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On PDE analysis for flows of quasi-incompressible fluids

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Theory of mixtures—1

Fluid of mixture with N constituents
$(\varrho_\alpha, \mathbf{v}_\alpha, e_\alpha)$, $\alpha = 1, \cdots, N$:

- balance of mass for each α . . . . N equations,
- balance of linear momentum for each α . . . . dN equations,
- balance of energy for each α . . . . N equations.

Whole mixture $(\varrho, \mathbf{v}, e)$

- balance of mass for the whole mixture . . . . 1 equation,
- balance of linear momentum for the whole mixture d equations,
- balance of energy for the whole mixture . . . . 1 equation.

Theory of mixtures—2

In applications, we consider combinations such as:

Combination 1:

- balance of mass for each α .... N equations,
- balance of linear momentum for each α .... dN equations,
- balance of energy for the whole mixture ....1 equation.

Combination 2:

- balance of mass for each α .... N equations,
- balance of linear momentum for the whole mixture d equations,
- balance of energy for the whole mixture ....1 equation.

Two constituents under Combination 2

Given $(\varrho^1, \varrho^2, \mathbf{v}, e)$. **Introduce**

$$\varrho = \varrho^1 + \varrho^2 \text{ (mass additivity)}, \quad c := \frac{\varrho_1}{\varrho} \text{ (concentration)}.$$

There arises:

$$\begin{aligned}
\dot{\varrho} &= -\varrho \operatorname{div} \mathbf{v}, \\
\varrho \dot{c} &= -\operatorname{div} \mathbf{j}, \\
\varrho \dot{\mathbf{v}} &= \operatorname{div} \mathbf{T} + \varrho \mathbf{b}, \\
\varrho\left(e + \frac{1}{2}|\mathbf{v}|^2\right)^{\cdot} &= \operatorname{div}\left(\mathbf{T}\,\mathbf{v} - \mathbf{q}_E\right) + \varrho \mathbf{b} \cdot \mathbf{v},
\end{aligned} \tag{1}$$

where $\dot{z} := \partial_t z + \mathbf{v} \cdot \nabla_x z$.

Goal

Now the unknown is

$$(\varrho, c, \mathbf{v}, e).$$

There are still three 'unknowns' (constitutive quantities):

$$(\mathbf{T}, \mathbf{j}, \mathbf{q}_E).$$

To derive a model of PDEs and boundary conditions that is compatible with thermodynamical principles and is solvable.

We focus on the fluids under quasi-incompressibility constraint. This makes the constitutive process more delicate.

Quasi-incompressibility assumption:

Quasi-incompressibility:

$$\varrho = \varrho(c).$$

Volume additivity constraint:

$1 = \varphi^1 + \varphi^2, \quad \varrho^\alpha = \varphi^\alpha \rho_m^\alpha, \quad \varrho_m^\alpha$ are the true densities and are constants,

one has

$$\varrho(c) = \frac{\varrho_m^1 \varrho_m^2}{(1-c)\varrho_m^1 + c\varrho_m^2}.$$

Quasi-incompressible fluids

For quasi-incompressible fluids:

$$\operatorname{div} \mathbf{v} = R(c) \operatorname{div} \mathbf{j}, \quad R(c) := \frac{\varrho'(c)}{\varrho^2(c)}.$$

With volume additivity:

$$R(c) = r^* := \frac{\varrho_m^1 - \varrho_m^2}{\varrho_m^1 \varrho_m^2}.$$

If $\varrho_m^1 = \varrho_m^2$, the fluids become homogeneous (ϱ becomes constant) and is then incompressible $\operatorname{div} \mathbf{v} = 0$.

Compatible constitutive theory

The modelling approach is also well presented by Rajagopal and Srinivasa in [7], Heida, Málek and Rajagopal in [3] and [4].

