = \frac{1}{\langle|\boldsymbol{G}\cdot\boldsymbol{u}|\rangle_0}\left\langle\frac{(\boldsymbol{G}\cdot\boldsymbol{u})\otimes(\boldsymbol{G}\cdot\boldsymbol{u})}{|\boldsymbol{G}\cdot\boldsymbol{u}|}\right\rangle_0 - \frac{1}{d}\boldsymbol{\delta}
$$

the brackets $\langle\cdot\rangle_0$ corresponding to the average over the isotropic distribution of unit vectors $\boldsymbol{u}\in\mathbb{S}^{d-1}$.

Theorem 3

For all time $T > 0$, the two dimensional Doi-Edwards model admits a strong solution on the interval time $[0, T]$.

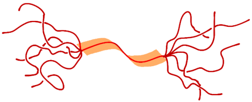
$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p - \Delta \boldsymbol{u} = \operatorname{div} \boldsymbol{\tau} \\ \operatorname{div} \boldsymbol{u} = 0 \\ \boldsymbol{\tau}(t, \boldsymbol{x}) = \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{S}(t, \boldsymbol{x}, \ell) \, \mathrm{d}\ell \\ \boldsymbol{S}(t, \boldsymbol{x}, \ell) = - \displaystyle\int_0^{+\infty} \partial_s K(s, t, \boldsymbol{x}, \ell) \, \mathcal{S}(\boldsymbol{G}(s, t, \boldsymbol{x})) \, \mathrm{d}s \\ \partial_t \boldsymbol{G} + \boldsymbol{u} \cdot \nabla \boldsymbol{G} + \partial_s \boldsymbol{G} = \boldsymbol{G} \cdot \nabla \boldsymbol{u} \\ \partial_t K + \boldsymbol{u} \cdot \nabla K + \partial_s K + \left(\nabla \boldsymbol{u} : \displaystyle\int_0^{\ell} \boldsymbol{S} \right) \partial_\ell K - \partial_\ell^2 K = 0 \end{cases}$$

where $\quad \mathcal{S}(\boldsymbol{G}) = \dfrac{1}{\langle |\boldsymbol{G} \cdot \boldsymbol{u}| \rangle_0} \left\langle \dfrac{(\boldsymbol{G} \cdot \boldsymbol{u}) \otimes (\boldsymbol{G} \cdot \boldsymbol{u})}{|\boldsymbol{G} \cdot \boldsymbol{u}|} \right\rangle_0 - \dfrac{1}{d} \boldsymbol{\delta}$

Other models?

✓ "Rolie-Poly" model
$\rightsquigarrow$ ROuse LInear Entangled POLYmers

✓ "Pom-Pom" model
$\rightsquigarrow$ developed in order to take into account the morphology of branched polymer melts



✓ $\cdots$

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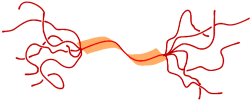

✓ ...

Question

What are the physically relevant models for which we know to show the global existence of a regular solution?

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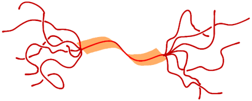

✓ ...

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THANK YOU.