On power-law fluids with the power-law index proportional to the pressure

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joint work with J. Málek and K.R. Rajagopal

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- Cauchy stress tensor

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classical Navier-Stokes fluids	$\mu \equiv \mu_0 > 0$
generalized Navier-Stokes fluids	$\mu = \mu(p, \varrho, \theta, c, \lvert\boldsymbol{D}\boldsymbol{v}\rvert^2, \ldots)$
our case	$\mu = \mu(p, \lvert\boldsymbol{D}\boldsymbol{v}\rvert^2)$

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$\Longrightarrow$ We consider a pressure- and shear-rate-dependent viscosity.

Shear-rate-dependent viscosity

Viewed purely as a function of the shear rate, i.e.

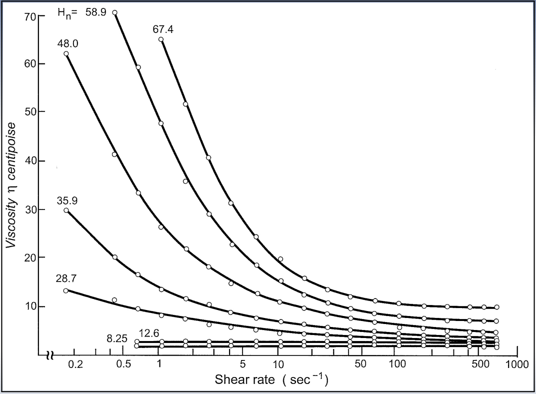
$$\mu = \mu(|\boldsymbol{D}\boldsymbol{v}|^2),$$

the most popular relation characterizes the power-law fluids:

$$\mu(|\boldsymbol{D}\boldsymbol{v}|^2) = \mu_0(1 + |\boldsymbol{D}\boldsymbol{v}|^2)^{(r-2)/2}, \quad \mu_0 > 0, r > 1,$$

where r is the power-law index that itself can vary with the concentration or the pressure.

Digression: $r = r(c)$



A shear-thinning experiment on synovial fluid over a wide range of physiological concentration of hyaluronan. Viscosity vs. the shear rate. (Petra Pustějovská, PhD Thesis, 2012)

Power-law viscosity

Power-law relation

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$r > 2$: Shear-thickening/dilatant behavior: Viscosity goes up with relative deformation, e.g. **cornstarch in water** (oobleck).

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- In the studies concerned with $\mu = \mu(p, |\boldsymbol{Dv}|^2)$, the viscosity has generally satisfied
  1. $|\mu(p, |\boldsymbol{Dv}|^2)| \leq C(|\boldsymbol{Dv}|^2)$ for any p,
  2. $\mu(p, |\boldsymbol{Dv}|^2) = \mu_1(p) \cdot \mu_2(|\boldsymbol{Dv}|^2)$,

  both of which are physically lacking.

$\Longrightarrow$ The existing theory for $\mu(p, |\boldsymbol{Dv}|^2)$ leaves a lot to be desired.

Our viscosity

Wishing to combine the power law and Barus' law, i.e.

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$$\mu(p, |\boldsymbol{Dv}|^2) = \mu_0 \exp\left(\frac{p-2}{2}\ln(1 + |\boldsymbol{Dv}|^2)\right) = \mu_0(1 + |\boldsymbol{Dv}|^2)^{(p-2)/2}, \quad \mu_0 > 0.$$

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$\implies$ A power-law fluid with pressure as the power-law exponent

Our equations

With the viscosity of the form

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the investigated system becomes

$$\varrho\,(\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})) - \operatorname{div}\left[\mu_0 (1 + |\boldsymbol{D}\boldsymbol{v}|^2)^{(p-2)/2} \boldsymbol{D}\boldsymbol{v}\right] + \nabla p = \varrho \boldsymbol{f},$$
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No chance to develop existence theory for this level of generality!

First glance

$$\varrho\,(\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})) - \operatorname{div}\left[\mu_0(1 + |\boldsymbol{D}\boldsymbol{v}|^2)^{(p-2)/2}\boldsymbol{D}\boldsymbol{v}\right] + \nabla p = \varrho \boldsymbol{f},$$
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- Mathematical tools have not been developed yet to tackle that problem.
- Therefore we consider a simplification, which allows us to compute the pressure explicitly making the power-law exponent still non-constant but at least known!

Simplification

- Let $\Omega := \mathbb{R}^2 \times (0, d)$ be a layer of a given depth $d > 0$.

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- Denoting the Cartesian coordinates in $\mathbb{R}^3$ by x, y and z, we will further assume that
$$\boldsymbol{v} = \boldsymbol{v}(t, z).$$
- Let the boundary and initial conditions be of the form
$$\left.\begin{aligned} \boldsymbol{v}(0, z) &= (f(z), 0, 0), \\ \boldsymbol{v}(t, 0) &= (g_0(t), 0, 0), \\ \boldsymbol{v}(t, d) &= (g_d(t), 0, 0), \\ p(t, d) &= p_0, \end{aligned}\right\}$$
for every $t > 0$, $z \in \mathbb{R}$, some smooth functions f, g_0, g_d and a constant reference pressure p_0.