Constitutive assumption:

$$\eta = \eta(e, \varrho, c), \quad \frac{\partial \eta}{\partial e} > 0, \quad \eta \text{ is the entropy.}$$

Then

$$e = e(\eta, \varrho, c).$$

Modelling

Thermodynamic temperature, pressure and the chemical potential:

$$\theta := \frac{\partial e}{\partial \eta}, \quad p := \frac{\partial e}{\partial \varrho}, \quad \mu := \frac{\partial e}{\partial c}.$$

With volume additivity,

$$\varrho\dot{\eta} + \operatorname{div}\left(\frac{\mathbf{q}_\eta}{\theta}\right) = \frac{1}{\theta}\left(\mathbf{T}^d : \mathbf{D}^d - \mathbf{j}\cdot\nabla_x(\mu + r_*(m+p)) - \frac{\mathbf{q}_\eta \cdot \nabla_x\theta}{\theta}\right) =: \zeta\,.$$

where $m := \mathrm{tr}\mathbf{T}/3, \mathbf{q}_\eta := \mathbf{q}_E - \big(\mu + mR(c)\big)\,\mathbf{j}$ and

$$\mathbf{D} := \frac{1}{2}\big(\nabla\mathbf{v} + (\nabla\mathbf{v})^t\big), \quad \mathbf{T}^d := \mathbf{T} - \frac{1}{3}\mathrm{tr}\mathbf{T}, \quad \mathbf{D}^d := \mathbf{D} - \frac{1}{3}\mathrm{tr}\mathbf{D}.$$

Modelling

For the isothermal processes, i.e. θ is constant,

$$\zeta = \mathbf{T}^d : \mathbf{D}^d - \mathbf{j} \cdot \nabla_x(\mu + r_*(m+p)) \,. \tag{2}$$

We impose the linear relations

$$\begin{aligned} \mathbf{T}^d &= 2\nu_* \mathbf{D}^d \quad \nu_* \in (0,\infty)\,, \\ \mathbf{j} &= -\beta_* \nabla_x(\mu + r_*(m+p)) \quad \beta_* \in (0,\infty)\,, \end{aligned}$$

Then

$$\zeta \geq 0$$

and

$$\begin{aligned} \dot{\varrho} &= -\varrho \,\mathrm{div}\mathbf{v}, \\ \varrho \dot{\mathbf{v}} &= \nabla m + \nu_* \Delta \mathbf{v} + \frac{\nu_*}{3} \nabla \mathrm{div}\mathbf{v} + \varrho \mathbf{b}, \\ -\Delta(m + p + r_*^{-1}\mu) &= r_*^{-2} \beta_* \mathrm{div}\mathbf{v}\,. \end{aligned}$$

Modelling–Boundary conditions–1

We assume Dirichlet boundary condition for the velocity

$$\mathbf{v} = 0 \quad \text{on} \quad \partial\Omega \times (0, T).$$

If $\mathbf{b} = 0$, boundary condition

$$\mathbf{q}_E \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \times (0, T)$$

is sufficient to make sure the conservation of the energy

$$\mathcal{E}(t) := \int_\Omega \varrho\Big(e + \frac{1}{2}|\mathbf{v}|^2\Big)dx.$$

Modelling–Boundary conditions–2

If we consider total entropy

$$S(t) := \int_{\Omega} \varrho\eta \, dx,$$

then for the rate of entropy production, that is dS/dt, there arises the boundary integral

$$\int_{\partial\Omega} \frac{\mathbf{q}_\eta \cdot \mathbf{n}}{\theta} dS, \quad \mathbf{q}_\eta := \mathbf{q}_E - \big(\mu + mR(c)\big)\,\mathbf{j}. \tag{3}$$

Then boundary condition

$$\mathbf{j} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \times (0, T)$$

is sufficient to make sure that there is no contribution to the rate of entropy production due to the boundary integral (3). This implies

$$\frac{\partial(\mu + r_*(m + p))}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega \, .$$

Modelling–The chemical concentration

Recall $0 < c < 1$, Following the ansatz corresponding to the mixture of ideal gases (see Müller [4]) it is reasonable to choose $\mu(c)$ of the form

$$\mu(c) = \frac{1}{2}\big(\ln c - \ln(1-c)\big).$$

By volume additivity with $\varrho_m^1 = 1, \varrho_m^2 = \varrho_m$,

$$c(\varrho) = \frac{\varrho - \varrho_m}{\varrho(1-\varrho_m)} = \frac{\varrho r_* + (\varrho - 1)}{\varrho r_*}.$$

Then

$$\mu(c(\varrho)) = \ln\big((1+r_*)\varrho - 1\big) - \ln(1-\varrho).$$

This implies, unconditionally

$$\frac{1}{1+\varrho_*} \leq \varrho \leq 1.$$

A selected problem—preparation

We set

$$\beta_* = \nu_* = 1, \quad \mathbf{b} = 0.$$

We introduce the standard Helmholtz decomposition

$$\mathbf{z} = \mathbf{H}[\mathbf{z}] + \mathbf{H}^{\perp}[\mathbf{z}]\,, \quad \mathbf{H}^{\perp}[\mathbf{z}] := \nabla(\Delta_N)^{-1}\mathrm{div}\mathbf{z}.$$

Here $(\Delta_N)^{-1}$ the solution operator of the homogeneous Neumann problem associated to the Laplace operator.

A selected problem—the equations

Then

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{v}) = 0,$$
$$\partial_t(\varrho \mathbf{v}) + \mathrm{div}(\varrho \mathbf{v} \otimes \mathbf{v}) + \frac{1}{r_*}\nabla q(\varrho) = 2\mathrm{div}\mathbf{D}^d - \frac{1}{r_*^2}\mathbf{H}^{\perp}[\mathbf{v}], \tag{4}$$

Here q is defined as

$$q(\varrho) := r_* p(\varrho) + \mu(c(\varrho)).$$

Then

$$\lim_{\varrho \to \frac{1}{1+r_*}+} q(\varrho) = -\infty, \quad \lim_{\varrho \to 1-} q(\varrho) = +\infty.$$

A selected problem—initial and boundary data

Boundary condition

$$\mathbf{v} = 0 \quad \text{on } [0, T] \times \Omega. \tag{5}$$

Initial data

$$\mathbf{v}_0 \in L^2(\Omega; \mathbb{R}^3), \quad \frac{1}{1 + r_*} < \varrho_0 < 1, \quad \int_\Omega Q(\varrho_0) dx < \infty. \tag{6}$$

Pressure potential Q:

$$Q(\varrho) := \varrho \int_{\varrho_*}^{\varrho} \frac{q(z)}{z^2} dz,$$

where ϱ_* is the only zero point of $q(\cdot)$.

Questions

1, Existence of solution?

2, Asymptotic behavior as $r_* \to 0$? From quasi-incompressible fluids to incompressible fluids?

Definition of finite energy global weak solution

Definition

We say $[\varrho, \mathbf{v}]$ to be a global finite energy weak solution for problem (4), (5) and (6), if there holds for any $T > 0$:

- $\frac{1}{1+r_*} \le \varrho \le 1$ a.e. in $(0, T) \times \Omega$, $\mathbf{v}|_{(0,T)\times\partial\Omega} = 0$ and

  $$q(\varrho) \in L^1((0,T)\times\Omega), \quad \mathbf{v} \in C_w(0,T; L^2(\Omega)) \cap L^2(0,T; W_0^{1,2}(\Omega)).$$