Solution

Lemma

Assume that there is a smooth solution $\boldsymbol{v} = (u, v, w)$ *and* p *to our simplified problem. Then*

$$v \equiv w \equiv 0 \quad \textit{and} \quad p(t, x, y, z) = p_0 + \varrho g(d - z).$$

Proof of the lemma

- Solenoidality and the simplifying assumptions read

$$
\begin{aligned}
(u, v, w) &= \boldsymbol{v}(t, z), \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= \operatorname{div} \boldsymbol{v} = 0, \\
\Rightarrow \frac{\partial w}{\partial z} &= 0.
\end{aligned}
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Therefore $w = w(t)$.

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- $\boldsymbol{v}(t, 0) = (g_0(t), 0, 0)$ then implies

$$w(t) = w(t, 0) = 0 \quad \text{for all } t.$$

Proof of the lemma

- We then observe

$$\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = \frac{\partial}{\partial x} \begin{pmatrix} u \cdot u \\ v \cdot u \\ w \cdot u \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} u \cdot v \\ v \cdot v \\ w \cdot v \end{pmatrix} + \frac{\partial}{\partial z} \begin{pmatrix} u \cdot w \\ v \cdot w \\ w \cdot w \end{pmatrix} = \mathbf{0}$$

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- It also holds that

$$\boldsymbol{D}\boldsymbol{v} = \frac{1}{2}\begin{pmatrix} 0 & 0 & \frac{\partial u}{\partial z} \\ 0 & 0 & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & 0 \end{pmatrix}$$

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- Our system

$$\varrho \partial_t \boldsymbol{v} = -\varrho \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \nabla p + 2 \operatorname{div}(\mu(p, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v}) + \varrho \boldsymbol{f}$$

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can therefore be written as

$$\varrho\,\partial_t u = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\mu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\frac{\partial u}{\partial z}\right)$$

$$\varrho\,\partial_t v = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(\mu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\frac{\partial v}{\partial z}\right)$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\mu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left(\mu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\frac{\partial v}{\partial z}\right) - \varrho g$$

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- But then

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)p = 0.$$

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- Poisson's formula therefore yields

$$p(t, x, y, d - z) = \frac{1}{2\pi z^2} \int_{B_t(x',z)} \frac{p_0 z + \varrho g z^2}{(z^2 - |x' - \bar{x}'|^2)^{1/2}} \, d\bar{x}' = p_0 + \varrho g z,$$

so that

$$\boxed{p(z) = p_0 + \varrho g(d - z).}$$

Proof of the lemma

- Since $p = p(z)$, from the equation

$$\varrho\,\partial_t v = \underbrace{-\frac{\partial p}{\partial y}}_{=0} + \frac{\partial}{\partial z}\left(\mu(p, |\boldsymbol{D}\boldsymbol{v}|^2)\frac{\partial v}{\partial z}\right),$$

we deduce

$$\frac{\varrho}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(0,d)} = -\int_0^d \mu(p, |\boldsymbol{D}\boldsymbol{v}|^2)\left|\frac{\partial v}{\partial z}\right|^2 dz \le 0,$$

leading to $v \equiv 0$, because $\boldsymbol{v}(0) = (f(z), 0, 0) = (u(0), v(0), w(0))$. □

Implications

Due to the lemma, the original equation

$$\varrho\partial_t \boldsymbol{v} = -\varrho\,\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \nabla p + 2\,\mathrm{div}(\mu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v}) + \varrho\boldsymbol{f}$$

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becomes

$$\begin{aligned}
\varrho\, \partial_t u &= 2 \frac{\partial}{\partial z} \left(\mu(p, |\partial_z u|^2)\partial_z u\right) && \text{in } (0,\infty) \times (0,d),\\
p &= p_0 + \varrho g(d - z) && \text{in } (0,\infty) \times (0,d),\\
u(0,z) &= f(z) && \text{in } (0,d),\\
u(t,0) &= g_0(t) && \text{in } (0,\infty),\\
u(t,d) &= g_d(t) && \text{in } (0,\infty).
\end{aligned}$$

That is, a PDE for a scalar function of one spatial and one temporal variable.

Existence for the original problem

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Theorem

Let us further assume $p_0 > 1$ so that $\inf_z p(z) > 1$. There is a unique weak solution to the above equation, i.e. a function u satisfying

$$u \in L^\infty(0,T;L^2(0,d)) \cap L^{p(\cdot)}(0,T;W^{1,p(\cdot)}(0,d)),$$

$$\partial_t u \in \left(L^{p(\cdot)}(0,T;W^{1,p(\cdot)}(0,d))\right)^*,$$

$$\lim_{t\to 0_+} \|u(t) - f\|_{L^2(0,d)} = 0$$

and solving the balance equation in the sense of distribution.

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- However, a major detour from the standard theory are the Lebesgue and Sobolev spaces with variable exponents, i.e. $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$, respectively.
- When the power law exponent $p(\cdot)$ satisfies the so-called log-Hölder condition, in particular when it is linear like here, then the resulting function spaces with a variable exponent behave much like their standard counterparts with respect to reflexivity, separability, density of smooth functions etc.

Thank you for your attention!