- For any $0 \le \tau \le T$ and any test functions

  $$\varphi \in C_c^\infty([0,T] \times \Omega), \quad \psi \in C_c^\infty([0,T] \times \Omega; \mathbb{R}^3),$$

  there holds

Definition of finite energy global weak solution

$$\int_0^\tau \int_\Omega \varrho \partial_t \varphi + \varrho \mathbf{v} \cdot \nabla_x \varphi \, dx \, dt + \int_\Omega \varrho_0 \varphi(0, \cdot) \, dx = \int_\Omega \varrho(\tau, \cdot) \varphi(\tau, \cdot) \, dx \tag{7}$$

and

$$\int_0^\tau \int_\Omega \varrho \mathbf{v} \cdot \partial_t \psi + \varrho \mathbf{v} \otimes \mathbf{v} : \nabla_x \psi + \frac{1}{r_*} q(\varrho) \mathrm{div} \psi \, dx \, dt + \int_\Omega \varrho_0 \mathbf{v}_0 \cdot \psi(0, \cdot) \, dx$$
$$= \int_0^\tau \int_\Omega 2\mathbf{D}^d(\mathbf{v}) : \nabla_x \psi + \frac{1}{r_*^2} \mathbf{H}^\perp[\mathbf{v}] \cdot \psi \, dx \, dt + \int_\Omega \varrho \mathbf{v}(\tau, \cdot) \cdot \psi(\tau, \cdot) \, dx. \tag{8}$$

Definition of finite energy global weak solution

For a.a. $\tau \in (0, T)$, there holds the energy inequality for any constant $\bar{\varrho}$:

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{v}|^2 + \frac{1}{r_*} (Q(\varrho) - Q(\bar{\varrho}) - Q'(\bar{\varrho})(\varrho - \bar{\varrho})) \right) (\tau, \cdot) \, dx \\
&\quad + \int_0^\tau \int_\Omega 2 |\mathbf{D}^d(\mathbf{v})|^2 + \frac{1}{r_*^2} |\mathbf{H}^\perp[\mathbf{v}]|^2 \, dx \, dt \qquad (9) \\
&\leq \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{v}_0|^2 + \frac{1}{r_*} (Q(\varrho_0) - Q(\bar{\varrho}) - Q'(\bar{\varrho})(\varrho_0 - \bar{\varrho})) \right) dx.
\end{aligned}
$$

Energy inequality

By the continuity equation and the Drichlet boundary condition

$$\int_{\Omega} \varrho(\tau, \cdot)\, dx = \int_{\Omega} \varrho_0\, dx, \quad \forall \tau \in [0, T].$$

Then (9) is equivalent to

$$\int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{v}|^2 + \frac{1}{r_*} Q(\varrho) \right) (\tau, \cdot)\, dx + \int_0^{\tau} \int_{\Omega} 2 |\mathbf{D}^d(\mathbf{v})|^2 + \frac{1}{r_*^2} |\mathbf{H}^{\perp}[\mathbf{v}]|^2\, dx\, dt$$
$$\leq \int_{\Omega} \left(\frac{1}{2} \varrho_0 |\mathbf{v}_0|^2 + \frac{1}{r_*} Q(\varrho_0) \right) dx.$$

However, it is more convenient to use (9) in studying the asymptotic behavior of the solution $\mathbf{v}$ as $r_* \to 0$.

Existence of weak solution

Theorem

Suppose for some $\beta_0 > 5/2$:

$$\liminf_{\varrho \to \frac{1}{1+r_*}+} \left| q(\varrho)(\varrho - \frac{1}{1+r_*})^{\beta_0} \right| > 0, \quad \liminf_{\varrho \to 1-} \left| q(\varrho)(1-\varrho)^{\beta_0} \right| > 0. \tag{10}$$

Then there exists a finite energy global weak solution $[\varrho, \mathbf{v}]$.

Proof–Approximate solutions

We employ the idea by Feireisl and Zhang [2]

Regularized pressure for $\alpha > 0$ small and $\gamma > 3/2$ large:

$$q_\alpha(\varrho) := \begin{cases} q(\frac{1}{1+r_*}+\alpha), & \varrho \le \frac{1}{1+r_*}+\alpha, \\ q(\varrho), & \frac{1}{1+r_*}+\alpha \le \varrho \le 1-\alpha, \\ q(1-\alpha)+(\varrho-2)_+^\gamma, & \varrho \ge 1-\alpha. \end{cases}$$

By replacing the pressure term q by q_α in (4), we obtain an approximate system:

$$\partial_t \varrho_\alpha + \mathrm{div}(\varrho_\alpha \mathbf{v}_\alpha) = 0,$$
$$\partial_t(\varrho_\alpha \mathbf{v}_\alpha) + \mathrm{div}(\varrho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha) + \frac{1}{r_*}\nabla_x q_\alpha(\varrho_\alpha) = 2\mathrm{div}\mathbf{D}^d(\mathbf{v}_\alpha) - \frac{1}{r_*^2}\mathbf{H}^\perp[\mathbf{v}_\alpha].$$

Proof–Approximate solutions

Existence for approximate system

Armed with same initial data and boundary condition, the global existence of weak solution $[\varrho_\alpha, \mathbf{v}_\alpha]$ to this approximate system is known (see [5] and [1]).

Uniform bound

$$\{\varrho_\alpha|\mathbf{v}_\alpha|^2\}_{0<\alpha<\alpha_0} \text{ bounded in } L^\infty(0,T;L^1(\Omega;\mathbb{R}^3)),$$
$$\{Q_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0} \text{ bounded in } L^\infty(0,T;L^1(\Omega;\mathbb{R}^3)),$$
$$\{\mathbf{v}_\alpha\}_{0<\alpha<\alpha_0} \text{ bounded in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)).$$

Proof–passing to the limit

Weak convergence

$$\varrho_\alpha \to \varrho \text{ weakly(*) in } L^\infty(0,T;L^\gamma(\Omega;\mathbb{R}^3)),$$
$$\mathbf{v}_\alpha \to \mathbf{v} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)).$$

For the nonlinear terms in the approximate system (at least in the sense of distribution):

$$\varrho_\alpha \mathbf{v}_\alpha \to \varrho \mathbf{v}, \quad \varrho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha \to \varrho \mathbf{v} \otimes \mathbf{v}.$$

The difficulty is to show

$$q_\alpha(\varrho_\alpha) \to q(\varrho) \text{ weakly in } L^1?$$

Proof–Uniform bound for the pressure

Uniform bound of $Q_\alpha(\varrho_\alpha)$ does not imply uniform bound for $q_\alpha(\varrho_\alpha)$, not even in $L^1((0,T)\times\Omega)$.

Indeed, for singular $q(\varrho)$ such that

$$q(\varrho) = O\Big(\frac{1}{(\varrho - \frac{1}{1+r_*})^{\beta_1}}\Big) \quad \text{near} \quad \frac{1}{1+r_*}+,$$
$$q(\varrho) = O\Big(\frac{1}{(1-\varrho)^{\beta_2}}\Big) \quad \text{near} \quad 1-,$$

where β_1 and β_2 are numbers larger than $5/2$, the pressure potential functional Q is less singular:

$$Q(\varrho) = O\Big(\frac{1}{(\varrho - \frac{1}{1+r_*})^{\beta_1-1}}\Big) \quad \text{near} \quad \frac{1}{1+r_*}+,$$
$$Q(\varrho) = O\Big(\frac{1}{(1-\varrho)^{\beta_2-1}}\Big) \quad \text{near} \quad 1- .$$

Proof–Uniform L^1 estimate of the pressure

To show the integrability of $q_\alpha(\varrho_\alpha)$, introduce

$$\varphi = \psi(t)\mathcal{B}(\varrho_\alpha - \langle \varrho_\alpha \rangle), \quad \langle \varrho_\alpha \rangle := \frac{1}{|\Omega|}\int_\Omega \varrho_\alpha \, dx,$$

with $\psi \in C_c^\infty(0, T)$ and $\mathcal{B}$ a bounded linear operator from $\{g \in L^p(\Omega), \quad \langle g \rangle = 0\}$ to $W_0^{1,p}(\Omega; \mathbb{R}^3)$ for $1 < p < \infty$ such that

$$\mathrm{div}\mathcal{B}(g) = g, \quad \mathcal{B}(g)|_{\partial\Omega} = 0.$$

Taking φ as a test function implies

$$\{q_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0} \text{ bounded in } L^1((0, T) \times \Omega).$$

Proof–Equi-integrability of the pressure

Introduce

$$\varphi = \psi(t)\mathcal{B}(\eta_\alpha(\varrho_\alpha) - \langle \eta_\alpha(\varrho_\alpha)\rangle),$$

where $\psi \in C_c^\infty(0, T)$ and

$$\eta_\alpha(s) = \begin{cases} \log(s - \frac{1}{1+r*}) - \log(1-s), & \frac{1}{1+r_*} + \alpha \le s \le 1-\alpha, \\ -\log(\alpha), & s \ge 1-\alpha, \\ \log(\alpha), & s \le \frac{1}{1+r_*} - \alpha. \end{cases}$$

Taking φ as a test function implies

$$\{q_\alpha(\varrho_\alpha)\eta_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0} \quad \text{bounded in} \quad L^1((0,T)\times\Omega).$$

This gives the equi-integrability of the pressure. Then

$$q_\alpha(\varrho_\alpha) \to \overline{q(\varrho)} \text{ weakly in } L^1((0,T)\times\Omega)).$$

Proof–Growth assumption for the pressure-1

The growth assumption (10) is to control

$$I := \int_0^T \int_\Omega \psi \varrho_\alpha \mathbf{v}_\alpha \mathcal{B}\left(\eta_\alpha'(\varrho_\alpha)\varrho_\alpha \mathrm{div}\mathbf{v}_\alpha - \langle \eta_\alpha'(\varrho_\alpha)\varrho_\alpha \mathrm{div}\mathbf{v}_\alpha \rangle\right)\, dx\, dt.$$

A direct fact is

$$Q_\alpha(\varrho_\alpha) \geq c_1 |\eta_\alpha'(\varrho_\alpha)|^{\beta_0 - 1} - c_2, \quad |q_\alpha(\varrho_\alpha)| \geq c_1 |\eta_\alpha'(\varrho_\alpha)|^{\beta_0} - c_2.$$

Then

$$\eta_\alpha'(\varrho_\alpha) \text{ uniformy bounded in } L^\infty L^{\beta_0} \cap L^{\beta_0} L^{\beta_0}.$$

We need $\beta_0 > 5/2$ to get the uniform bound of I.

Proof–Growth assumption for the pressure-2

Recall

$$\varrho_\alpha \in L^\infty L^{\infty-}, \quad \sqrt{\varrho_\alpha}\mathbf{v}_\alpha \in L^\infty L^2, \quad \mathbf{v}_\alpha \in L^2 W_0^{1,2}.$$

We choose $\beta_0 > 5/2$, then

$$\eta_\alpha'(\varrho_\alpha)\varrho_\alpha \mathrm{div}\mathbf{v}_\alpha \in L^{\frac{10}{9}+}(L^{\frac{10}{9}+})$$

which implies

$$\mathcal{B}\left(\eta_\alpha'(\varrho_\alpha)\varrho_\alpha \mathrm{div}\mathbf{v}_\alpha - \langle \eta_\alpha'(\varrho_\alpha)\varrho_\alpha \mathrm{div}\mathbf{v}_\alpha \rangle\right) \in L^{\frac{10}{9}+}(W_0^{1,\frac{10}{9}+})$$

and Sobolev embedding gives

$$\mathcal{B}\left(\eta_\alpha'(\varrho_\alpha)\varrho_\alpha \mathrm{div}\mathbf{v}_\alpha - \langle \eta_\alpha'(\varrho_\alpha)\varrho_\alpha \mathrm{div}\mathbf{v}_\alpha \rangle\right) \in L^{\frac{10}{9}+}(L^{\frac{30}{17}+}).$$

Proof–Growth assumption for the pressure-3

On the other hand

$$\{\varrho_\alpha \mathbf{v}_\alpha\}_{0<\alpha<\alpha_0} \quad \text{bounded in} \quad L^\infty(L^{2-}) \cap L^2(L^{6-}).$$

By interpolation,

$$\{\varrho_\alpha \mathbf{v}_\alpha\}_{0<\alpha<\alpha_0} \quad \text{bounded in} \quad L^{10}(L^{\frac{30}{13}-}).$$

Then the quantity I is uniformly bounded with respect to α.

Proof–Strong convergence of density

By employing the arguments of Lions [3], we can obtain

$$\int_\Omega \overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \leq 0.$$

This imples

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho),$$

and furthermore

$$\varrho_\alpha \to \varrho \ a.e \text{ in } (0, T) \times \Omega.$$

Then

$$\overline{q(\varrho)} = q(\varrho).$$

Convergence for $r_* = \varepsilon \to 0$

The first observation is that, unconditionally,

$$\sup_{t \in (0,T)} \|\varrho_\varepsilon(\tau, \cdot) - 1\|_{L^\infty(\Omega)} \leq \varepsilon. \tag{11}$$

We then assume initial data satisfies

$$\|\mathbf{v}_{0,\varepsilon}\|_{L^2(\Omega;\mathbb{R}^3)} \leq c, \quad \frac{1}{|\Omega|} \int_\Omega Q(\varrho_{0,\varepsilon}) \, dx \leq Q(\overline{\varrho}_{0,\varepsilon}) + \varepsilon c.$$

This implies the right-hand side of energy inequality is uniformly bounded. Then we have uniform bound for the solution $[\varrho_\varepsilon, \mathbf{v}_\varepsilon]$.

Then

$\mathbf{v}_\varepsilon \to \mathbf{U}$ weakly in $L^2(0,T;W_0^{1,2}(\Omega)$ and weakly-* in $L^\infty(0,T;L^2(\Omega))$

and moreover,

$$\mathbf{H}^\perp[\mathbf{v}_\varepsilon] \to 0 \text{ in } L^2((0,T)\times\Omega).$$

For any $\varphi \in C_c^1(\Omega;\mathbb{R}^3),\ \mathrm{div}\ \varphi = 0,$

$$t \mapsto \int_\Omega \mathbf{v}_\varepsilon \cdot \varphi \, dx \text{ precompact in } C([0,T]). \tag{12}$$

Then, by means of the standard Lions-Aubin argument,

$$\mathbf{H}[\mathbf{v}_\varepsilon] \to \mathbf{U} \text{ in } L^2((0,T)\times\Omega;\mathbb{R}^3), \tag{13}$$

From quasi-incompressible to incompressible fluids

Consequently

$$\mathbf{v}_\varepsilon \to \mathbf{U} \text{ in } L^2((0,T)\times\Omega;\mathbb{R}^3),$$

where $\mathbf{U}$ is a *weak* solution to the incompressible Navier-Stokes system

$$\mathrm{div}\mathbf{U} = 0, \tag{14}$$

$$\partial_t \mathbf{U} + \nabla\mathbf{U}\cdot\mathbf{U} + \nabla P = \Delta\mathbf{U}, \ \mathbf{U}|_{\partial\Omega} = 0, \tag{15}$$

supplemented with the initial condition

$$\mathbf{U}(0,\cdot) = \mathbf{H}[\mathbf{v}_0], \tag{16}$$

where $\mathbf{v}_0$ is a weak limit of $\mathbf{v}_{0,\varepsilon}$ in $L^2(\Omega;\mathbb{R}^3)$.

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Thank you for your attention!